

Online Appendix for “Up then Down: The Bid-Price Trends in Revenue Management”

The online appendix provides detailed analysis for several extensions and discussions summarized in the conclusion section.

Appendix A: Heuristic Bid-Price Trends

It is known that fixed-price heuristics of the deterministic fluid model can provide an upper bound on the expected revenue of the stochastic model with time-invariant customer valuations, and the approximation given by the heuristics is provably good when the volumes of expected sales and capacity are large (Gallego and van Ryzin 1994). It is natural to ask the question whether the bid-price processes under the heuristics also have the same trends as that of optimal policies.

For simplicity, we assume that the consumer valuation distribution is time-invariant. For convenience we drop time subscription from r_t , Φ_t and ψ_t . As shown by Gallego and van Ryzin (1994), the optimal fixed-price policy can be obtained by solving the following nonlinear program (NLP):

$$\begin{aligned} J(t, x) &= \max_{d \in [0,1]} a(t)r(d) \\ \text{subject to } &a(t)d \leq x, \end{aligned}$$

where $a(t) = \int_t^T \Lambda_s ds$ which is a decreasing function in t .

The optimal price, denoted by $\hat{p}(t, x)$, and the associated bid price (Lagrange multiplier), denoted by $\gamma(t, x)$, can be obtained by solving the associated Lagrange problem:

$$\min_{\gamma \geq 0} \max_{d \in [0,1]} \{a(t)r(d) - \gamma(a(t)d - x)\}. \quad (1)$$

Solving the problem yields $\hat{d}(t, x) = \min(d^*, x/a(t))$ (i.e., $\hat{p}(t, x) = \max(\psi(d^*), \psi(x/a(t)))$) and $\gamma(t, x) = \max(r'(x/a(t)), 0)$, where $d^* = \arg \max_d r(d)$. Since $a(t)$ is decreasing in t , $\hat{d}(t, x)$ is increasing in t , and r is concave (i.e., r' is a decreasing function), we know that $\gamma(t, x)$ is decreasing in t and x .

Williamson (1992) uses simulation to show that the performance of the static heuristic can be enhanced by frequent re-optimization. In practice RM systems often re-optimize the heuristic bid prices frequently to approximate the optimal bid prices. That is, approximate the optimal bid price $\Delta V(t, n)$ with the heuristic bid price $\gamma(t, n)$ for all (t, n) instead of sticking to the initial static bid price $\gamma(0, C)$ throughout the selling horizon. Then, in a stochastic system, as time goes by, the inventory state changes and the heuristic bid price $\gamma(t, N(t))$ also varies over time. It is interesting to know whether the resulting heuristic bid-price processes have the similar trends as that of the optimal bid-price processes.

We now investigate the trends of the re-optimization heuristic bid-price process.

PROPOSITION 1 (HEURISTIC BID-PRICE TRENDS). *The heuristic bid-price process satisfies the following properties.*

- (a) If $r'''(x) \geq 0$ for all x , then for $N(t) \geq 2$ and $t < s < T$, $\gamma(t, N(t)) \leq E[\gamma(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t]$.
- (b) For $N(t) = 1$ and $t < s < T$, $\gamma(t, N(t)) \geq E[\gamma(s, N(s)) | \mathcal{F}_t]$

Proof. Note that $\gamma(t, n) = r'(\hat{d}(t, n))$ where $\hat{d}(t, n) = \min(n/a(t), d^*)$. Clearly, $\gamma(t, n)$ is a constant when $n/a(t) \geq d^*$, and $\gamma(t, n)$ is differentiable in t when $n/a(t) < d^*$ and $\frac{d}{dt}\gamma(t, n) = r''(n/a(t)) \frac{\Lambda_t n}{a_t^2}$. For any $n \geq 2$, define operator \mathcal{L}_r such that

$$\mathcal{L}_r \gamma(t, n) = r''(n/a(t)) \frac{\Lambda_t n}{a(t)^2} 1_{\{n/a(t) < d^*\}} - \Lambda_t \hat{d}(t, n) [r'(\hat{d}(t, n)) - r'(\hat{d}(t, n-1))].$$

