E-Companion to
"Contingent Stimulus in Crowdfunding"

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OA.1. Proofs

Proof of Lemma 1. (i) Taking derivative of $H(q)$ w.r.t. q, we have

$$\frac{dH(q)}{dq} = \frac{c}{\theta q^2} f\left(\frac{1}{\theta}\left(p + c\cdot\left(\frac{1}{q}-1\right)\right)\right) > 0.$$

(ii) Assumption 1(ii) is guaranteed by the fact that the support of the distribution $F(\cdot)$ is unbounded.

(iii) We prove Assumption 1(iii) by contradiction. Taking derivative of $H(\alpha q)/H(q)$ w.r.t. q, we have

$$\frac{d}{dq}\left(\frac{H(\alpha q)}{H(q)}\right) = \frac{H(\alpha q)}{H(q)}\left[\frac{c}{\theta\alpha q^2}\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q}-1)))} - \frac{c}{\theta q^2}\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q}-1)))}\right].$$

Suppose there exists a q' such that $\frac{d}{dq}\left(\frac{H(\alpha q')}{H(q')}\right) \le 0$, which implies that $\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))} \le \alpha\cdot\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}$. Coupling with the IGFR property that $\frac{1}{\theta}\left[p+c(\frac{1}{\alpha q'}-1)\right]\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))} \ge \frac{1}{\theta}\left[p+c(\frac{1}{q'}-1)\right]\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}$, we have $p+c(\frac{1}{\alpha q'}-1) \ge \frac{1}{\alpha}\left[p+c(\frac{1}{q'}-1)\right]$. A direct consequence of the preceding inequality is that $(p-c) \ge \frac{p-c}{\alpha}$, which contradicts with $0<\alpha<1$ and $p>c$. Thus, we obtain the desired result. $\square$

Proof of Proposition 1. Suppose that a backer arrives with time-to-go $t>0$ and pledges needed $n \ge 1$. This focal backer would decide whether or not to pledge based on her expected project's success rate conditional on her pledging, i.e., $Q_t(n-1)$. Consider what happens in a small time interval δ, and we have

$$Q_t(n) = (1-\delta\lambda_t H(Q_t(n-1)))\cdot Q_{t-\delta}(n) + \delta\lambda_t H(Q_t(n-1))\cdot Q_{t-\delta}(n-1) + o(\delta).$$

Rearranging and taking the limit as $\delta \to 0$, we obtain Equation (2). With the boundary conditions, the solution to Equation (2), which is an ordinary differential equation solved by induction, is unique. $\square$

Proof of Theorem 1. (i) We prove this by induction. First when $n=1$, because $Q_t(0)=1$, it is easy to verify that $Q_t(1) = 1-\exp\left(-\int_0^t \lambda_s H(1)ds\right)$ is the unique solution of Equation (2). Hence $Q_t(1)$ increases in t, and $Q_t(1) < Q_t(0)$.

Now assume the statement is true for $n-1$ ($n\ge 2$), then for n:

$$\frac{\partial}{\partial t}\Big[Q_t(n-1) - Q_t(n)\Big] = \lambda_t\Big[H(Q_t(n-2))\left(Q_t(n-2) - Q_t(n-1)\right) - H(Q_t(n-1))\left(Q_t(n-1) - Q_t(n)\right)\Big].$$

Since $Q_t(n-2) - Q_t(n-1) > 0$, $\frac{\partial}{\partial t}\Big[Q_t(n-1) - Q_t(n)\Big] > \lambda_t H(Q_t(n-1))\big[Q_t(n-1) - Q_t(n)\big]$. Based on Grönwall's Inequality and the fact that $Q_t(n-1) - Q_t(n)\Big|_{t=0} = 0$, we have $Q_t(n-1) - Q_t(n) > 0$ for any $t > 0$. This also implies that $\frac{\partial Q_t(n)}{\partial t} > 0$. Therefore the statement is also true for n.

(ii) The inequality is equivalent to $\frac{Q_t(n)}{Q_t(n-1)} \le 1 - e^{-\bar{\lambda}t}$. Consider the function $e^{\bar{\lambda}t} Q_t(n)$. Taking the derivative w.r.t. t, we have

$$\frac{\partial(e^{\bar{\lambda}t}Q_t(n))}{\partial t} = \bar{\lambda}e^{\bar{\lambda}t}Q_t(n) + e^{\bar{\lambda}t}\frac{\partial Q_t(n)}{\partial t} \le \bar{\lambda}e^{\bar{\lambda}t}Q_t(n) + \bar{\lambda}e^{\bar{\lambda}t}[Q_t(n-1) - Q_t(n)] = \bar{\lambda}e^{\bar{\lambda}t}Q_t(n-1),$$

where the inequality is due to $\frac{\partial Q_t(n)}{\partial t} > 0$ and $\frac{\partial Q_t(n)}{\partial t} \le \bar{\lambda}[Q_t(n-1) - Q_t(n)]$, as implied by Equation (2). Integrating from 0 to t on both sides, we have

$$Q_t(n) \le \int_0^t \bar{\lambda}e^{-\bar{\lambda}(t-s)}Q_s(n-1)ds \le \bar{\lambda}Q_t(n-1)\int_0^t e^{-\bar{\lambda}(t-s)}ds = (1 - e^{-\bar{\lambda}t})Q_t(n-1).$$

where the second inequality is due to the increasing monotonicity of $Q_t(n-1)$ in t as shown in Theorem 1(i). Therefore, we conclude that $\frac{Q_t(n)}{Q_t(n-1)} \le 1 - e^{-\bar{\lambda}t}$.

(iii) We will prove that $\frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t and $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t by induction. Consider first when $n = 1$. Because $\frac{Q_t(1)}{Q_t(0)} = Q_t(1)$ and $\frac{H(Q_t(1))}{H(Q_t(0))} = \frac{H(Q_t(1))}{H(1)}$, the monotonicity is guaranteed by part (i) and Assumption 1(i). Now assume that the monotonicity in t holds for $n-1$. We next show that $r_t(n) \equiv \frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t and $\varphi_t(n) \equiv \frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t. First from part (i), we observe that $0 < r_t(n) < 1$ for $t > 0$. Taking the derivative of $r_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial r_t(n)}{\partial t} &= \frac{\lambda_t H(Q_t(n-1))\,[Q_t(n-1) - Q_t(n)]}{Q_t(n-1)} - \frac{Q_t(n)\lambda_t H(Q_t(n-2))\,[Q_t(n-2) - Q_t(n-1)]}{Q_t^2(n-1)} \\
&= \lambda_t \left[H(Q_t(n-1))\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right) - \frac{Q_t(n)}{Q_t(n-1)}H(Q_t(n-2))\left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right)\right] \\
&= \lambda_t \frac{Q_t(n)}{Q_t(n-1)}H_t(Q_t(n-2))\left[\frac{H(Q_t(n-1))}{H(Q_t(n-2))}\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right)\right] \\
&= \lambda_t r_t(n) H(Q_t(n-2))\left[\varphi_t(n-1)\left(\frac{1}{r_t(n)} - 1\right) - \left(\frac{1}{r_t(n-1)} - 1\right)\right].
\end{aligned}$$

Suppose that there exists some t_1 such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_1} \le 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_2} > 0$. Otherwise, if $\frac{\partial r_t(n)}{\partial t} \le 0$ for all $t < t_1$, then $\lim_{t\to 0} r_t(n) = 0 \ge r_{t_1}(n)$, which contradicts with the fact that $Q_t(n) > 0$. Due to the continuity of $\frac{\partial r_t(n)}{\partial t}$, there exists some $t_3 \in [t_2, t_1)$, such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_3} = 0$. That is,

$$\varphi_{t_3}(n-1)\left(\frac{1}{r_{t_3}(n)} - 1\right) - \left(\frac{1}{r_{t_3}(n-1)} - 1\right) = 0.$$

Because $\varphi_t(n-1)$ strictly increases in t and $r_t(n-1)$ increases in t, and $r_t(n)$ decreases in t between $[t_3, t_1]$, we have

$$\varphi_{t_1}(n-1)\left(\frac{1}{r_{t_1}(n)}-1\right)-\left(\frac{1}{r_{t_1}(n-1)}-1\right)>\varphi_{t_3}(n-1)\left(\frac{1}{r_{t_3}(n)}-1\right)-\left(\frac{1}{r_{t_3}(n-1)}-1\right)=0,$$

which implies that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1}>0$. However, this contradicts with the preceding statement that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1}\leq 0$. Therefore, we conclude that $\frac{\partial r_t(n)}{\partial t}>0$ for any $t>0$.

Next we show that $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t. For any $t'>t$, we have

$$H(Q_{t'}(n))=H\Big(\frac{Q_{t'}(n)}{Q_{t'}(n-1)}Q_{t'}(n-1)\Big)\geq H\Big(\frac{Q_t(n)}{Q_t(n-1)}Q_{t'}(n-1)\Big),$$

where the inequality is due to the increasing monotonicity of $\frac{Q_t(n)}{Q_t(n-1)}$ in t and Assumption 1(i). Due to Assumption 1(iii) and Theorem 1(i), we have

$$\frac{H(Q_{t'}(n))}{H(Q_{t'}(n-1))}\geq\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_{t'}(n-1)\right)}{H(Q_{t'}(n-1))}\geq\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_t(n-1)\right)}{H(Q_t(n-1))}=\frac{H(Q_t(n))}{H(Q_t(n-1))}.$$

We hence prove the increasing monotonicity of $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ in t.

For the monotonicity in n, because we have shown that $\frac{\partial r_t(n)}{\partial t}>0$ for any $t>0$, $\varphi_t(n-1)\big(\frac{1}{r_t(n)}-1\big)-\big(\frac{1}{r_t(n-1)}-1\big)>0$. Since $\varphi_t(n-1)\leq 1$, we have $r_t(n)<r_t(n-1)$, i.e., $\frac{Q_t(n)}{Q_t(n-1)}<\frac{Q_t(n-1)}{Q_t(n-2)}$. A direct consequence is that $\frac{H(Q_t(n))}{H(Q_t(n-1))}=\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_t(n-1)\right)}{H(Q_t(n-1))}<\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-1)\right)}{H(Q_t(n-1))}$. Due to Assumption 1(iii) and part (i), we have

$$\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-1)\right)}{H(Q_t(n-1))}\leq\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-2)\right)}{H(Q_t(n-2))}=\frac{H(Q_t(n-1))}{H(Q_t(n-2))}.$$

Therefore, we conclude that $\frac{H(Q_t(n))}{H(Q_t(n-1))}\leq\frac{H(Q_t(n-1))}{H(Q_t(n-2))}$ for any $t>0$.

(iv) For any $n\geq 1$, we have

$$\frac{Q_{t+h}(n)}{Q_t(n)}=\frac{Q_{t+h}(n)}{Q_{t+h}(n-1)}\cdot\frac{Q_{t+h}(n-1)}{Q_t(n-1)}\cdot\frac{Q_t(n-1)}{Q_t(n)}>\frac{Q_{t+h}(n-1)}{Q_t(n-1)},$$

where the inequality is due to $\frac{Q_{t+h}(n)}{Q_{t+h}(n-1)}>\frac{Q_t(n)}{Q_t(n-1)}$ as shown in Theorem 1(iii).

Last, we prove the monotonicty in t by induction. When $n=0$ the statement is obvious. Suppose that the statement is true for $n-1$, where $n\geq 1$. Then for any $t_2>t_1\geq 0$,

$$\frac{H(Q_{t_2}(n-1))}{H(Q_{t_2+h}(n-1))}=\frac{H\left(\frac{Q_{t_2}(n-1)}{Q_{t_2+h}(n-1)}Q_{t_2+h}(n-1)\right)}{H(Q_{t_2+h}(n-1))}\geq\frac{H\left(\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)}Q_{t_2+h}(n-1)\right)}{H(Q_{t_2+h}(n-1))}.$$

Based on Assumption 1(iii), we have

$$\frac{H(Q_{t_2}(n-1))}{H(Q_{t_2+h}(n-1))}\geq\frac{H\left(\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)}Q_{t_1+h}(n-1)\right)}{H(Q_{t_1+h}(n-1))}=\frac{H(Q_{t_1}(n-1))}{H(Q_{t_1+h}(n-1))},$$

due to $Q_{t_2+h}(n-1) \geq Q_{t_1+h}(n-1)$ and $\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)} \leq 1$. Thus $\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}$ increases in t. Next we take derivative of $\frac{Q_{t+h}(n)}{Q_t(n)}$ w.r.t. t:

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} &= \frac{H_{t+h}(n)\left[Q_{t+h}(n-1)-Q_{t+h}(n)\right]}{Q_t(n)} - \frac{Q_{t+h}(n)}{Q_t(n)}\frac{H_t(n)\left[Q_t(n-1)-Q_t(n)\right]}{Q_t(n)} \\
&= H_{t+h}(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{Q_{t+h}(n-1)}{Q_t(n-1)} - \frac{Q_{t+h}(n)}{Q_t(n)}\left(\frac{Q_t(n)}{Q_t(n-1)} + \frac{H_t(n)}{H_{t+h}(n)}\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)\right)\right] \\
&= H_{t+h}(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{Q_{t+h}(n-1)}{Q_t(n-1)} - \frac{Q_{t+h}(n)}{Q_t(n)}\left[1-\left(1-\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}\right)\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)\right]\right].
\end{aligned}
$$

Note that $\frac{Q_{t+h}(n)}{Q_t(n)} \to 1$ when $t \to \infty$, and $\frac{Q_{t+h}(n)}{Q_t(n)} > 1$ for any finite t. Thus $\frac{Q_{t+h}(n)}{Q_t(n)}$ decreases in t when t is sufficiently large. Suppose that $\frac{Q_{t+h}(n)}{Q_t(n)}$ is not monotonically decreasing in t. Then there must exist a $t_3 > t_2 > t_1$ such that $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)}\Big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} > 0$ for any $t \in (t_2,t_3)$. However, $\frac{Q_{t+h}(n-1)}{Q_t(n-1)}$ decreases in t by the induction assumption. We also know that $\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}$ increases in t, which would imply that $1-\left(1-\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}\right)\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)$ increases in t over (t_2,t_3). Consequently $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} \leq \frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)}\Big|_{t=t_2} = 0$ for $t \in (t_2,t_3)$, which contradicts with $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} > 0$ for any $t \in (t_2,t_3)$. We thus obtain the announced results. □

Proof of Lemma 2. (i) Taking derivative of $\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}$ w.r.t. q, we have

$$
\frac{\partial}{\partial q}\left(\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\right) = \frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\frac{c}{q^2}\Big[\frac{1}{\theta_a}\frac{f\left(\frac{1}{\theta_a}(p+c(\frac{1}{q}-1))\right)}{\bar{F}\left(\frac{1}{\theta_a}(p+c(\frac{1}{q}-1))\right)} - \frac{1}{\theta_b}\frac{f\left(\frac{1}{\theta_b}(p+c(\frac{1}{q}-1))\right)}{\bar{F}\left(\frac{1}{\theta_b}(p+c(\frac{1}{q}-1))\right)}\Big].
$$

Because $\theta_a < \theta_b$ and Assumption 1, we conclude that $\frac{\partial}{\partial q}\left(\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\right) > 0$. Thus, we obtain the announced results.

(ii) Taking derivative of $\frac{H^{p_a}(q)}{H^{p_b}(q)}$ w.r.t. q, we have

$$
\begin{aligned}
\frac{\partial}{\partial q}\left(\frac{H^{p_a}(q)}{H^{p_b}(q)}\right) &= \frac{1}{[H^{p_b}(q)]^2}\Bigg[\frac{c}{\theta q^2}f\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\bar{F}\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right) \\
&\quad -\frac{c}{\theta q^2}f\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\bar{F}\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\Bigg] \\
&= \frac{H^{p_a}(q)}{H^{p_b}(q)}\frac{c}{\theta q^2}\left[\frac{f\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}{\bar{F}\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)} - \frac{f\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}{\bar{F}\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}\right].
\end{aligned}
$$

Due to $p_a > p_b$ and that $\frac{f(v)}{\bar{F}(v)}$ increases in v, we conclude that $\frac{\partial}{\partial q}\left(\frac{H^{p_a}(q)}{H^{p_b}(q)}\right) > 0$. □

Proof of Proposition 2. Denote $x_t(n) = \frac{Q_t^a(n)}{Q_t^b(n)}$ and $\gamma_t(n) = \frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))}$. We first prove that $x_t(n)$ and $\gamma_t(n)$ increase in t by induction. When $n=0$, $x_t(0)=1$ and $\gamma_t(0) = \frac{H^a(1)}{H^b(1)}$, and thus the monotonicity holds trivially. Now suppose that the statement is true for $n-1$. Taking the derivative of $x_t(n)$ w.r.t. t, we have

$$
\frac{dx_t(n)}{dt} = \frac{\lambda_t H^a(Q_t^a(n-1))\left[Q_t^a(n-1)-Q_t^a(n)\right]}{Q_t^b(n)} - \frac{Q_t^a(n)\lambda_t H^b(Q_t^b(n-1))\left[Q_t^b(n-1)-Q_t^b(n)\right]}{[Q_t^b(n)]^2}
$$

$$
\begin{aligned}
&= \lambda_t \frac{Q_t^a(n)}{Q_t^b(n)} \left[H^a(Q_t^a(n-1)) \left(\frac{Q_t^a(n-1)}{Q_t^a(n)} - 1 \right) - H^b(Q_t^b(n-1)) \left(\frac{Q_t^b(n-1)}{Q_t^b(n)} - 1 \right) \right] \\
&= \lambda_t \left[H^a(Q_t^a(n-1)) \left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)} \frac{Q_t^b(n-1)}{Q_t^b(n)} - \frac{Q_t^a(n)}{Q_t^b(n)} \right) - H^b(Q_t^b(n-1)) \frac{Q_t^a(n)}{Q_t^b(n)} \left(\frac{Q_t^b(n-1)}{Q_t^b(n)} - 1 \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \cdot \\
&\quad \left[\frac{Q_t^a(n-1)}{Q_t^b(n-1)} - \frac{Q_t^b(n)}{Q_t^b(n-1)} \frac{Q_t^a(n)}{Q_t^b(n)} - \frac{H^b(Q_t^b(n-1))}{H^a(Q_t^a(n-1))} \frac{Q_t^a(n)}{Q_t^b(n)} \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \left[x_t(n-1) - \frac{Q_t^b(n)}{Q_t^b(n-1)} x_t(n) - \frac{x_t(n)}{\gamma_t(n-1)} \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \left[x_t(n-1) - x_t(n) - \left(\frac{1}{\gamma_t(n-1)} - 1 \right) \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) x_t(n) \right].
\end{aligned}
$$

Denote $L(t) = x_t(n-1) - \left[1 + \left(\frac{1}{\gamma_t(n-1)} - 1 \right) \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) \right] x_t(n)$. Next we show that if there exists some t_1 such that $L(t_1) < 0$, there must exist some $t_2 \in (0, t_1)$ such that $L(t_2) \geq 0$. Consider the following two cases.

(1) $\lim_{t \to 0} \gamma_t(n-1) = 0$. Using L' Hopital's rule, we have

$$
\begin{aligned}
\lim_{t \to 0} x_t(n) &= \lim_{t \to 0} \frac{\frac{\partial Q_t^a(n)}{\partial t}}{\frac{\partial Q_t^b(n)}{\partial t}} = \lim_{t \to 0} \frac{\lambda_t H^a(Q_t^a(n-1)) \left(Q_t^a(n-1) - Q_t^a(n) \right)}{\lambda_t H^b(Q_t^b(n-1)) \left(Q_t^b(n-1) - Q_t^b(n) \right)} \\
&= \lim_{t \to 0} \gamma_t(n-1) \cdot \frac{Q_t^a(n-1) \left[1 - \frac{Q_t^a(n)}{Q_t^a(n-1)} \right]}{Q_t^b(n-1) \left[1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right]} = 0.
\end{aligned}
$$

Suppose there exists some $t_1 > 0$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_1} < 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_2} \geq 0$; otherwise $x_t(n)$ decreases within $(0, t_1]$, which implies that $x_{t_1} \leq \lim_{t \to 0} x_t(n) = 0$. This contradicts with the fact that $x_t(n) > 0$ for $t > 0$.

(2) $\lim_{t \to 0} \gamma_t(n-1) > 0$. Because of $\lim_{t \to 0} \frac{Q_t^b(n)}{Q_t^b(n-1)} = 0$ as shown in Theorem 1(ii), $\lim_{t \to 0} L(t) = \lim_{t \to 0} x_t(n-1) - \frac{x_t(n)}{\gamma_t(n-1)}$. Again using L' Hopital's rule, we have

$$
\lim_{t \to 0} L(t) = \lim_{t \to 0} x_t(n-1) - \lim_{t \to 0} \frac{1}{\gamma_t(n-1)} \frac{H^a(Q_t^a(n-1)) \cdot Q_t^a(n-1) \left[1 - \frac{Q_t^a(n)}{Q_t^a(n-1)} \right]}{H^b(Q_t^b(n-1)) \cdot Q_t^b(n-1) \left[1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right]} = 0.
$$

Suppose there exists some $t_1 > 0$ such that $\left.\frac{\partial x_t(n)}{\partial t}\right|_{t=t_1} < 0$, i.e., $L(t_1) < 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_2} \geq 0$; otherwise, $x_t(n)$ decreases within $(0, t_1]$. Combined with the results that $x_t(n-1)$, $\gamma_t(n-1)$ and $\frac{Q_t^b(n)}{Q_t^b(n-1)}$ all increase in t, we have that $L(t)$ increases in $(0, t_1]$, which suggests that $L(t_1) \geq \lim_{t \to 0} L(t) = 0$. This contradicts with the preceding argument that $L(t_1) < 0$.

Therefore, if there exists some t_1 such that $L(t_1) < 0$, there must exist a $t_2 \in (0, t_1)$ such that $L(t_2) \geq 0$. Coupling with the continuity of $L(t)$, there exists a $t_3 \in [t_2, t_1)$ such that $L(t_3) = 0$.

This implies that $x_t(n)$ strictly decreases within $(t_3, t_1]$. Combined with the results that $x_t(n-1)$, $\gamma_t(n-1)$ and $\frac{Q_t^b(n)}{Q_t^b(n-1)}$ all increase in t, we have that $L(t)$ increases in $(t_3, t_1]$, which suggests that $L(t_1) \geq L(t_3) = 0$. This contradicts with the preceding argument that $L(t_1) < 0$. Therefore, we conclude that $\frac{\partial x_t(n)}{\partial t} \geq 0$ for all $t > 0$.

Given that $x_t(n)$ increases in t, for any $\delta > 0$, we have

$$\frac{H^b(Q_{t+\delta}^a(n))}{H^b(Q_{t+\delta}^b(n))} = \frac{H^b\left(x_{t+\delta}(n)Q_{t+\delta}^b(n)\right)}{H^b(Q_{t+\delta}^b(n))} \geq \frac{H^b\left(x_t(n)Q_{t+\delta}^b(n)\right)}{H^b(Q_{t+\delta}^b(n))} \geq \frac{H^b\left(x_t(n)Q_t^b(n)\right)}{H^b(Q_t^b(n))} = \frac{H^b(Q_t^a(n))}{H^b(Q_t^b(n))},$$

where the second inequality is a result of Assumption 1(iii). Hence $\frac{H^b(Q_t^a(n))}{H^b(Q_t^b(n))}$ increases in t. Combining with the assumption that $\frac{H^a(q)}{H^b(q)}$ increases in q, we conclude that $\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))}$ increases in t.

Next we prove that $x_t(n)$ and $\gamma_t(n)$ decrease in n. Because $x_t(n)$ increases in t, we have $L(t) = x_t(n-1) - x_t(n) - \left(\frac{1}{\gamma_t(n-1)} - 1\right)\left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)}\right) x_t(n) > 0$ for any $t > 0$. Coupling with the results that $\gamma_t(n-1) \leq 1$, $x_t(n) \geq 0$ and Theorem 1(i), we thus have that $x_t(n-1) > x_t(n)$.

Given that $\frac{Q_t^a(n)}{Q_t^b(n)}$ decreases in n, we have

$$\frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))} = \frac{H^a\left(\frac{Q_t^a(n)}{Q_t^b(n)}Q_t^b(n)\right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)}Q_t^b(n)\right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)}Q_t^b(n-1)\right)}{H^a(Q_t^b(n-1))} = \frac{H^a(Q_t^a(n-1))}{H^a(Q_t^b(n-1))},$$

where the second inequality is a result of Assumption 1(iii). Moreover, $\frac{H^a(Q_t^b(n))}{H^b(Q_t^b(n))} \leq \frac{H^a(Q_t^b(n-1))}{H^b(Q_t^b(n-1))}$ because of the assumption that $\frac{H^a(q)}{H^b(q)}$ increases in q. Therefore, we have

$$\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))} = \frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))}\frac{H^a(Q_t^b(n))}{H^b(Q_t^b(n))} \leq \frac{H^a(Q_t^a(n-1))}{H^a(Q_t^b(n-1))}\frac{H^a(Q_t^b(n-1))}{H^b(Q_t^b(n-1))} = \frac{H^a(Q_t^a(n-1))}{H^b(Q_t^b(n-1))}.$$

We thus complete the proof. $\square$

For notational convenience, we denote $H_t(n) \equiv H(Q_t(n-1))$ in the following proofs.

Proof of Theorem 2. Denote $J_t^s(n)$ as the optimal expected profit at state (t, n) assuming that the seeding stimulus has not been activated. We prove that $\tau^s(n)$ is given by

$$\begin{aligned}\tau^s(n) = \sup\Big\{t : {} & H_t((n-n_0)^+) \cdot Q_t((n-n_0-1)^+) - \left[H_t((n-n_0)^+) - H_t(n)\right] \cdot Q_t((n-n_0)^+) \\ & \geq H_t(n)\frac{J_t^s(n-1)}{G+B-R}\Big\}.\end{aligned} \tag{OA.1}$$

Expected profit $J_t^s(n)$ at state (t, n) is given by

- when $n \geq 1$ and $t \leq \tau^s(n)$, $J_t^s(n) = (G+B-R) \cdot Q_t((n-n_0)^+)$;
- when $t > \tau^s(n)$, $J_t^s(n)$ is given by

$$\frac{\partial J_t^s(n)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1) - J_t^s(n)\right], \tag{OA.2}$$

with boundary conditions $J_{\tau^s(n)}^s(n) = (G+B-R) \cdot Q_{\tau^s(n)}((n-n_0)^+)$ and $J_t^s(0) = G+B$.

Denote $l_t(n) \equiv \frac{J_t^s(n)}{Q_t((n-n_0)^+)}$. We add to the statement that $l_t(n)$ increases in t, and prove by induction. When $n \le n_0$, the optimal expected profit is given by $J_t^s(n) = (G+B)\cdot Q_t(n) + (G+B-R)\cdot(1-Q_t(n))$. That is, the creator's optimal policy is to hold off until right before the deadline, and to activate "seeding" if no backer pledges by then. It is not hard to verify that it is the unique solution to the differential equation characterized by Equation (OA.2). We thus conclude that $l_t(n) = J_t^s(n)$ increases in t for $n \le n_0$.