Since $r''' \geq 0$, i.e., r' is convex, we have $r'(n/a(t)) - r'((n-1)/a(t)) \leq r''(n/a(t))/a(t)$. Hence, if $n/a(t) < d^*$, we have

$$\begin{aligned} \mathcal{L}_r \gamma(t, n) &= r''(n/a(t)) \frac{\Lambda_t n}{a(t)^2} - \Lambda_t (n/a(t)) [r'(\hat{d}(t, n)) - r'(\hat{d}(t, n-1))] \\ &= \frac{\Lambda_t n}{a(t)} [r''(n/a(t))/a(t) - [r'(\hat{d}(t, n)) - r'(\hat{d}(t, n-1))]] \geq 0. \end{aligned}$$

If $n/a(t) \geq d^*$, we have

$$\mathcal{L}_r \gamma(t, n) = -\Lambda_t d^* [r'(d^*) - r'(\min((n-1)/a(t), d^*))] \geq 0.$$

On the other hand, since r' and r'' are bounded, $\int_0^{s \wedge \tau_1} \mathcal{L}_r \gamma(\xi, N(\xi)) d\xi$ is well-defined. It follows from Dynkin's Lemma (Rogers and Williams 1987) and optional sampling theorem of martingale (Karatzas and Shreve 1988) that $\gamma(s \wedge \tau_1, N(s \wedge \tau_1)) - \int_0^{s \wedge \tau_1} \mathcal{L}_r \gamma(\xi, N(\xi)) d\xi$ is an \mathcal{F}_t -martingale. Then, if $r''' \geq 0$, $E[\gamma(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t]$ is increasing in s ; if $r''' \leq 0$, $E[\gamma(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t]$ is decreasing in s .

For $N(t) = 1$ and $t < s < T$, $\gamma(t, N(t)) \geq \gamma(s, N(s))$. Then, $\gamma(t, N(t)) \geq E[\gamma(s, N(s)) | \mathcal{F}_t]$. Q.E.D.

Proposition 1 shows that when $r'''(x) \geq 0$, the heuristic bid-price process has the same trend as that of the optimal bid-price process: Moves upward before the inventory level falls to one and then moves downward. Note that the downward trend after the inventory level falls to one does not depend on the condition.

PROPOSITION 2 (HEURISTIC PRICE TRENDS). *The heuristic price process satisfies the following properties.*

- (a) If ψ is convex, then for $N(t) \geq 2$ and $t \leq s \leq T$, $\hat{p}(t, N(t)) \leq E[\hat{p}(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t]$.
- (b) For $N(t) = 1$ and $t < s < T$, $\hat{p}(t, N(t)) \geq E[\hat{p}(s, N(s)) | \mathcal{F}_t]$.

Proof. For $n \geq 2$, define operator \mathcal{L}_h such that

$$\mathcal{L}_h \hat{p}(t, n/a(t)) = \psi'(n/a(t)) \frac{\Lambda_t n}{a(t)^2} 1_{\{n/a(t) < d^*\}} - \Lambda_t \hat{d}(t, n) [\hat{p}(t, n/a(t)) - \hat{p}(t, (n-1)/a(t))].$$

When ψ is convex, i.e., $\psi'' \geq 0$, we have $\psi(n/a(t)) - \psi((n-1)/a(t)) \leq \psi'(n/a(t))/a(t)$. Then, if $n/a(t) < d^*$, we have

$$\mathcal{L}_h \hat{p}(t, n/a(t)) = \psi'(n/a(t)) \frac{\Lambda_t n}{a(t)^2} - \frac{\Lambda_t n}{a(t)} [\psi(n/a(t)) - \psi((n-1)/a(t))] \geq 0.$$

If $n/a(t) \geq d^*$, we have

$$\mathcal{L}_h \hat{p}(t, n/a(t)) = -\Lambda d^* [\psi(d^*) - \psi(\hat{d}(t, n))] \geq 0,$$

where the inequality is due to the fact that ψ is decreasing and $d^* \geq \hat{d}(t, n)$.

Applying Dynkin's formula and the optional sampling theorem of martingale, we know that $E[\hat{p}(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t]$ is increasing in s if ψ is convex and for any $t \leq s \leq T$,

$$\hat{p}(t) \leq E[\hat{p}(s \wedge \tau_1, N(s \wedge \tau_1)) | \mathcal{F}_t].$$

For $N(t) = 1$ and $t < s < T$, $\hat{p}(t, N(t)) \geq \hat{p}(s, N(s))$. Then, $\hat{p}(t, N(t)) \geq E[\hat{p}(s, N(s)) | \mathcal{F}_t]$. Q.E.D.

Proposition 2 shows that, when ψ is convex, the heuristic price exhibits exactly the same pattern as that of the heuristic bid price when ψ is convex: the heuristic price process has an upward trend before the inventory level falls to one and then moves downward on expectation, which is consistent with the pattern of the heuristic bid-price process. Note that the downward trend after the inventory level falls to one does not depend on the convexity of ψ .

The following table examines the conditions of Propositions 1 and 2 for some common demand models. One can observe that the linear, exponential and iso-elastic demand models satisfy the sufficient conditions of part (a) in Propositions 1 and 2.

Table 1: Some Common Demand Models ($a > 0, b > 0$)

Demand model	$\bar{\Phi}(p)$	$\psi(d)$	$\psi''(d)$	$r(d)$	$r'''(d)$
linear	$a - bp$	$(a - d)/b$	0	$(a - d)d/b$	0
exponential	e^{a-bp}	$\frac{a - \ln(d)}{b}$	$1/(bd^2)$	$\frac{(a - \ln(d))d}{b}$	$\frac{1}{bd^2}$
iso-elastic ($b > 1$)	ap^{-b}	$(d/a)^{-1/b}$	$a^{1/b} \frac{b+1}{b^2} d^{-1/b-2}$	$d(d/a)^{-1/b}$	$\frac{a^{1/b}}{b} \left(1 - \frac{1}{b^2}\right) d^{-2-1/b}$

Appendix B: Fluid Approximation From A Martingale Perspective

Xu and Hopp (2006) provide a closed-form solution in a stochastic flow RM model with an iso-elastic demand function and show that the optimal-price process is a martingale. It is not hard to show that with an iso-elastic demand function (i.e., $\lambda_t(p) = \Lambda_t p^{-\alpha}$, $\alpha > 1$) the optimal price is linear in the bid price (i.e., $P(t) = \frac{\alpha}{\alpha-1} B(t)$, where $B(t)$ is the bid price). This implies that the bid price is also a martingale in Xu and Hopp's model. They argue that the martingale optimal-price process in the stochastic fluid model (Xu and Hopp 2006) and the submartingale price process in the discrete state system (Xu and Hopp 2009) are due to the nature of the customer arrivals: on the one hand, because the flow demand model with the customer arrival rate being a geometric Brownian motion always provides a positive customer flow, the seller can count on future customer arrivals; on the other hand, with Poisson demand arrivals, the seller is not certain that there will be incoming customers, and hence tends to set price lower at the beginning of the selling season. However, this cannot explain the downward trend of the price process as shown in Xu and Hopp (2009). In the following discussion, we provide an alternative interpretation from the perspective of bid prices.

Note that bid price or shadow price measures the marginal loss or opportunity cost of reducing one unit of inventory. When the inventory level at time t is n , the marginal contribution of having one more unit of inventory can be measured by $\Delta V(t, n+1) = V(t, n+1) - V(t, n)$. The next proposition characterizes the trend of the marginal contribution of having one more unit of inventory.