Assume that the statement is true for $n-1$, where $n \ge n_0+1$. Next, we seek to derive $J_t^s(n)$ by showing that the creator's optimal policy is to "seed" immediately when $t \le \tau_t^s(n)$ and to hold off when $t > \tau_t^s(n)$. We can rewrite the inequality within the curly brackets in Equation (OA.1) as follows.

$$1 + \left[\frac{H_t(n-n_0)}{H_t(n)} - 1\right]\left[1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right] \ge \frac{J_t^s(n-1)}{(G+B-R)\cdot Q_t(n-n_0-1)}.$$

RHS of the inequality increases in t because $l_t(n-1)$ increases in t, while LHS decreases in t due to Theorem 1(iii). Therefore, for any $t \le \tau^s(n)$, the inequality within the curly brackets in Equation (OA.1) holds; whereas the direction of the inequality is flipped for any $t > \tau^s(n)$.

Suppose there exists some $t_1 > \tau^s(n)$ such that the creator's optimal policy is to activate the seeding stimulus immediately, i.e., $J_{t_1}^s(n) = (G+B-R)\cdot Q_{t_1}(n-n_0)$. Comparing the case without activating the stimulus at time t_1, we have

$$\begin{aligned} J_{t_1}^s(n) &\ge \lambda_{t_1} H_{t_1}(n)\delta \cdot J_{t_1-\delta}^s(n-1) + (1-\lambda_{t_1}H_{t_1}(n)\delta)\cdot J_{t_1-\delta}^s(n) + o(\delta) \\ &\ge \lambda_{t_1} H_{t_1}(n)\delta \cdot J_{t_1-\delta}^s(n-1) + (1-\lambda_{t_1}H_{t_1}(n)\delta)\cdot (G+B-R)\cdot Q_{t_1-\delta}(n-n_0) + o(\delta). \end{aligned}$$

Plugging $Q_{t_1}(n-n_0) = (1-\lambda_{t_1}H_{t_1}(n-n_0)\delta)\cdot Q_{t_1-\delta}(n-n_0) + \lambda_{t_1}H_{t_1}(n-n_0)\delta \cdot Q_{t_1-\delta}(n-n_0-1) + o(\delta)$ into $J_{t_1}^s(n)$ in the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$(G+B-R)\Big[H_{t_1}(n-n_0)Q_{t_1}(n-n_0-1) - (H_{t_1}(n-n_0) - H_{t_1}(n))Q_{t_1}(n-n_0)\Big] \ge H_{t_1}(n)J_{t_1}^s(n-1).$$

This contradicts with the fact that $t_1 > \tau^s(n)$. Therefore, the creator's optimal policy is to hold off when $t > \tau^s(n)$, i.e., $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$. Consider what happens in a small time interval δ, we have

$$J_t^s(n) = (1-\delta\lambda_t H_t(n))\cdot J_{t-\delta}^s(n) + \delta\lambda_t H_t(n)\cdot J_{t-\delta}^s(n-1) + o(\delta).$$

Rearranging and taking the limit as $\delta \to 0$, we obtain Equation (OA.2).

We next show that the creator's optimal policy is to "seed" immediately when $t < \tau^s(n)$. Suppose that there exists some $t_2 < \tau^s(n)$, such that $J_t^s(n) = (G+B-R)\cdot Q_t(n-n_0)$ for any $t \le t_2$, and

$J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t \in (t_2, t_2+h]$. (Because $J_0^s(n) = 0$ for any $n > n_0$, we can always find some t_2 such that $J_t^s(n) = (G+B-R)\cdot Q_t(n-n_0)$ for any $t \le t_2$.) Then, for any $t \in (t_2, t_2+h]$

$$J_{t+\delta}^s(n) = (1-\lambda_{t+\delta} H_{t_2+\delta}(n)\delta)\cdot J_t^s(n) + \lambda_{t+\delta} H_{t+\delta}(n)\delta \cdot J_t^s(n-1) + o(\delta).$$

Let $\delta \to 0$, we obtain $\frac{\partial J_t^s(n)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1) - J_t^s(n)\right]$ over interval $(t_2, t_2+h]$. According to Equation (OA.1), $J_t^s(n-1) \le \frac{G+B-R}{H_t(n)}\left[H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\,Q_t(n-n_0)\right]$. Also because $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t \in (t_2, t_2+h]$, we have

$$\begin{aligned}
&\frac{\partial J_t^s(n)}{\partial t} \\
&< \lambda_t(G+B-R)\left[H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\,Q_t(n-n_0) - H_t(n)Q_t(n-n_0)\right] \\
&= \lambda_t(G+B-R)\cdot H_t(n-n_0)\left[Q_t(n-n_0-1) - Q_t(n-n_0)\right].
\end{aligned}$$

However, we know from Equation (2) that $\frac{\partial}{\partial t}\left[(G+B-R)\cdot Q_t(n-n_0)\right] = \lambda_t(G+B-R)\cdot H_t(n-n_0)\left[Q_t(n-n_0-1) - Q_t(n-n_0)\right]$. Therefore, $\frac{\partial}{\partial t}\left[J_t^s(n) - (G+B-R)\cdot Q_t(n-n_0)\right] < 0$ for any $t \in (t_2, t_2+h]$. Since $\left[J_t^s(n) - (G+B-R)\cdot Q_t(n-n_0)\right]\Big|_{t=t_2} = 0$, we obtain that $J_t^s(n) < (G+B-R)\cdot Q_t(n-n_0)$ when $t \in (t_2, t_2+h]$. This contradicts with the assumption we made earlier. Hence, the creator's optimal policy is to "seed" immediately for any $t < \tau_t^s(n)$, i.e., $J_t^s(n) = (G+B-R)\cdot Q_t(n-n_0)$ for any $t < \tau^s(n)$.

Lastly, we show that $l_t(n)$ is an increasing function of t. This is obvious when $t \le \tau^s(n)$, as $\frac{J_t^s(n)}{Q_t(n-n_0)} = G+B-R$. When $t > \tau^s(n)$, taking the derivative of $l_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial l_t(n)}{\partial t} &= \frac{\lambda_t H_t(n)\left[J_t^s(n-1) - J_t^s(n)\right]}{Q_t(n-n_0)} - \frac{\lambda_t H_t(n-n_0)J_t^s(n)\left[Q_t(n-n_0-1) - Q_t(n-n_0)\right]}{[Q_t(n-n_0)]^2} \\
&= \lambda_t\left\{H_t(n)\left[\frac{J_t^s(n-1)}{Q_t(n-n_0-1)}\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)} - \frac{J_t^s(n)}{Q_t(n-n_0)}\right] - H_t(n-n_0)\frac{J_t^s(n)}{Q_t(n-n_0)}\left[\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)} - 1\right]\right\} \\
&= \lambda_t H_t(n)\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)}\left\{l_t(n-1) - l_t(n)\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)} - l_t(n)\frac{H_t(n-n_0)}{H_t(n)}\left[1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right]\right\} \\
&= \lambda_t H_t(n)\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)}\left[l_t(n-1) - l_t(n) - \left(\frac{H_t(n-n_0)}{H_t(n)} - 1\right)\left(1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right)l_t(n)\right].
\end{aligned}$$

Notice that $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t > \tau^s(n)$, and thus $l_t(n) > G+B-R$ when $t > \tau^s(n)$. Suppose that there exists some $t_3 > \tau^s(n)$ such that $\frac{\partial l_t(n)}{\partial t}\Big|_{t=t_3} < 0$. Then, there must be some $t_4 \in (\tau^s(n), t_3)$, such that $\frac{\partial l_t(n)}{\partial t}|_{t=t_4} \ge 0$; otherwise, $\frac{\partial l_t(n)}{\partial t} < 0$ for any $\tau_s(n) < t \le t_3$, leading to $l_{t_3} < l_{\tau^s(n)}(n) = G+B-R$, which contradicts with the result that $l_t(n) > (G+B-R)$ when $t > \tau^s(n)$.

Due to the continuity of $\frac{\partial l_t(n)}{\partial t}$, there exists some $t_5 \in [t_4, t_3)$ such that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_5} = 0$, and $\frac{\partial l_t(n)}{\partial t} < 0$ on $(t_5, t_3]$. That is,

$$l_{t_5}(n-1) - l_{t_5}(n) - \left(\frac{H_{t_5}(n-n_0)}{H_{t_5}(n)} - 1\right)\left(1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0-1)}\right) l_{t_5}(n) = 0.$$

According to Theorem 1(iii), $\frac{H_t(n-1)}{H_t(n)}$ decreases in t and $\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}$ increases in t. Coupling with the result that $l_t(n)$ strictly decreases within $(t_5, t_3]$, we have

$$\begin{aligned} & l_{t_3}(n-1) - l_{t_3}(n) - \Big(\frac{H_{t_3}(n-n_0)}{H_{t_3}(n)} - 1\Big)\Big(1 - \frac{Q_{t_3}(n-n_0)}{Q_{t_3}(n-n_0-1)}\Big) l_{t_3}(n) \\ > \; & l_{t_5}(n-1) - l_{t_5}(n) - \Big(\frac{H_{t_5}(n-1)}{H_{t_5}(n)} - 1\Big)\Big(1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0-1)}\Big) l_{t_5}(n) = 0. \end{aligned}$$

This implies that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_3} > 0$ and contradicts with our assumption that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_3} < 0$. We thus complete the proof. □

Proof of Corollary 1. (i) We prove by induction. When $n = n_0 + 1$, it is straightforward that $\tau^s(n_0+1) \geq \tau^s(n_0) = \cdots = \tau^s(1) = 0$. Now assume the statement is true for $n-1$, i.e., $\tau^s(1) \leq \cdots \leq \tau^s(n-1)$ for some $n > n_0$. We prove $\tau^s(n-1) \leq \tau^s(n)$ by showing that for any $t < \tau^s(n-1)$, the creator's optimal action is not to activate the seeding stimulus at state (t, n). Suppose this is not true, then $t > \tau^s(n)$. From Equation (OA.1), we have

$$H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\, Q_t(n-n_0) < H_t(n)\frac{J_t^s(n-1)}{G+B-R}.$$

Because $t < \tau^s(n-1)$, $J_t^s(n-1) = (G+B-R) \cdot Q_t(n-n_0-1)$. Plugging $J_t^s(n-1)$ into the inequality above, we have

$$\begin{aligned} & H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\, Q_t(n-n_0) < H_t(n)Q_t(n-n_0-1) \\ \Rightarrow \; & (H_t(n-n_0) - H_t(n))\,(Q_t(n-n_0-1) - Q_t(n-n_0)) < 0. \end{aligned}$$

However, it contradicts with Theorem 1(i) and Assumption 1(i). We thus obtain the announced results.

(ii) Denote $Y_t(n; B, R) = \frac{J_t^s(n;B,R)}{G+B-R}$. Here, we use the notation $J_t^s(n; B, R) \equiv J_t^s(n)$ to emphasize the dependence of $J_t^s(n)$ on B and R. Similarly, we denote $\tau^s(n; B, R) \equiv \tau^s(n)$. We add to the statement that $Y_t(n; B, R)$ decreases in B and increases in R, and prove by induction. For any $n \leq n_0$, the statement is obviously true since $\tau^s(n; B, R) = 0$ and $Y_t(n; B, R) = \frac{(G+B)\cdot Q_t(n) + (G+B-R)\cdot(1-Q_t(n))}{G+B-R} = 1 + \frac{R \cdot Q_t(n)}{G+B-R}$ decreases in B and increases in R. Now suppose $\tau^s(n; B_1, R) \geq \tau^s(n; B_2, R)$ and $Y_t(n; B_1, R) \leq Y_t(n; B_2, R)$, for any $n \leq n_0$ and $B_1 > B_2 \geq 0$. From Equation (OA.1), for any $t > \tau^s(n+1; B_1, R)$,

$$1 + \left[\frac{H_t(n+1-n_0)}{H_t(n+1)} - 1\right]\left[1 - \frac{Q_t(n+1-n_0)}{Q_t(n+1-n_0-1)}\right] < \frac{Y_t(n; B_1, R)}{Q_t(n+1-n_0)} \leq \frac{Y_t(n; B_2, R)}{Q_t(n+1-n_0)}.$$

Therefore, $\tau^s(n+1;B,R)$ increases in B. Similarly we can show that $\tau^s(n+1;B,R)$ decreases in R.

Now we show the monotonicity of $Y_t(n+1;B,R)$ w.r.t B and R. For any $t \le \tau^s(n+1;B_2,R)$, $Y_t(n+1;B_1,R) = Y_t(n+1;B_2,R) = Q_t(n+1-n_0)$.

When $t \in (\tau^s(n+1;B_2,R), \tau^s(n+1;B_1,R)]$, $Y_t(n+1;B_1,R) = Q_t(n+1-n_0)$ whereas the $Y_t(n+1;B_2,R) \ge Q_t(n+1-n_0)$ because of the definition of $J_t^s(n)$.

When $t > \tau^s(n+1;B_1,R)$, $Y_t(n+1;B_i,R)$ is the solution of

$$\frac{\partial y}{\partial t} = \lambda_t H_t(n+1)[Y_t(n;B_i,R) - y],$$

with the boundary condition $y_{\tau^s(n+1;B_1,R)} = Y_{\tau^s(n+1;B_1,R)}(n+1;B_i,R)$ where $i=1,2$. Note that RHS of the equation decreases in B based on the induction hypothesis of n. Coupling with the fact that $Y_{\tau^s(n+1;B_1,R)}(n+1;B_1,R) \le Y_{\tau^s(n+1;B_1,R)}(n+1;B_2,R)$, $Y_t(n+1;B,R)$ decreases in B when $t > \tau^s(n+1;B_1,R)$. In a similar fashion, we can show that $Y_t(n+1;B,R)$ increases in R. We thus obtain the announced results. □

Proof of Theorem 3. (i) Since $J_{T,N}^b = (G+B)\cdot Q_T(N)$ and $J_{T,N}^s = J_T^s(N)$, it is sufficient to show that $\frac{Q_t(n)}{J_t^s(n)}$ increases in t.

When $n=0$, the statement is obvious as $Q_t(0)=1$ and $J_t^s(0) = G+B$. Now assume that $\frac{Q_t(n-1)}{J_t^s(n-1)}$ weakly increases in t. In that case:

When $t < \tau^s(n)$, $J_t^s(n) = (G+B-R)\cdot Q_t((n-n_0)^+)$. Therefore $\frac{Q_t(n)}{J_t^s(n)} = \frac{Q_t(n)}{(G+B-R)\cdot Q_t((n-n_0)^+)}$. According to Theorem 1, it increases in t.

When $t \ge \tau^s(n)$,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^s(n)} &= \frac{\lambda_t H_t(n)\,[Q_t(n-1) - Q_t(n)]}{J_t^s(n)} - \frac{Q_t(n)}{J_t^s(n)}\frac{\lambda_t H_t(n)\,[J_t^s(n-1) - J_t^s(n)]}{J_t^s(n)} \\ &= \lambda_t H_t(n)\frac{Q_t(n)}{J_t^s(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - \frac{J_t^s(n-1)}{J_t^s(n)}\right] = \lambda_t H_t(n)\frac{J_t^s(n-1)}{J_t^s(n)}\left[\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)}\right].\end{aligned}$$

When $t = \tau^s(n)$, because $J_t^s(n) = (G+B-R)\cdot Q_t(n-1)$,

$$\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{(G+B-R)\cdot Q_t(n-1)}.$$

Also, according to Equation (OA.1), at $t = \tau^s(n)$,

$$J_t^s(n-1) = (G+B-R)\cdot\left[\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)\right].$$

Hence,

$$\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{1}{G+B-R}\frac{Q_t(n-1)}{\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)} - \frac{Q_t(n)}{(G+B-R)\cdot Q_t(n-1)}$$

$$= \frac{1}{G+B-R} \frac{Q_t(n)}{\frac{H_t(n-1)}{H_t(n)} Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right) Q_t(n-1)} \cdot$$

$$\left[\frac{G+B-R}{Q_t(n)} - \frac{\frac{H_t(n-1)}{H_t(n)} Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right) Q_t(n-1)}{Q_t(n-1)} \right]$$

$$= \frac{1}{G+B-R} \frac{Q_t(n)}{\frac{H_t(n-1)}{H_t(n)} Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right) Q_t(n-1)} \left[\left(\frac{Q_t(n-1)}{Q_t(n)} - 1 \right) - \frac{H_t(n-1)}{H_t(n)} \left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1 \right) \right].$$

Recall that in the proof of Theorem 1, we have shown that for any $t > 0$, $\frac{H_t(n)}{H_t(n-1)} \left(\frac{Q_t(n-1)}{Q_t(n)} - 1 \right) - \left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1 \right) > 0$. Therefore $\left. \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \right|_{t=\tau^s(n)} > 0$.

Suppose that there exists a $t' > \tau^s(n)$ such that $\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} < 0$, then because of continuity, there must exists a $\tau^s(n) < t_1 < t'$ such that $\left. \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \right|_{t=t_1} = 0$ and $\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} < 0$ when $t \in (t_1, t')$. This also means that $\frac{Q_t(n)}{J_t^s(n)}$ decreases in t over the interval. However, since $\frac{Q_t(n-1)}{J_t^s(n-1)}$ increases in t, $\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)}$ must be increasing in t within (t_1, t'). This indicates $\left. \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \right|_{t=t'} \geq 0$, which leads to contradiction. Therefore $\frac{Q_t(n)}{J_t^s(n)}$ increases in t for any $t > 0$.

(ii) Note that $J_{T,N}^b \leq J_{T,N}^s \leq (G+B) \cdot Q_T((N-n_0)^+)$. Consequently, we have

$$0 \leq J_{T,N}^s - J_{T,N}^b \leq (G+B) \cdot \left[Q_T((N-n_0)^+) - Q_T(N) \right].$$

Letting $T \to \infty$ or $T \to 0$, we thus obtain the announced results. $\square$

Proof of Theorem 4. We show that $\tau^u(n)$ is given by

$$\tau^u(n) = \sup \left\{ t : \tilde{H}_t(n) \tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n)) \tilde{Q}_t(n) \geq H_t(n) \frac{J_t^u(n-1)}{G+B-K} \right\}, \tag{OA.3}$$

where $J_t^u(n)$ is the expected profit at state (t, n). It is given by

- when $t \leq \tau^u(n)$, $J_t^u(n) = (G+B-K)\tilde{Q}_t(n)$;
- when $t > \tau^u(n)$,

$$\frac{\partial J_t^u(n)}{\partial t} = \lambda_t H_t(n) \left[J_t^u(n-1) - J_t^u(n) \right], \tag{OA.4}$$

with boundary conditions $J_{\tau^u(n)}^u(n) = (G+B-K)\tilde{Q}_t(n)$, and $J_t^u(0) = G+B$.

First we show that if $\frac{J_t^u(n-1)}{\tilde{Q}_t(n-1)}$ increases in t, then the creator would activate the stimulus if and only if $t \leq \tau^u(n)$. To see that, we can rewrite the inequality in the bracket in Equation (OA.3) as follows.

$$1 + \left(\frac{\tilde{H}_t(n)}{H_t(n)} - 1 \right) \left(1 - \frac{\tilde{Q}_t(n)}{\tilde{Q}_t(n-1)} \right) \geq \frac{J_t^u(n-1)}{(G+B-K)\tilde{Q}_t(n-1)}.$$

According to Theorem 1(iii), LHS of the above inequality strictly decreases in t; while RHS increases in t due to our induction hypothesis. Therefore, for any $t < \tau^u(n)$, the inequality holds; whereas the direction of the inequality is flipped for any $t > \tau^u(n)$.

Suppose that there exists some $t_1 > \tau^u(n)$, such that the creator's optimal policy is to upgrade immediately, i.e., $J^u_{t_1}(n) = (G+B-K)\tilde{Q}_{t_1}(n)$. Then, we have

$$\begin{aligned}(G+B-K)\tilde{Q}_{t_1}(n) &> (1-\delta\lambda_{t_1}H_{t_1}(n))J^u_{t_1-\delta}(n) + \delta\lambda_{t_1}H_{t_1}(n)J^u_{t_1-\delta}(n-1) + o(\delta)\\ &\geq (1-\delta\lambda_{t_1}H_{t_1}(n))(G+B-K)\tilde{Q}_{t_1-\delta}(n) + \delta\lambda_{t_1}H_{t_1}(n)J^u_{t_1-\delta}(n-1) + o(\delta).\end{aligned}$$

Plugging $\tilde{Q}_{t_1}(n) = (1-\delta\lambda_{t_1}\tilde{H}_{t_1}(n))\tilde{Q}_{t_1-\delta}(n) + \delta\lambda_{t_1}\tilde{H}_{t_1}(n)\tilde{Q}_{t_1-\delta}(n-1) + o(\delta)$ into the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$\tilde{H}_{t_1}(n)\tilde{Q}_{t_1}(n-1) - \left(\tilde{H}_{t_1}(n) - H_{t_1}(n)\right)\tilde{Q}_{t_1}(n) \geq \frac{H_{t_1}(n)J^u_{t_1}(n-1)}{G+B-K}.$$

This contradicts with our assumption that $t_1 > \tau^u(n)$. Therefore, the creator would not upgrade when $t > \tau^u(n)$, i.e., $J^u_t(n) > (G+B-K)\tilde{Q}_t(n)$ for any $t > \tau^u(n)$. Consider what happens in a small time interval δ, we have

$$J^u_t(n) = (1-\delta\lambda_t H_t(n))J^u_{t-\delta}(n) + \delta\lambda_t H_t(n)J^u_{t-\delta}(n-1) + o(\delta).$$

Rearranging and taking the limit as $\delta \to 0$, we thus obtain Equation (OA.4).

We next show that the creator's optimal policy is to upgrade immediately when $t < \tau^u(n)$. Suppose that there exists some $t_2 < \tau^u(n)$, such that $J^u_t(n) = (G+B-K)\tilde{Q}_t(n)$ for all $t \leq t_2$, and $J^u_t(n) > (G+B-K)\tilde{Q}_t(n)$ for $t \in (t_2, t_2+\delta]$. Then, we have

$$\begin{aligned}(G+B-K)\tilde{Q}_{t_2+\delta}(n) &< J^u_{t_2+\delta}(n) = (1-\delta\lambda_{t_2+\delta}H_{t_2+\delta}(n))J^u_{t_2}(n) + \delta\lambda_{t_2+\delta}H_{t_2+\delta}(n)J^u_{t_2}(n-1) + o(\delta)\\ &= (1-\delta\lambda_{t_2+\delta}H_{t_2+\delta}(n))(G+B-K)\tilde{Q}_{t_2}(n) + \delta\lambda_{t_1+\delta}H_{t_2+\delta}(n)J^u_{t_2}(n-1) + o(\delta).\end{aligned}$$

Plugging $\tilde{Q}_{t_2+\delta}(n) = (1-\delta\lambda_{t_2+\delta}\tilde{H}_{t_2+\delta}(n))\tilde{Q}_{t_2}(n) + \delta\lambda_{t_2+\delta}\tilde{H}_{t_2+\delta}(n)\tilde{Q}_{t_2}(n-1) + o(\delta)$ into the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$(G+B-K)\left[\tilde{H}_{t_2}(n)\tilde{Q}_{t_2}(n-1) - \left(\tilde{H}_{t_2}(n) - H_{t_2}(n)\right)\tilde{Q}_{t_2}(n)\right] \leq H_{t_2}(n)J^u_{t_2}(n-1).$$

This contradicts with the assumption that $t_2 < \tau^u(n)$. Therefore, the creator would upgrade immediately when $t < \tau^u(n)$, i.e., $J^u_t(n) = (G+B-K)\tilde{Q}_t(n)$ for any $t < \tau^u(n)$.

Therefore, to prove Theorem 4, it is sufficient to show that $\frac{J^u_t(n)}{\tilde{Q}_t(n)}$ increases in t. We do this by induction. For $n=0$, the statement is obvious since $\frac{J^u_t(0)}{\tilde{Q}_t(0)} = G+B$.

Now assume that the statement is true for $n-1$, and consider the case n. It is trivial when $t \leq \tau^u(n)$ because $\frac{J^u_t(n)}{\tilde{Q}_t(n)} = G+B-K$. Consider next when $t > \tau^u(n)$. Suppose that there exists some $t_3 > \tau^u(n)$ such that $\frac{\partial}{\partial t}\frac{J^u_t(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} < 0$. Then, there must exist some $t_4 \in (\tau^u(n), t_3)$ such that

$\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_4} \geq 0$; otherwise, $\frac{J_{t_3}^u(n)}{\tilde{Q}_{t_3}(n)} < \frac{J_{\tau^u(n)}^u(n)}{\tilde{Q}_{\tau^u(n)}(n)} = G+B-K$, which contradicts with the result that $J_t^u(n) > (G+B-K)\cdot\tilde{Q}_t(n)$ for any $t > \tau^u(n)$. Due to the continuity of $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}$, there exists some $t_5 \in [t_4, t_3)$, such that $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_5} = 0$. That is,

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\bigg|_{t=t_5} &= \frac{\lambda_{t_5}H_{t_5}(n)\left[J_{t_5}^u(n-1)-J_{t_5}^u(n)\right]}{\tilde{Q}_{t_5}(n)} - \frac{\lambda_{t_5}\tilde{H}_{t_5}(n)J_{t_5}^u(n)\left[\tilde{Q}_{t_5}(n-1)-\tilde{Q}_{t_5}(n)\right]}{[\tilde{Q}_{t_5}(n)]^2} \\
&= \lambda_{t_5}H_{t_5}(n)\frac{\tilde{Q}_{t_5}(n-1)}{\tilde{Q}_{t_5}(n)}\left[\frac{J_{t_5}^u(n-1)}{\tilde{Q}_{t_5}(n-1)} - \frac{J_{t_5}^u(n)}{\tilde{Q}_{t_5}(n)} - \left(\frac{\tilde{H}_{t_5}(n)}{H_{t_5}(n)}-1\right)\left(1-\frac{\tilde{Q}_{t_5}(n)}{\tilde{Q}_{t_5}(n-1)}\right)\frac{J_{t_5}^u(n)}{\tilde{Q}_{t_5}(n)}\right] = 0.
\end{aligned}$$

Because $\frac{\tilde{Q}_t(n)}{\tilde{Q}_t(n-1)}$ increases in t, $\frac{\tilde{H}_t(n)}{H_t(n)}$ decreases in t as shown in Theorem 1(iii), and the induction hypothesis that $\frac{J_t^u(n-1)}{\tilde{Q}_t(n-1)}$ increases in t, we have $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} \geq 0$, which contradicts with the assumption that $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} < 0$. Therefore, $\frac{J_t^u(n)}{\tilde{Q}_t(n)}$ increases in t for any $t > \tau^u(n)$, and we thus complete the proof. $\square$

Proof of Corollary 2. (i) Suppose that there exists an n, such that $\tau^u(n) < \tau^u(n-1)$. For any $t \in (\tau^u(n), \tau^u(n-1))$, $J_t^u(n-1) = (G+B-K)\tilde{Q}_t(n-1)$. Using the definition of $\tau^u(n)$, we have

$$\begin{aligned}
&(G+B-K)\left[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n)-H_t(n))\tilde{Q}_t(n)\right] < H_t(n)J_t^u(n-1) \\
\Rightarrow\ &(\tilde{H}_t(n)-H_t(n))(\tilde{Q}_t(n-1)-\tilde{Q}_t(n)) < 0.
\end{aligned}$$

This contradicts with Theorem 1(i) and Assumption 1(i). We thus obtain the announced results.