PROPOSITION 3. *For any $t \in [0, T)$ and $s \in (t, T]$, $\Delta V(t, N(t) + 1) \geq E[\Delta V(s \wedge \tau_0, N(s \wedge \tau_0) + 1) | \mathcal{F}_t]$.*

Proof. From the HJB equation (1), for any $n \geq 1$, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V(t, n) + \Lambda_t d(t, n) [\psi(d(t, n)) - \Delta V(t, n)], \\ 0 &= \frac{\partial}{\partial t} V(t, n+1) + \Lambda_t d(t, n+1) [\psi(d(t, n+1)) - \Delta V(t, n+1)] \\ &\geq \frac{\partial}{\partial t} V(t, n+1) + \Lambda_t d(t, n) [\psi(d(t, n)) - \Delta V(t, n+1)]. \end{aligned}$$

Subtracting (5) from (4) yields

$$0 \geq \frac{\partial}{\partial t} \Delta V(t, n+1) + \Lambda_t d(t, n) [\Delta V(t, n) - \Delta V(t, n+1)].$$

Using the similar arguments in the proof of Theorem 1, we can show that $E[\Delta V(s \wedge \tau_0, N(s \wedge \tau_0) + 1) | \mathcal{F}_t]$ is a supermartingale. Q.E.D.

This proposition shows that the marginal contribution of having one more unit of inventory has a downward trend. This implies that the marginal value of having more inventories declines over time. Combining Theorem 1 and Proposition 3, we can provide an alternative explanation for the fact that the bid-price process in a fluid approximation model becomes a martingale.

Our analysis can be readily extended to the setting with the arrival rate driven by an exogenous Markov process (e.g., geometric Brownian motion). The corresponding fluid approximation is a stochastic flow model (e.g., Xu and Hopp 2006 and Akan and Ata 2009). Let s be the state of the exogenous Markov process $S(t)$, x be the continuous inventory level, and $V(t, x, s)$ be the expected revenue function. Suppose $V(t, x, s)$ is differentiable in x . Then the bid price at time t is $\frac{\partial}{\partial x} V(t, x, s) = \lim_{\delta \rightarrow 0} \frac{V(t, x, s) - V(t, x - \delta, s)}{\delta} = \lim_{\delta \rightarrow 0} \frac{V(t, x + \delta, s) - V(t, x, s)}{\delta}$. As an implication of Theorem 1 and Proposition 3, we can show that the bid-price process $\frac{\partial}{\partial x} V(t, X(t), S(t))$ must be a martingale, where $X(t)$ is the inventory process with $X(0) = C$. This implies that it is the discrete nature of the capacity in RM systems that drives the non-martingale structure of the bid-price processes.

Note that Xu and Hopp (2006) in fact consider an iso-elastic demand model with a constant price elasticity and hence the optimal price is linear in the bid price, which implies that the optimal-price process is also a martingale. Similarly, one can easily see that when the valuation distribution is exponential with a constant mean, the optimal-price process must also be martingale. However, in general, such a martingale property does not hold as the optimal price may be nonlinear in the bid price and the valuation distribution may be time varying. Using the analysis in the proof Theorem 2 and the martingale property of the bid-price process, we can show that if the Mills ratio $m_t(p)$ is convex in p and $h_t(p)$ is increasing in t then the optimal-price process has an upward trend; if the Mills ratio $m_t(p)$ is concave in p and $h_t(p)$ is decreasing in t then the optimal-price process has a downward trend.

Appendix C: Capacity Rationing Models

In capacity rationing models, also called quantity-based RM models (see, e.g., Lautenbacher and Stidham 1999 and Feng and Xiao 2001), customers are segmented into several distinct fare classes with predetermined prices (fares). An incoming booking request of a fare class is accepted only if its fare exceeds the bid-price (or opportunity cost) of the remaining inventory. This type of control is also called bid price control.