(ii) Denote $Z_t(n;B,K) = \frac{J_t^u(n;B,R)}{G+B-K}$. Here, we use the notation $J_t^u(n;B,R) \equiv J_t^u(n)$ to emphasize the dependence of $J_t^u(n)$ on B and K. Similarly, we denote $\tau^u(n;B,K) \equiv \tau^u(n)$. Note that Equation (OA.3) can be rewritten as:

$$\tau^u(n;B,K) = \sup\left\{t:\ \tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n)-H_t(n))\tilde{Q}_t(n) \geq H_t(n)Z_t(n-1;B,K)\right\}.$$

Only the RHS of the above inequality depends on B and K. Thus, $\tau^u(n;B,K)$ increases in B and decreases in K if and only if $Z_t(n-1;B,K)$ decreases in B and increases in K. We prove the monotonicity of $Z_t(n;B,K)$ w.r.t. B and K for any $n \geq 0$ by induction.

First when $n=0$, $Z_t(0;B,K) = \frac{G+B}{G+B-K}$. The statement is obvious. Now suppose it is true for any $m \leq n-1$. This implies that $\tau^u(m;B,K)$ increases in B and decreases in K for any $m \leq n$. Thus, we have $\tau^u(n;B_1,K) > \tau^u(n;B_2,K)$ for any $B_1 > B_2 \geq 0$. Consider the following cases w.r.t. t:

When $t \leq \tau^u(n;B_2,K)$, creators of both projects would upgrade project features immediately. Hence $Z_t(n;B_1,K) = Z_t(n;B_2,K) = \tilde{Q}_t(n)$.

When $\tau^u(n;B_2,K) < t \le \tau^u(n;B_1,K)$, only the project with a long-term profit of B_1 would use the stimulus. Therefore, $Z_t(n;B_1,K) = \tilde{Q}_t(n)$ whereas $Z_t(n;B_2,K) \ge \tilde{Q}_t(n) = Z_t(n;B_1,K)$.

When $t > \tau^u(n;B_1,K)$, neither projects activates the stimulus policy. For $i = 1,2$, $Z_t(n;B_i,K)$ is the solution of

$$\frac{dz}{dt} = \lambda_t H_t(n)\left(Z_t(n-1;B_i,K) - z\right),$$

with boundary condition $z(\tau^u(n;B_1,K)) = Z_{\tau^u(n;B_1,K)}(n,B_i,K)$. RHS of the above equation decreases in B_1. Coupling with the inequality $Z_{\tau^u(n;B_1,K)}(n,B_1,K) \le Z_{\tau^u(n;B_1,K)}(n,B_2,K)$, we have $Z_t(n;B_1,K) \le Z_t(n;B_2,K)$.

In a similar fashion, we can show that $Z_t(n;B,K)$ increases in K. This completes the proof. □

Proof of Theorem 5. (i) Since $J_t^b(n) = (G+B)\cdot Q_t(n)$. It is equivalent to show that $\frac{Q_t(n)}{J_t^u(n)}$ increases in t.

When $n = 0$, the statement is obvious as $Q_t(n) = 1$ and $J_t^u(n) = G+B$. Now assume that it's true for $n-1$. Then for n:

When $t < \tau^u(n)$, $J_t^u(n) = (G+B-K)\tilde{Q}_t(n)$. Hence $\frac{Q_t(n)}{J_t^u(n)} = \frac{1}{G+B-K}\frac{Q_t(n)}{\tilde{Q}_t(n)}$. According to Proposition 2, $\frac{Q_t(n)}{J_t^u(n)}$ increases in t.

When $t \ge \tau^u(n)$,

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} &= \frac{\lambda_t H_t(n)\left[Q_t(n-1) - Q_t(n)\right]}{J_t^u(n)} - \frac{Q_t(n)}{J_t^u(n)}\frac{\lambda_t H_t(n)\left[J_t^u(n-1) - J_t^u(n)\right]}{J_t^u(n)} \\
&= \lambda_t H_t(n)\frac{Q_t(n)}{J_t^u(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - \frac{J_t^u(n-1)}{J_t^u(n)}\right] = \lambda_t H_t(n)\frac{J_t^u(n-1)}{J_t^u(n)}\left[\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)}\right].
\end{aligned}$$

At $t = \tau^u(n)$, because $J_t^u(n) = (G+B-K)\tilde{Q}_t(n)$,

$$\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} = \frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{(G+B-K)\tilde{Q}_t(n)}.$$

Also according to Equation (OA.3), $J_t^u(n-1) = \frac{G+B-K}{H_t(n)}[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)]$ at $t = \tau^u(n)$. Hence,

$$\begin{aligned}
&\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} = \frac{H_t(n)\cdot Q_t(n-1)}{(G+B-K)\left[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)\right]} - \frac{Q_t(n)}{(G+B-K)\tilde{Q}_t(n)} \\
&= \frac{1}{G+B-K}\left[\frac{H_t(n)Q_t(n-1)}{\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)} - \frac{Q_t(n)}{\tilde{Q}_t(n)}\right] \\
&= \frac{1}{G+B-K}\frac{Q_t(n)}{\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)}\left[H_t(n)\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \tilde{H}_t(n)\left(\frac{\tilde{Q}_t(n-1)}{\tilde{Q}_t(n)} - 1\right)\right].
\end{aligned}$$

In the proof of Proposition 2, we have shown that for any $t > 0$, $H_t(n)\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \tilde{H}_t(n)\left(\frac{\tilde{Q}_t(n-1)}{\tilde{Q}_t(n)} - 1\right) > 0$. Therefore, $\left.\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\right|_{t=\tau^u(n)} > 0$. Suppose there exists some $t' > \tau^u(n)$ such that $\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} < 0$, then according to the continuity of the functions, there must exist some $\tau^u(n) < t_0 < t'$, such that $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\Big|_{t=t_0} = 0$ and $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} < 0$ in the interval $(t_0, t']$. However, since $\frac{Q_t(n-1)}{J_t^u(n-1)}$ increases in t, $\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)}$ must strictly increase in the interval $(t_0, t']$, implying $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\Big|_{t=t'} > 0$. This leads to contradiction. Therefore, $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} \geq 0$ for any $t > 0$.

(ii) Following a similar approach as the proof for Theorem 3(ii), we can show that $\lim_{T\to\infty} J_{T,N}^u - J_{T,N}^b = \lim_{T\to 0} J_{T,N}^u - J_{T,N}^b = 0$ for any $N \geq 1$. □

Proof of Theorem 6. Denote $A_t(n)$ the optimal expected profit at state (t,n) assuming that the creator has not ended LTO yet. The optimal expected profit over the course of the entire pledging process is denoted by $J_t^l(n)$. We show that $\tau^l(n)$ is given by

$$\tau^l(n) = \sup\Big\{t:\ A_t(n) \geq [G + B - (N-n)k]\cdot Q_t(n)\Big\}, \tag{OA.5}$$

where $A_t(n)$ is the solution of

$$\frac{\partial A_t(n)}{\partial t} = \lambda_t \hat{H}_t(n)\left[J_t^l(n-1) - A_t(n)\right], \tag{OA.6}$$

with boundary conditions $A_0(n) = 0$ for any $n \geq 1$, and $A_t(0) = G + B - Nk$.

Expected profit $J_t^l(n)$ at state (t,n) is given by

$$J_t^l(n) = \begin{cases} A_t(n), & \text{if } t < \tau^l(n) \\ [G+B-(N-n)k]\cdot Q_t(n), & \text{if } t \geq \tau^l(n) \end{cases}.$$

Denote $d_t(n) = \frac{A_t(n)}{Q_t(n)}$. We add to the statement that $d_t(n)$ decreases in t and prove by induction. It's trivial when $n = 0$ because $d_t(0) = \frac{A_t(0)}{Q_t(0)} = G + B - Nk$. Suppose that the statement is true for $n-1$. Taking the derivative of $d_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial d_t(n)}{\partial t} &= \frac{\lambda_t \hat{H}_t(n)\left[J_t^l(n-1) - A_t(n)\right]}{Q_t(n)} - \frac{\lambda_t H_t(n) A_t(n)\left[Q_t(n-1) - Q_t(n)\right]}{[Q_t(n)]^2} \\
&= \lambda_t\left(\hat{H}_t(n)\left[\frac{J_t^l(n-1)}{Q_t(n-1)}\frac{Q_t(n-1)}{Q_t(n)} - \frac{A_t(n)}{Q_t(n)}\right] - H_t(n)\frac{A_t(n)}{Q_t(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - 1\right]\right) \\
&= \lambda_t \hat{H}_t(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right)\right] d_t(n)\right].
\end{aligned}$$

Taking the limit as $t \to 0$ and using L'Hopital's rule, we have

$$\lim_{t\to 0} d_t(n) = \lim_{t\to 0}\frac{\lambda_t \hat{H}_t(n)\left[J_t^l(n-1) - A_t(n)\right]}{\lambda_t H_t(n)\left[Q_t(n-1) - Q_t(n)\right]} = \lim_{t\to 0}\frac{\hat{H}_t(n)}{H_t(n)}\frac{J_t^l(n-1)}{Q_t(n-1)}.$$

By the induction hypothesis, we know that $\frac{J_t^l(n-1)}{Q_t(n-1)}$ decreases in t since $J_t^l(n-1)$ is equal to either $A_t(n-1)$ or $[G+B-(N-n)k]\cdot Q_t(n-1)$. From Theorem 1(iii), $\frac{\hat{H}_t(n)}{H_t(n)}$ decreases in t. Therefore

$\lim_{t\to 0} d_t(n) = \lim_{t\to 0} \frac{\hat{H}_t(n)}{H_t(n)} \frac{J_t^l(n-1)}{Q_t(n-1)}$ exists. Next we show that, if there exists some t_1 such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_1} > 0$, there must be some $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_2} < 0$. Consider the following two cases.

(1) $\lim_{t\to 0} d_t(n) = \infty$. If there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_1} > 0$, then there must exist a $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_2} \le 0$; Otherwise $d_{t_1} \ge \lim_{t\to 0} d_t(n) = \infty$, which is impossible.

(2) $\lim_{t\to 0} d_t(n) < \infty$. This implies that $\lim_{t\to 0} \frac{J_t^l(n-1)}{Q_t(n-1)} < \infty$ and $\lim_{t\to 0} \frac{H_t(n)}{\hat{H}_t(n)} > 0$. Let $S(t) = \frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right)\right] d_t(n)$. Because $\lim_{t\to 0} 1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right) = \lim_{t\to 0} \frac{H_t(n)}{\hat{H}_t(n)} > 0$ and $\lim_{t\to 0} d_t(n) = \lim_{t\to 0} \frac{\hat{H}_t(n)}{H_t(n)} \frac{J_t^l(n-1)}{Q_t(n-1)} < \infty$, we have $\lim_{t\to 0} S(t) = 0$.

Recall that $\frac{\partial d_t(n)}{\partial t} = \lambda_t \hat{H}_t(n) \frac{Q_t(n-1)}{Q_t(n)} \cdot S(t)$. Suppose there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_1} > 0$, then there must exist a $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_2} < 0$; Otherwise, $d_t(n)$ increases within $[0, t_1]$. Coupling with the results that $\frac{H_t(n)}{\hat{H}_t(n)}$ and $\frac{Q_t(n)}{Q_t(n-1)}$ both increase in t, we conclude that $S(t)$ decreases in t. A direct consequence is that $S(t_1) \le S(0) = 0$, which contradicts with the assumption that $\frac{\partial d_t(n)}{\partial t} > 0$.

Consequently, if there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_1} > 0$, there must exist a $t_2 \in [0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_2} \le 0$. Due to the continuity of $\frac{\partial d_t(n)}{\partial t}$, there exists some $t_3 \in [t_2, t_1)$ such that $S(t_3) = 0$, and $S(t) > 0$ for any $t \in (t_3, t_1]$. However, because $d_t(n)$, $\frac{H_t(n)}{\hat{H}_t(n)}$ and $\frac{Q_t(n)}{Q_t(n-1)}$ increase in t, and $\frac{J_t^l(n-1)}{Q_t(n-1)}$ decreases in t for any $t \in (t_3, t_1)$, $S(t)$ should decrease in t, which contradicts with the preceding result. Therefore, $d_t(n)$ must decrease in t for any $t > 0$. Moreover, because $\frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t, $S(t) \ne 0$ for any t. Therefore, for any $t > 0$, $S(t) < 0$ and $d_t(n)$ strictly decreases in t. As a result, $A_t(n) > [G + B - (N-n)k] \cdot Q_t(n)$ for any $t < \tau^l(n)$, and the direction of the inequality is flipped for any $t > \tau^l(n)$.