More specifically, we assume that there are K fare classes, indexed by $i = 1, \dots, K$. The fare of class k is p_k which is fixed during the selling season. Without loss of generality, we assume that $0 < p_1 < p_2 < \dots < p_K$. Customer arrivals of fare class k follow a non-homogenous Poisson process whose intensity at t is $\lambda_t^k > 0$. The control is represented by $u = (u_1, \dots, u_K)$ where

$$u_k = \begin{cases} 1 & \text{if booking request of class } k \text{ is accepted,} \\ 0 & \text{otherwise.} \end{cases}$$

The unit sales revenue of class k under a policy u is denoted by $r_t^k(u)$. Then, $r_t^k(u) = \begin{cases} p_k & \text{if } u_k = 1, \\ 0 & \text{if } u_k = 0. \end{cases}$

It follows by Feng and Xiao (2001) that the optimal revenue function, $V(t, n)$, satisfies the following HJB equation:

$$0 = \frac{\partial}{\partial t} V(t, n) + \sum_{k=1}^K \lambda_t^k [p_k - \Delta V(t, n)]^+, \quad n \geq 0, \quad (2)$$

with the boundary conditions $V(t, 0) = 0$ and $V(T, n) = 0$ for all t and n . The optimal policy is that an incoming booking request of class k is accepted if and only if the corresponding fare is greater than the bid price, i.e., $p_k > \Delta V(t, n)$. Note that the fare structure is nested, hence if a request of class $k < K$ is accepted, then all the requests from classes with higher fares should be accepted. By Theorem 1, the optimal policy tends to reject more lower fare classes as time goes by before the inventory level falls to one but afterwards it accepts more lower fare classes.

Similar to Theorem 1, we have the following results.

PROPOSITION 4 (STRICT BID-PRICE TRENDS). *For any $t \in [0, T)$ and $s \in (t, T]$,*

- (i) *if $N(t) > 1$, then $B(t) < E[B(s \wedge \tau_1) | \mathcal{F}_t]$;*
- (ii) *if $N(t) = 1$, then $B(t) > E[B(s) | \mathcal{F}_t]$.*

Proposition 4 shows that the optimal bid-price process of the capacity rationing model has the same trends as that in the dynamic pricing models. The strict inequality implies that the bid-price process is not a martingale. Note that the strict inequalities are ensured by the assumption that the arrival rates are strictly positive. We can also obtain strict inequalities in the dynamic pricing models when the market size Λ_t is strictly positive for all t .

Appendix D: Bid-Price Trends under Dynamic Price Competition

We now consider an oligopoly model with dynamic price competition (see, e.g., Gallego and Hu 2013). Suppose that there are k firms in a market competing on prices, indexed by $i = 1, \dots, k$, $k > 1$. All firms' products in this market are differentiated and substitutable. The information is complete to all the firms. Let $\mathbf{p} = (p_1, \dots, p_k)' \in [\underline{p}, \bar{p}]^k$ denote the pricing decision vector. The arrival rate for firm i depends on the price vector, denoted by $\lambda_i(\mathbf{p})$. Assume that $\lambda_i(\mathbf{p})$ is decreasing in p_i but increasing in $p_j, j \neq i$. At any point in time, each firm observes not only its own inventory level but also the competitors' inventory levels. Denote by $\mathbf{n} = (n_1, \dots, n_k)'$ the vector of the inventory levels and $\mathbf{N}(t) = (N_1(t), \dots, N_k(t))'$ the corresponding vector of inventory processes. Then the profit-to-go of each firm i , denoted by $V_i(t, \mathbf{n})$, depends not only on the time and its inventory level but also on the inventory levels of its competitors.