Next we show that the creator's optimal policy is to end the limited-time offer if and only if $t > \tau^l(n)$. Suppose that there exists some $t_4 < \tau^l(n)$, such that the creator's optimal decision is to end the limited-time offer immediately, i.e., $J_{t_4}^l(n) = [G + B - (N-n)k] \cdot Q_{t_4}(n)$. Then, we have

$$\begin{aligned}
&[G + B - (N-n)k] \cdot Q_{t_4}(n) \\
> &\lambda_{t_4} \hat{H}_{t_4}(n) \delta J_{t_4-\delta}^l(n-1) + (1 - \lambda_{t_4} \hat{H}_{t_4}(n)\delta) J_{t_4-\delta}^l(n) + o(\delta) \\
\ge &\lambda_{t_4} \hat{H}_{t_4}(n) \delta J_{t_4-\delta}^l(n-1) + \left(1 - \lambda_{t_4} \hat{H}_{t_4}(n)\delta\right) A_{t_4-\delta}(n) + o(\delta) = A_{t_4}(n) + o(\delta),
\end{aligned}$$

which contradicts with $t_4 < \tau^l(n)$. Therefore, the creator would not end the limited-time offer for any $t \le \tau^l(n)$. Consider what happens in a small time interval δ, we have

$$J_t^l(n) = (1 - \delta\lambda_t \hat{H}_t(n)) J_{t-\delta}^l(n) + \delta\lambda_t \hat{H}_t(n) J_{t-\delta}^l(n-1) + o(\delta).$$

Rearranging, taking the limit as $\delta \to 0$, and replacing $J_t^l(n)$ with $A_t(n)$, we thus have

$$\frac{\partial A_t(n)}{\partial t} = \lambda_t \hat{H}_t(n) \left[J_t^l(n-1) - A_t(n)\right].$$

Suppose that there exists some $t_5 \geq \tau^l(n)$ such that $J_t^l(n) = [G+B-(N-n)k]\cdot Q_t(n)$ for any $t \leq t_5$ but $J_t^l(n) > [G+B-(N-n)k]\cdot Q_t(n)$ for any $t \in (t_5, t_5+\delta]$. (Because $J^l_{\tau^l(n)} = A_{\tau^l(n)}(n) = [G+B-(N-n)k]\cdot Q_{\tau^l(n)}(n)$, we can always find such $t_5 \geq \tau^l(n)$.) Thus, we have

$$\begin{aligned}
&[G+B-(N-n)k]\cdot Q_{t_5+\delta}(n) \\
<\ & \lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta J^l_{t_5}(n-1) + \left(1-\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta\right) J^l_{t_5}(n) + o(\delta) \\
=\ & \lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta J^l_{t_5}(n-1) + \left(1-\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta\right) [G+B-(N-n)k]\cdot Q_{t_5}(n) + o(\delta).
\end{aligned}$$

Plugging $Q_{t_5+\delta}(n) = \lambda_{t_5+\delta}H_{t_5+\delta}(n)\delta Q_{t_5}(n-1) + (1-\lambda_{t_5+\delta}H_{t_5+\delta}(n)\delta)Q_{t_5}(n) + o(\delta)$ into the inequality above, rearranging, and taking the limit as $\delta \to 0$, we have

$$[G+B-(N-n)k]\cdot\left(H_{t_5}(n)Q_{t_5}(n-1) + (\hat{H}_{t_5}(n)-H_{t_5}(n))Q_{t_5}(n)\right) \leq \hat{H}_{t_5}(n)J^l_{t_5}(n-1).$$

Because that $S(t) = \frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1-\left(1-\frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)\right] d_t(n) < 0$ for any t, we have

$$\hat{H}_{t_5}(n)J^l_{t_5}(n-1) < \left[H_{t_5}(n)Q_{t_5}(n-1) + (\hat{H}_{t_5}(n)-H_{t_5}(n))Q_{t_5}(n)\right] d_{t_5}(n).$$

Combining the preceding two inequalities, we have that $d_{t_5}(n) = \frac{A_{t_5}(n)}{Q_{t_5}(n)} > G+B-(N-n)k$, which contradicts with $t_5 > \tau^l(n)$. Therefore, the creator's optimal policy is to end the limited-time offer for any $t > \tau^l(n)$, i.e., $J_t^l(n) = [G+B-(N-n)k]\cdot Q_t(n)$ for any $t > \tau^l(n)$. We thus obtain the announced results. $\square$

Proof of Corollary 3. Denote $W_t(n;B,k) = \frac{A_t(n)}{G+B-(N-n)k}$. We prove by induction that $W_t(n;B,k)$ increases in B and decreases in k. When $n=0$, this statement is trivial as $W_t(0;B,k) = 1$. Now suppose it is true for $n-1$ and consider the case n:

$$\frac{\partial W_t(n;B,k)}{\partial t} = \lambda_t\hat{H}_t(n)\left[\frac{J_t^l(n-1)}{G+B-(N-n+1)k}\frac{G+B-(N-n+1)k}{G+B-(N-n)k} - W_t(n;B,k)\right],$$

with boundary condition $W_0(n;B,k) = 0$ for any $n > 0$. Since $J_t^l(n-1) = \max\{A_t(n-1), [G+B-(N-n+1)k]Q_t(n-1)\}$, $\frac{J_t^l(n-1)}{G+B-(N-n+1)k}$ increases in B and decreases in k based on the induction hypothesis for $n-1$. It is also obvious that $\frac{G+B-(N-n+1)k}{G+B-(N-n)k}$ increases in B and decreases in k. Therefore, the RHS of the equation above increases in B and decreases in k, which implies that $W_t(n;B,k)$ increases in B and decreases in k. We thus proved the statement for n.

Denote $\tau^l(n;B,k) \equiv \tau^l(n)$ to emphasize the dependence of $\tau^l(n)$ on B and k. From Equation (OA.5), we have $\tau^l(n;B,k) = \sup\{t : W_t(n;B,k) \geq Q_t(n)\}$. Consider any $B_1 > B_2$. For any $t \leq \tau^l(n;B_2,k)$, we have $W_t(n;B_1,k) \geq W_t(n;B_2,k) \geq Q_t(n)$. Therefore, $\tau^l(n;B_1,k) \geq \tau^l(n;B_2,k)$. Similarly we can show that $\tau^l(n;B,k)$ decreases in k. We thus obtain the announced result. $\square$

Proof of Theorem 7. (i) follows directly from the proof of Theorem 6, where we show that $\frac{J_t^l(n)}{Q_t(n)}$ decreases in t.

Next we prove (ii). When $T \to 0$, both $J_{T,N}^l$ and $J_{T,N}^b \to 0$. On the other hand, when $T \geq \tau^l(N)$, the creator ends the LTO immediately, so we have $J_{T,N}^l = J_{T,N}^b$. Thus, $\lim_{T\to\infty} J_{T,N}^l - J_{T,N}^b = \lim_{T\to 0} J_{T,N}^l - J_{T,N}^b = 0$, and thus we obtain the announced results. $\square$

OA.2. Extension: Multiple Rounds of Stimulus

For analytical tractability, we restrict our attention to the circumstance where a creator can apply the stimulus only once in the main text. However, as we can see from Table 2, creators typically update their projects rather frequently in practice, especially for those successful projects. In this section, we extend the model in Section 4 to consider multiple rounds of stimulus offerings for the two reactive stimulus policies: seeding and feature upgrade. We show that the optimal strategies still follow the threshold structure in the sense that the creator should adopt the stimuli if and only the time-to-go is shorter than a cutoff, and that the cutoff increases in pledge-to-go n. For limited-time offers, when there are multiple LTOs in effect, the decision to end one of them would depend on the total funds collected at a given time, which makes the problem significantly more complicated. While we hypothesize that the optimal strategy is a threshold policy, the proof is beyond the scope of this paper, which we leave for future research.

OA.2.1. Seeding

Suppose that the creator is able to offer up to $n_0 \geq 1$ seeds, potentially in multiple rounds. Denote $0 \leq m \leq n_0$ as the number of seeds left at a given point during the crowdfunding campaign. At the state of time-to-go t, pledges needed n and seeds left m, the expected profit is denoted as $J_t^s(n,m)$. The cost of the ith seed is assumed to be $R_i \geq 0$ for any $1 \leq i \leq n_0$.

Proposition OA.1. *For any (n,m), there exists a $0 \leq \tau^s(n,m) \leq \infty$, such that:*

- *When $t \leq \tau^s(n,m)$, the creator will activate seeding stimulus right away. That is, $J_t^s(n,m) = J_t^s(n-1,m-1)$ for any $t \leq \tau^s(n,m)$;*
- *When $t > \tau^s(n,m)$, the creator is better-off withholding seeding stimulus. The expected profit $J_t^s(n,m)$ in this case is given by:*

$$\frac{\partial J_t^s(n,m)}{\partial t} = \lambda_t H_t(n) \left(J_t^s(n-1,m) - J_t^s(n,m) \right),$$

with boundary condition $J_{\tau^s(n,m)}^s(n,m) = J_{\tau^s(n,m)}^s(n-1,m-1)$, $J_t^s(n,0) = (G+B-\sum_{i=1}^{n_0} R_i)Q_t(n)$, $J_0^s(0,m) = G+B-\sum_{i=m+1}^{n_0} R_i$, and $J_0^s(n,m) = 0$ for all $n > 0$.

Moreover, $\tau^s(n,m)$ increases in n.

Proof of Proposition OA.1. First for any (n,m) and $t>0$,

$$J^s_{t+\delta}(n,m) \geq (1-\lambda_t H_t(n)\delta) J^s_t(n,m) + \lambda_t H_t(n)\delta J^s_t(n-1,m) + o(\delta).$$

Let $\delta \to 0$, we get the following inequality:

$$\frac{\partial J^s_t(n,m)}{\partial t} \geq \lambda_t H_t(n) \left[J^s_t(n-1,m) - J^s_t(n,m) \right]. \tag{OA.7}$$

We add the following statements to the proposition and prove by induction.

(1) For any $1 \leq j \leq m$, $\frac{J^s_t(n,m)}{J^s_t(n-j,m-j)}$ increases in t.

(2) $\frac{J^s_t(n,m)}{J^s_t(n-1,m)}$ increases in t when $t \geq \tau^s(n,m)$.

First when $m=1$, we already prove the threshold structure and the monotonicity of the thresholds in Theorem 2. We also show that statement (1) holds for $m=1$ in the proof of Theorem 2. In addition, in the proof of Theorem 3, we show that $\frac{J^s_t(n,0)}{J^s_t(n,1)}$ increases in t for any n. Therefore $\frac{J^s_t(n,1)}{J^s_t(n-1,1)} = \frac{J^s_t(n,1)}{J^s_t(n-1,0)} \frac{J^s_t(n-1,0)}{J^s_t(n-1,1)}$ increases in t. Thus the statements are true for $m=1$.

Now consider $m>1$. Suppose the statements are true for any n and $m-i$ where $i \geq 1$. When $n=1$, it is obvious that the optimal strategy is to wait to use the seeding stimulus right before time expires, i.e., $\tau^s(1,m)=0$. So the statements are true for $n=1$.

Assume that the statements are true for some $n-1$, where $n>1$. We prove the threshold structure for n by contradiction. Suppose it is not true, then there exists an time interval $(\underline{t}, \underline{t}+h)$ over which the stimulus will not be used. Because of the continuity of $J^s_t(n,m)$, $J^s_{\underline{t}}(n,m) = J^s_{\underline{t}}(n-1,m-1)$ and $J^s_{\underline{t}+h}(n,m) = J^s_{\underline{t}+h}(n-1,m-1)$. For any $t \in (\underline{t}, \underline{t}+h)$, $J^s_t(n,m) > J^s_t(n-1,m-1)$. Now for every $t \in [\underline{t}, \underline{t}+h]$, we find $j = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq t\}$. We collect all those unique j's, and denote them as $j_0 > j_1 > \cdots > j_\kappa$, where $j_0 = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq \underline{t}\}$ and $j_\kappa = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq \underline{t}+h\}$. For any $\tau^s(n-j_i,m-j_i) \leq t \leq \tau^s(n-j_{i+1},m-j_{i+1})$, $J^s_t(n,m) = J^s_t(n-1,m-1) = \cdots = J^s_t(n-j_i,m-j_i)$.

Since the optimal strategy is not to use the stimulus at $\underline{t}+\delta$ and $J^s_{\underline{t}}(n,m) = J^s_{\underline{t}}(n-j_0,m-j_0)$,

$$\begin{aligned} J^s_{\underline{t}+\delta}(n,m) &= (1-\lambda_{\underline{t}} H_{\underline{t}}(n)\delta) J^s_{\underline{t}}(n,m) + \lambda_{\underline{t}} H_{\underline{t}}(n)\delta J^s_{\underline{t}}(n-1,m) + o(\delta) \\ &= (1-\lambda_{\underline{t}} H_{\underline{t}}(n)\delta) J^s_{\underline{t}}(n-j_0,m-j_0) + \lambda_{\underline{t}} H_{\underline{t}}(n)\delta J^s_{\underline{t}}(n-1,m) + o(\delta) \\ &> (1-\lambda_{\underline{t}} H_{\underline{t}}(n-j_0)\delta) J^s_{\underline{t}}(n-j_0,m-j_0) + \lambda_{\underline{t}} H_{\underline{t}}(n-j_0)\delta J^s_{\underline{t}}(n-j_0-1,m-j_0) + o(\delta) \end{aligned}$$

Let $\delta \to 0$, we have

$$H_t(n-j_0) \left[J^s_t(n-j_0-1,m-j_0) - J^s_t(n-j_0,m-j_0) \right] < H_t(n) \left[J^s_t(n-1,m) - J^s_t(n-j_0,m-j_0) \right],$$

at $t = \underline{t}$. Rearrange the terms:

$$1 + \left[\frac{H_t(n-j_0)}{H_t(n)} - 1\right]\left[1 - \frac{J^s_t(n-j_0, m-j_0)}{J^s_t(n-j_0-1, m-j_0)}\right] < \frac{J^s_t(n-1,m)}{J^s_t(n-j_0-1,m-j_0)},$$

at $t = \underline{t}$. According to our induction assumptions, the LHS decreases in t for any $t \geq \tau^s(n-j_0, m-j_0)$ and RHS increases in t. Thus the inequality holds for any $t > \underline{t}$. Also for any $\underline{t} \leq t < \tau^s(n-j_1, m-j_1)$, $J^s_t(n-j_1, m-j_1) = J^s_t(n-j_0, m-j_0)$. According to Inequality (OA.7), we have

$$\begin{aligned}
&\frac{\partial J^s_t(n, m-j_1)}{\partial t} = \frac{\partial J^s_t(n, m-j_0)}{\partial t} \\
&= H_t(n-j_0)\left[J^s_t(n-j_0-1, m-j_0) - J^s_t(n-j_0, m-j_0)\right] \\
&\geq H_t(n-j_1)\left[J^s_t(n-j_1-1, m-j_1) - J^s_t(n-j_1, m-j_1)\right]
\end{aligned}$$

Thus, the following inequality holds for any $\underline{t} \leq t \leq \tau^s(n-j_1, m-j_1)$.

$$H_t(n-j_1)\left[J^s_t(n-j_1-1, m-j_1) - J^s_t(n-j_1, m-j_1)\right] < H_t(n)\left[J^s_t(n-1, m) - J^s_t(n-j_0, m-j_0)\right].$$

Note that $J^s_t(n-j_0, m-j_0) = J^s_t(n-j_1, m-j_1)$ at $t = \tau^s(n-j_1, m-j_1)$. Thus,

$$H_t(n-j_1)\left[J^s_t(n-j_1-1, m-j_1) - J^s_t(n-j_1, m-j_1)\right] < H_t(n)\left[J^s_t(n-1, m) - J^s_t(n-j_1, m-j_1)\right],$$

at $t = \tau^s(n-j_1, m-j_1)$. In a similar manner, we can show that for any $t > \tau^s(n-j_\kappa, m-j_\kappa)$,

$$H_t(n-j_\kappa)\left[J^s_t(n-j_\kappa-1, m-j_\kappa) - J^s_t(n-j_\kappa, m-j_\kappa)\right] < H_t(n)\left[J^s_t(n-1, m) - J^s_t(n-j_\kappa, m-j_\kappa)\right].$$

On the other hand, at $t = \underline{t} + h$, the optimal strategy is to activate the stimulus, which means that

$$\begin{aligned}
&J^s_{\underline{t}+h}(n,m) = J^s_{\underline{t}+h}(n-j_\kappa, m-j_\kappa) \\
&= \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n-j_\kappa)\delta\right) J^s_{\underline{t}+h-\delta}(n-j_\kappa, m-j_\kappa) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n-j_\kappa)\delta J^s_{\underline{t}+h-\delta}(n-j_\kappa-1, m-j_\kappa) + o(\delta) \\
&\geq \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta\right) J^s_{\underline{t}+h-\delta}(n,m) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta J^s_{\underline{t}+h-\delta}(n-1,m) + o(\delta) \\
&\geq \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta\right) J^s_{\underline{t}+h-\delta}(n-j_\kappa, m-j_\kappa) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta J^s_{\underline{t}+h-\delta}(n-1,m) + o(\delta).
\end{aligned}$$

This would imply that

$$H_t(n-j_\kappa)\left[J^s_t(n-j_\kappa-1, m-j_\kappa) - J^s_t(n-j_\kappa, m-j_\kappa)\right] \geq H_t(n)\left[J^s_t(n-1, m) - J^s_t(n-j_\kappa, m-j_\kappa)\right],$$

and therefore leads to contradiction.

Next we show that $\tau^s(n,m) < \tau^s(n-1,m)$. Suppose this is not true. Then for any $\tau^s(n,m) \leq t \leq \tau^s(n-1,m)$, $J_t^s(n,m) > J_t^s(n-1,m-1)$ and $J_t^s(n-1,m) = J_t^s(n-2,m-1)$. Thus,

$$\frac{\partial J_t^s(n,m)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1,m) - J_t^s(n,m)\right] \leq \lambda_t H_t(n-1)\left[J_t^s(n-2,m-1) - J_t^s(n,m)\right].$$

On the other hand, $\frac{\partial J_t^s(n-1,m-1)}{\partial t} \geq \lambda_t H_t(n-1)[J_t^s(n-2,m-1) - J_t^s(n-1,m-1)]$ according to Inequality (OA.7). Since $J_{\tau^s(n,m)}^s(n,m) = J_{\tau^s(n,m)}^s(n-1,m-1)$, $J_t^s(n,m) \leq J_t^s(n-1,m-1)$ for any $t \in (\tau^s(n,m), \tau^s(n-1,m))$, which leads to contradiction.

Finally we prove statements (1) and (2) for n and m. For (1), if $t \leq \tau^s(n,m)$, $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} = \frac{J_t^s(n-1,m-1)}{J_t^s(n-j,m-j)}$ increases in t from the induction assumption. Thus all we need to show is that for a given $t > \tau^s(n,m)$, $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} \geq 0$ for any $j \leq \min\{j \geq 1 : \tau^s(n-j,m-j) < t\}$. This derivative is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} \\
=&\frac{\lambda_t H_t(n)[J_t^s(n-1,m) - J_t^s(n,m)]}{J_t^s(n-j,m-j)} - \frac{\lambda_t H_t(n-j) J_t^s(n,m)\left[J_t^s(n-j-1,m-j) - J_t^s(n-j,m-j)\right]}{[J_t^s(n-j,m-j)]^2} \\
=&\lambda_t \Bigg\{ H_t(n)\left[\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)}\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)} - \frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right] - \\
&H_t(n-j)\frac{J_t^s(n,m)}{J_t(n-j,m-j)}\left[\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)} - 1\right]\Bigg\} \\
=&\lambda_t H_t(n)\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)}\Bigg\{\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)} - \frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)} - \\
&\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\frac{H_t(n-j)}{H_t(n)}\left[1 - \frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}\right]\Bigg\} \\
=&\lambda_t H_t(n)\frac{J_t^s(n-2,m-1)}{J_t^s(n-1,m-1)}\Bigg\{\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)} - \\
&\left[1 + \left(\frac{H_t(n-j)}{H_t(n)} - 1\right)\left(1 - \frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}\right)\right]\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\Bigg\}.
\end{aligned}$$

Note that $\left.\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=\tau^s(n,m)} = 1$ and $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} > 1$ for any $t > \tau^s(n,m)$. Thus $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}$ increases in t initially. Now suppose there exists a t_1 such that $\left.\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=t_1} < 0$. We can then find a $t_2 < t_1$ such that $\left.\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=t_2} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} < 0$ for any $t \in (t_2, t_1]$. However, $\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)}$ increases in t for any $t \geq \tau^s(n-1,m)$, $\frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}$ increases in t for any $t \geq \tau^s(n-j,m-j)$, and $\frac{H_t(n-j)}{H_t(n)}$ decreases in t. This means that $\frac{\partial}{\partial t}\frac{J^s(n,m)}{J^s(n-j,m-j)} > 0$ over $(t_1, t_2]$, which leads to contradiction. Thus $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}$ increases in t when $t \geq \tau^s(n,m)$.

Finally we prove statement (ii). Since $\tau^s(n,m) \geq \tau^s(n-1,m)$, for any $t \geq \tau^s(n,m)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} =& \lambda_t H_t(n)\left(1-\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\right) - \lambda_t H_t(n-1)\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)\\ =& \lambda_t H_t(n)\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\left[\frac{J_t^s(n-1,m)}{J_t^s(n,m)}-1-\frac{H_t(n-1)}{H_t(n)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)\right].\end{aligned}$$

Note that $\frac{J_t^s(n,m)}{J_t^s(n-1,m)} < 1$ for any finite t and $\lim_{t\to\infty}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} = 1$. This means that $\frac{J_t^s(n,m)}{J_t^s(n-1,m)}$ approaches 1 from below. Thus, if it does not increase in t for any $t \geq \tau^s(n+1,m)$, there must exist $t_2 > t_1 \geq \tau^s(n,m)$ such that $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\Big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} \leq 0$ for any $t \in (t_1,t_2]$. However $\frac{H_t(n-1)}{H_t(n)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)$ strictly decreases in t from our induction assumption about $n-1$. This means that $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} > 0$ over $(t_1,t_2]$, which leads to contradiction. We thus prove the announced results. □

OA.2.2. Feature Upgrade

Suppose the creator can make at most n_u updates according to a predefined sequence (, which is possibly determined by the potential benefit of each update). We use $m(\leq n_u)$ to denote the number of feature upgrades that remains to be implemented. With m upgrades left, the corresponding pledge likelihood is denoted as $\tilde{H}^{(m)}(q)$. More upgrades make the project more attractive in the sense that $\tilde{H}^{(0)}(q) \geq \tilde{H}^{(1)}(q) \geq \cdots \geq \tilde{H}^{(n_u)}(q) = H(q)$. Similarly, the corresponding success probability $\tilde{Q}_t^m(n)$ satisfies the condition $\tilde{Q}_t^{(0)}(n) > \tilde{Q}_t^{(1)}(n) > \cdots > \tilde{Q}_t^{(n_u)}(n) = Q_t(n)$. As a direct extension of Assumption 1(iii), we assume that $\frac{\tilde{H}^{(m-1)}(q)}{\tilde{H}^{(m)}(q)}$ decreases in q for any m.

Cost of ith update is assumed to be K_i, $i=1,\ldots,n_u$. At the state of time-to-go t, pledges needed n and upgrades remaining m, the expected profit is denoted as $J_t^u(n,m)$.

Proposition OA.2. *For any (n,m), there exists a $0 \leq \tau^u(n,m) \leq \infty$, such that:*

- *When $t \leq \tau^u(n,m)$, the optimal strategy is to upgrade project features right away. That is, $J_t^u(n,m) = J_t^u(n,m-1)$ for any $t \leq \tau^u(n,m)$;*
- *When $t > \tau^u(n,m)$, the creator is better-off withholding feature upgrades. The expected profit $J_t^u(n,m)$ in this case is given by:*

$$\frac{\partial J_t^u(n,m)}{\partial t} = \lambda_t \tilde{H}_t(n,m)\left(J_t^u(n-1,m) - J_t^u(n,m)\right),$$

with boundary conditions $J^u_{\tau^u(n,m)}(n,m) = J^u_{\tau^u(n,m)}(n,m-1)$, $J_t^u(n,0) = (G+B-\sum_{i=1}^{n_u}K_i)\tilde{Q}_t^{(0)}(n)$, $J_0^u(0,m) = G+B-\sum_{i=m+1}^{n_u}K_i$, and $J_0^u(n,m) = 0$ for all $n>0$.

Moreover, $\tau^u(n,m)$ increases in n.

Proof of Proposition OA.2. First, for any (n,m) and $t>0$, we have

$$J^u_{t+\delta}(n,m) \geq \left[1-\lambda_t\tilde{H}_t(n,m)\delta\right] J^u_t(n,m)+\lambda_t\tilde{H}_t(n,m)\delta J^u_t(n-1,m)+o(\delta).$$

Let $\delta\to 0$, we obtain the following inequality:

$$\frac{\partial J^u_t(n,m)}{\partial t} \geq \lambda_t\tilde{H}_t(n,m)\left[J^u_t(n-1,m)-J^u_t(n,m)\right]. \tag{OA.8}$$

For proof convenience, let $J^u_t(-1,m)=J^u_t(0,m)$ and $\tau^u_t(-1,m)=0$. It's easy to see that $J^u_t(0,m)$ is indeed the unique solution of the differential equation where $n=0$.

We add the following statements to the Proposition and prove by induction

(i) For any $0\leq j\leq m$, $\frac{J^u_t(n,m)}{J^u_t(n,m-j)}$ increases in t.

(ii) $\frac{J^u_t(n,m)}{J^u_t(n-1,m)}$ increases in t for $t\geq \tau^u(n,m)$.

First when $m=0$, the threshold structure and the monotonicity of the thresholds follow immediately from our earlier denotation. Statement (i) is trivial as j can only be 0, and Statement (ii) holds according to Proposition 2.