Let \mathbf{e}_i be the k -dimensional unit vector with i -th component being one and the others being zeros. For any t and \mathbf{n} , let $\mathbf{p}(t, \mathbf{n}) = (p_1(t, \mathbf{n}), \dots, p_k(t, \mathbf{n}))'$ denote the Nash equilibrium and $\lambda(t, \mathbf{n}) =$

$(\lambda_1(t, \mathbf{n}), \dots, \lambda_n(t, \mathbf{n}))'$ be the corresponding arrival rates such that $\lambda_i(t, \mathbf{n}) = \lambda_i(p_i(t, \mathbf{n}))$ for $i = 1, \dots, n$. Note that the price and demand processes of a firm stop when it runs out of stock. That is, $\lambda_i(t, \mathbf{n}) = 0$ if $n_i = 0$. Then, for $i = 1, \dots, k$ and $n_i \geq 1$, if there exists a Markov perfect equilibrium (MPE), the expected revenue function for firm i satisfies the following HJB equations:

$$0 = \frac{\partial V_i(t, \mathbf{n})}{\partial t} + \left\{ \lambda_i(t, \mathbf{n})[p_i(t, \mathbf{n}) - \Delta_i V_i(t, \mathbf{n})] - \sum_{j \neq i} \lambda_j(t, \mathbf{n}) \Delta_j V_i(t, \mathbf{n}) \right\},$$

where $\Delta_j V_i(t, \mathbf{n}) = V_i(t, \mathbf{n}) - V_i(t, \mathbf{n} - \mathbf{e}_j)$, $V_i(T, \mathbf{n}) = V_i(t, \mathbf{n} - n_i \mathbf{e}_i) = 0$. Here, for notational convenience, we let $\Delta_j V_i(t, \mathbf{n}) = 0$ if $n_j = 0$. Given any t and \mathbf{n} with $n_i \geq 1$, the equilibrium pricing decisions $\mathbf{p}(t, \mathbf{n})$ are obtained by solving the following equations simultaneously:

$$\max_{p_i \in [0, \infty)} \left\{ \lambda_i(\mathbf{p})[p_i - \Delta_i V_i(t, \mathbf{n})] - \sum_{j \neq i} \lambda_j(t, \mathbf{p}) \Delta_j V_i(t, \mathbf{n}) \right\}, i = 1, \dots, k. \quad (3)$$

REMARK 2 (EXISTENCE OF MPE). To ensure the existence of MPE, one needs to impose some regularity conditions on the demand functions or specify the demand functions. Typical demand models include linear, logit and Cobb-Douglas models (Bernstein and Federgruen 2003, 2004). In the dynamic price competition setting with discrete customer choices, Lin and Sibdari (2009) show that the MPE exists. Following the same approach of Lin and Sibdari (2009), we can also show the existence of MPE. We refer to Lin and Sibdari (2009) for the detailed proof of the equilibrium existence. The following discussion focuses on the probabilistic characterization of the bid-price trends under the equilibrium.

As discussed above, in a monopoly setting, the dynamics of bid prices are driven by the perishability and scarcity of the capacity. Resource scarcity is the dominant driving force of the bid-price trend before the inventory level falls to one. Perishability becomes the only driving force when there is only one unit left. In this section, competition becomes the third driving force. One may ask: Will competition change the bid-price trend? What is the relationship between competition and the other driving forces?

We attempt to investigate whether the equilibrium bid-price trends have a similar pattern characterized by Theorem 1. Unfortunately, when the inventory level is greater than one, we are not able to show the trend of the bid prices applying the same technique used in the proof of Theorem 1. Because under competition given the optimal price response functions of other firms, the arrival rate of each individual firm depends not only on its own price level but also on the inventory levels of all the firms, we may not be able to replicate the revenue and demand rates for different inventory levels of the individual firm. However, as shown in the following theorem, the second part of Theorem 1 can be generalized to the case with price competition.

THEOREM 5. *For firm i , if $N_i(s) = 1$ for some $s < T$. Then, the bid-price process $B_i(t) = V_i(t \wedge \tau_0^{(i)-}, \mathbf{N}(t \wedge \tau_0^{(i)-}))$ is a supermartingale in $[s, T]$, where $\tau_0^{(i)}$ is the hitting time the inventory of firm i falls to zero.*

Proof. For any $t \geq s$ with $N_i(s) = 1$, we have

$$\begin{aligned} 0 &= \frac{\partial V_i(t, \mathbf{n})}{\partial t} + \lambda_i(t, \mathbf{n})[p_i - V(t, \mathbf{n})] - \sum_{j \neq i} \lambda_j(t, \mathbf{n}) \Delta_j V_i(t, \mathbf{n}) \\ &\geq \frac{\partial V_i(t, \mathbf{n})}{\partial t} - \lambda_i(t, \mathbf{n})V(t, \mathbf{n}) - \sum_{j \neq i} \lambda_j(t, \mathbf{n}) \Delta_j V_i(t, \mathbf{n}). \end{aligned}$$

Then using the same arguments as that of the proof of Theorem 3, we can show that $V_i(t \wedge \tau_0^{(i)-}, \mathbf{N}(t \wedge \tau_0^{(i)-}))$ is a supermartingale. Q.E.D.

This theorem shows that a firm's bid price under price competition has a downward trend after the inventory level falls to one. It is the same as the prediction of the monopoly model. This implies that the perishability for each individual firm is still the main driving force of its bid price after its inventory level falls to one even under competition.

Now the next question is: Does the equilibrium price of firm i also have the downward trend as predicted by the monopoly model? To gain tractability, we consider a symmetric duopoly model.

As in the monopoly case, we define the optimal-price processes of firm i , $i = 1, 2$ as

$$P_i(t) \triangleq \begin{cases} p_i(t, \mathbf{N}(t)) & \text{if } t < \tau_0^{(i)}, \\ p_i(\tau_0^{(i)-}, \mathbf{N}(\tau_0^{(i)-})) & \text{if } t \geq \tau_0^{(i)}. \end{cases}$$

THEOREM 6 (PRICE TREND OF A DUOPOLY). *Consider a symmetric duopoly ($k = 2$). Assume that (1) when both firms are posting prices the arrival rate functions are $\lambda_i(p_i, p_{-i}) = (\alpha - \beta p_i + \gamma p_{-i})^+$, $i = 1, 2$, $\alpha, \beta, \gamma > 0$ and $\beta > \gamma$, and (2) when firm i has inventory left while the other firm runs out of stock, the arrival rate becomes $\lambda_i(p_i, p_0(p_i))$ where $p_0(p_i) = (\alpha + \gamma p_i)/\beta$ serves as a null price which ensures that the arrival rate for the firm running out of stock is zero. If $N_i(s) = 1$ for some $s < T$, $i = 1, 2$, then the equilibrium price processes $(P_1(t), P_2(t))$ are supermartingales in $[s, T]$.*

Proof. Due to the symmetry, we know that $p_1(t, n_1, n_2) = p_2(t, n_2, n_1)$ and $V_1(t, n_1, n_2) = V_2(t, n_2, n_1)$. Then, at time t with state $(n_1, n_2) = (1, 1)$, $p_1(t, n_1, n_2) = p_2(t, n_2, n_1) = \frac{\alpha + \beta V_1(t, 1, 1) - \gamma \Delta_2 V_1(t, 1, 1)}{2\beta - \gamma} = \frac{\alpha + \beta V_2(t, 1, 1) - \gamma \Delta_1 V_2(t, 1, 1)}{2\beta - \gamma}$. That is, the equilibrium prices are linear increasing in $V_1(t, 1, 1)$ but linearly decreasing in $\Delta_2 V_1(t, 1, 1)$.

It follows from Theorem 5 that $V_1(t \wedge \tau_0^{(1)-}, N_1(t \wedge \tau_0^{(1)-}), N_2(t \wedge \tau_0^{(1)-}))$ is a supermartingale. To show that the price process $p_1(t)$ has a downward trend, it suffices to show that $\Delta_2 V_1(t \wedge$

$(\tau_0^{(1)} -), \mathbf{N}(t \wedge (\tau_0^{(1)} -))$) has an upward trend (a submartingale) for $t \geq s$. Note that the assumptions (1) and (2) ensure that the arrival rate of a monopoly is always greater than that of a duopoly under the same own price, and the monopoly can always achieve any equilibrium arrival rate it may achieve under competition. And for any $\lambda_1(t, 1, 1)$, there exists a price $p_1 \geq p_1(t, 1, 1)$ such that $\lambda_1(p_1, \bar{p}) = \lambda_1(t, 1, 1)$, denoted by $\bar{p}_1(t, 1, 1)$. Then, by the HJB equation we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V_1(t, 1, 1) + \lambda_1(t, 1, 1)[p_1(t, 1, 1) - V_1(t, 1, 1)] - \lambda_2(t, 1, 1)\Delta_2 V_1(t, 1, 1), \\ 0 &= \frac{\partial}{\partial t} V_1(t, 1, 0) + \lambda_1(t, 1, 0)[p_1(t, 1, 0) - V_1(t, 1, 0)] \\ &\geq \frac{\partial}{\partial t} V_1(t, 1, 0) + \lambda_1(t, 1, 1)[\bar{p}_1(t, 1, 1) - V_1(t, 1, 0)]. \end{aligned}$$

where the inequality is due to the suboptimality of $\bar{p}_1(t, 1, 1)$.

Note that $\Delta_2 V_1(t, 0, 1) = \Delta_2 V_1(t, 1, 0) = 0$. A subtraction yields

$$0 \leq \frac{\partial}{\partial t} \Delta_2 V_1(t, 1, 1) - \lambda_1(t, 1, 1)\Delta_2 V_1(t, 1, 1) - \lambda_2(t, 1, 1)\Delta_2 V_1(t, 1, 1).$$

Then using the same arguments as that of the proof of Theorem 3, we can show that $\Delta_2 V_1(t \wedge \tau_0^{(1)} -), N_1(t \wedge \tau_0^{(1)} -), N_2(t \wedge \tau_0^{(1)} -)$) is a submartingale. The same analysis applies to $\Delta_1 V_2(t \wedge \tau_0^{(2)} -), N_1(t \wedge \tau_0^{(2)} -), N_2(t \wedge \tau_0^{(2)} -)$).

Combining with Theorem 5 and the linear equilibrium price form, the desired result holds. Q.E.D.

This theorem shows that the optimal-price process of each firm is a supermartingale if both firms' inventory levels fall below one. This can be explained intuitively. As each firm's opportunity cost (bid price) falls over time after its inventory level falls to one, the prices under equilibrium must be driven down over time. This result further enhances the prediction of the declining price trends for single-unit sellers, which can be tested empirically.

Recall that in the monopoly setting, the optimal price tends to increase over time as the inventory level declines gradually. Under price competition, although the aggregate inventory level in the market declines over time, each firm's pricing behavior is driven by the perishability of its own inventory when there is only one unit left. Therefore, the equilibrium price processes have the downward trends.

Interestingly, [Sweeting \(2012\)](#) conducts an empirical study on the dynamic pricing behavior of the secondary markets for Major League Baseball (MLB) tickets. In these markets, sellers are typically small and they tend to have only a single unit. He finds that the sellers cut their prices dramatically, by 40% or more, as an event approaches. He argues that the simple dynamic pricing models (without addressing the customers' strategic behavior explicitly) predicts the seller's behavior accurately. He also attempts to analyze a theoretical model to explain this empirical observation and argues that

if the expected opportunity cost of a sale is an increasing function of its own price, the expected opportunity cost falls over time. However, it is notable that the expected opportunity cost itself is a result of the equilibrium solution. It is not suitable to use the solution to define the condition. Our work provides a rigorous analysis and better theoretical justification for his findings.

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