Now suppose that the statements are true for any $n\geq 0$ and $m-1$ where $m\geq 1$. We next show that they must also hold for n and m. First it is obvious that the statements hold for $n=0$. Now assume that they hold for $n-1$ where $n\geq 1$. We first show that the optimal stimulus strategy is a threshold policy. If this is not true, then we can find a time interval $(\underline{t},\underline{t}+h)$, over which the optimal strategy is not to upgrade the project features, i.e., $J^u_t(n,m)>J^u_t(n,m-1)$ over $(\underline{t},\underline{t}+h)$. Because of the continuity of $J^u_t(n,m)$, we have $J^u_{\underline{t}}(n,m)=J^u_{\underline{t}}(n,m-1)$ and $J^u_{\underline{t}+h}(n,m)=J^u_{\underline{t}+h}(n,m-1)$. For each $t\in[\underline{t},\underline{t}+h]$, there exists a $j=\min\{i\geq 1:\tau^u(n,m-i)\leq t\}$. We collect all those unique j's and denote them as $j_0>j_1>\cdots>j_\kappa$, where $j_0=\min\{i\geq 1:\tau^u(n,m-i)\leq \underline{t}\}$ and $j_\kappa=\min\{i\geq 1:\tau^u(n,m-i)\leq \underline{t}+h\}$. When $t=\underline{t}+\delta$, the optimal strategy is not to upgrade project features, but $J^u_{\underline{t}}(n,m)=J^u_{\underline{t}}(n,m-j_0)$. Thus,

$$\begin{aligned}
J^u_{\underline{t}+\delta}(n,m) &= \left(1-\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m)\delta\right) J^u_{\underline{t}}(n,m)+\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m)\delta J^u_{\underline{t}}(n-1,m)+o(\delta)\\
&> \left(1-\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m-j_0)\delta\right) J^u_{\underline{t}}(n,m-j_0)+\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m-j_0)\delta J^u_{\underline{t}}(n-1,m-j_0)+o(\delta).
\end{aligned}$$

Let $\delta\to 0$, we have

$$\tilde{H}_t(n,m)\left[J^u_t(n-1,m)-J^u_t(n,m-j_0)\right] > \tilde{H}_t(n,m-j_0)\left[J^u_t(n-1,m-j_0)-J^u_t(n,m-j_0)\right],$$

at $t=\underline{t}$. Rearrange the terms, at $t=\underline{t}$, we have

$$\left(\frac{\tilde{H}_t(n,m-j_0)}{\tilde{H}_t(n,m)}-1\right)\left[1-\frac{J^u_t(n,m-j_0)}{J^u_t(n-1,m-j_0)}\right] \leq \frac{J^u_t(n-1,m)}{J^u_t(n-1,m-j_0)}.$$

According to our induction assumptions, LHS decreases in t for $t \geq \underline{t} \geq \tau^u(n, m-j_0)$ and RHS increases in t. Thus the above inequality holds for any $t > \underline{t}$. In addition, $J_t^u(n, m-j_1) = J_t^u(n, m-j_0)$ for any $\tau^u(n, m-j_0) < t \leq \tau^u(n, m-j_1)$, which leads to

$$\frac{\partial J_t^u(n, m-j_1)}{\partial t} = \frac{\partial J_t^u(n, m-j_0)}{\partial t} = \tilde{H}_t(n, m-j_0)\left[J_t^u(n-1, m-j_0) - J_t^u(n, m-j_0)\right].$$

According to Inequality (OA.8),

$$\tilde{H}_t(n, m-j_0)\left[J_t^u(n-1, m-j_0) - J_t^u(n, m-j_0)\right] \geq \tilde{H}_t(n, m-j_1)\left[J_t^u(n-1, m-j_1) - J_t^u(n, m-j_1)\right].$$

Coupling with the fact that $J_t^u(n, m-j_0) = J_t^u(n, m-j_1)$ for any $t \in [\tau^u(n, m-j_0), \tau^u(n, m-j_1)]$, we thus have

$$\tilde{H}_t(n, m)\left[J_t^u(n-1, m) - J_t^u(n, m-j_1)\right] > \tilde{H}_t(n, m-j_1)\left[J_t^u(n-1, m-j_1) - J_t^u(n, m-j_1)\right].$$

Similarly, we can show that for any $t > \tau^u(n, m-j_\kappa)$,

$$\tilde{H}_t(n, m)\left[J_t^u(n-1, m) - J_t^u(n, m-j_\kappa)\right] > \tilde{H}_t(n, m-j_\kappa)\left[J_t^u(n-1, m-j_\kappa) - J_t^u(n, m-j_\kappa)\right].$$

However for any $t > \underline{t} + h$,

$$\begin{aligned}
& J_t^u(n, m) = J_t^u(n, m-j_\kappa) \\
= & \left(1 - \lambda_{t-\delta}\tilde{H}_{t-\delta}(n, m-j_\kappa)\delta\right) J_{t-\delta}^u(n, m-j_\kappa) + \lambda_{t-\delta}\tilde{H}_{t-\delta}(n, m-j_\kappa)\delta J_{t-\delta}^u(n-1, m-j_\kappa) \\
\geq & \left(1 - \lambda_{t-\delta}\tilde{H}_{t-\delta}(n, m)\delta\right) J_{t-\delta}^u(n, m-j_\kappa) + \lambda_{t-\delta}\tilde{H}_{t-\delta}(n, m)\delta J_{t-\delta}^u(n-1, m) + o(\delta).
\end{aligned}$$

Let $\delta \to 0$,

$$\tilde{H}_t(n, m)\left[J_t^u(n-1, m) - J_t^u(n, m-j_\kappa)\right] \leq \tilde{H}_t(n, m-j_\kappa)\left[J_t^u(n-1, m-j_\kappa) - J_t^u(n, m-j_\kappa)\right],$$

which leads to contradiction. Thus a unique threshold $\tau^u(n, m)$ exists, such that the optimal policy is to upgrade the features when $t \leq \tau^u(n, m)$, and not to upgrade when $t > \tau^u(n, m)$.

Next we show $\tau^u(n, m) \geq \tau^u(n-1, m)$ by contradiction. Suppose this is not true. Then for any $\tau^u(n, m) < t \leq \tau^u(n-1, m)$, $J^u(n, m) > J_t^u(n, m-1)$ and $J_t^u(n-1, m) = J_t^u(n-1, m-1)$. Thus,

$$\begin{aligned}
\frac{\partial J_t^u(n, m)}{\partial t} & = \lambda_t \tilde{H}_t(n, m)\left[J_t^u(n-1, m) - J_t^u(n, m)\right] \\
& = \lambda_t \tilde{H}_t(n, m)\left[J_t^u(n-1, m-1) - J_t^u(n, m)\right] \\
& < \lambda_t \tilde{H}_t(n, m-1)\left[J_t^u(n-1, m-1) - J_t^u(n, m)\right].
\end{aligned}$$

On the other hand, $\frac{\partial J_t^u(n,m-1)}{\partial t} \geq \lambda_t \tilde{H}_t(n,m-1)[J_t^u(n-1,m-1) - J_t^u(n,m-1)]$ according to Inequality (OA.8). Since $J^u_{\tau^u(n,m)}(n,m) = J^u_{\tau^u(n,m)}(n,m-1)$, $J_t^u(n,m) < J_t^u(n,m-1)$ for any $\tau^u(n,m) < t \leq \tau^u(n-1,m)$, which leads to contradiction.

Now we show that statements (i) and (ii) are true for n. For any $t \leq \tau^u(n,m)$, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} = \frac{J_t^u(n,m-1)}{J_t^u(n,m-j)}$ increases in t based on the induction assumptions. Thus in order to prove statement (i), we only need to focus on $t > \tau^u(n,m)$. Without loss of generality, we assume that $\tau^u(n,m-j) < \tau^u(n,m)$ (Otherwise $J_t^u(n,m-j) = J_t^u(n,m-j')$, where $j' = \min\{i < j : \tau^u(n,m-i) < \tau^u(n,m)\}$ for any $t > \tau^u(n,m)$).

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} =& \frac{\lambda_t \tilde{H}_t(n,m)[J_t^u(n-1,m) - J_t^u(n,m)]}{J_t^u(n,m-j)} - \\
& \frac{\lambda_t \tilde{H}_t(n,m-j) J_t^u(n,m)\left[J_t^u(n-1,m-j) - J_t^u(n,m-j)\right]}{[J_t^u(n,m-j)]^2} \\
=& \lambda_t \Bigg\{ \tilde{H}_t(n,m)\left[\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)}\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)} - \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\right] - \\
& \tilde{H}_t(n,m-j)\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\left[\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)} - 1\right]\Bigg\} \\
=& \lambda_t \tilde{H}_t(n,m)\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)}\Bigg\{\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)} - \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)} - \\
& \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)}\left[1 - \frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)}\right]\Bigg\} \\
=& \lambda_t \tilde{H}_t(n,m)\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)}\Bigg\{\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)} - \\
& \left[1 + \left(\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)} - 1\right)\left(1 - \frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)}\right)\right]\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\Bigg\}.
\end{aligned}$$

When $t = \tau^u(n,m)$, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} = 1$, whereas $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} > 1$ for any $t > \tau^u(n,m)$. Thus, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)}$ increases in t at $t = \tau^u(n,m)$. Suppose there exists a $t_1 > \tau^u(n,m)$ such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} < 0$. Because of the continuity of $\frac{J_t^u(n,m)}{J_t^u(n,m-j)}$, there must exist a $t_2 < t_1$ such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\Big|_{t=t_2} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-1)} < 0$ for any $t \in (t_2, t_1]$. However, $\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)}$ strictly increases in t according to our induction assumption, and $\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)}$ decreases in t according to our assumption. This means that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} \geq 0$ over $(t_2, t_1]$, which leads to contradiction.

Finally we prove statement (ii) by contradiction. Suppose it is not true. Then there must exist $t_2 > t_1 \geq \tau^u(n,m)$, such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)}\big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)} < 0$ for any $t \in (t_1, t_2]$. Because $\tau^u(n,m) \geq \tau^u(n-1,m)$, we have

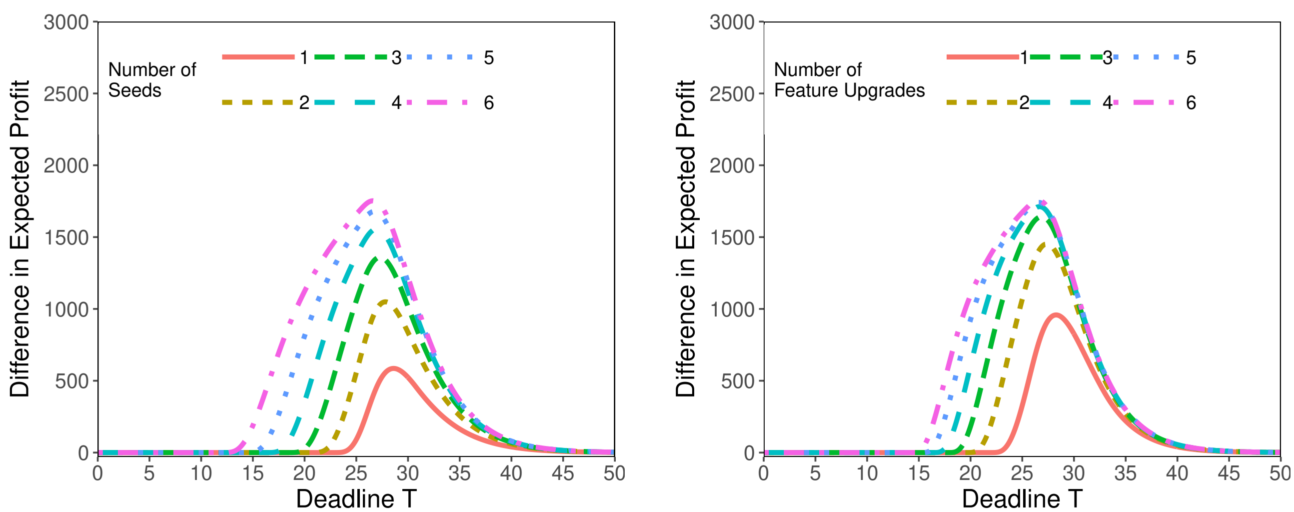
$$\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)} = \lambda_t\tilde{H}_t(n,m)\left(1-\frac{J_t^u(n,m)}{J_t^u(n-1,m)}\right) - \lambda_t\tilde{H}_t(n-1,m)\frac{J_t^u(n,m)}{J_t^u(n-1,m)}\left(\frac{J_t^u(n-2,m)}{J_t^u(n-1,m)}-1\right)$$
$$= \lambda_t\tilde{H}_t(n,m)\frac{J_t^u(n,m)}{J_t^u(n-1,m)}\left[\frac{J_t^u(n-1,m)}{J_t^u(n,m)} - 1 - \frac{\tilde{H}_t(n-1,m)}{\tilde{H}_t(n,m)}\left(\frac{J_t^u(n-2,m)}{J_t^u(n-1,m)}-1\right)\right].$$

However, $\frac{J_t^u(n-2,m)}{J_t^u(n-1,m)}$ decreases in t as $t \geq \tau^u(n,m) \geq \tau^u(n-1,m)$, and $\frac{\tilde{H}_t(n-1,m)}{\tilde{H}_t(n,m)}$ decreases in t according to Theorem 1(iii). Consequently, if $\frac{J_t^s(n-1,m)}{J_t^s(n,m)}$ decreases in t, then $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)} \geq 0$ over $(t_1, t_2]$, which leads to contradiction. We thus obtained the announced results. □

OA.2.3. Numerical Experiments

In this section, we complement our analytical results with a numerical analysis illustrating the benefit of multiple rounds of stimuli. Parameters of the numerical experiments are specified as follows. For seeding, we consider the case where each seeding stimulus allows the creator to acquire 1 pledge at a cost of \$120. For feature upgrade, we consider the case where each upgrade in project features costs the creator $K = \$120$, and improves the project quality by 0.1.

Figure OA.1 Benefits of Mutiple Rounds of Stimulus Policies



(a) Improvement in Expected Profit: *Seeding* (b) Improvement in Expected Profit: *Feature Upgrade*

Note: $V \sim \exp(\frac{1}{100})$, $p = \$120$, $\theta = 1$, $c = \$30$, $G = \$1,800$ (i.e., $N = 15$), $B = \$500$, $T = 30$ and $\lambda_t = 2$. The benchmark is the base model with no stimulus.

Figure OA.1 illustrates the change in the expected profit w.r.t. the number of stimuli and the deadline. While access to additional rounds of stimuli always improve the expected profit, the absolute benefit is non-monotonic w.r.t. the deadline T. When the deadline T is small, having more rounds of the stimuli helps little because the project has little chance to succeed even if multiple stimuli are applied. At the other end of the spectrum, when the deadline T is sufficiently large, again multiple rounds of stimuli render little benefit as the project is likely to reach the target without help of any stimulus policies. Similar to our observation from the numerical analysis in Section 4.4, we also observe that the stimulus policy with multiple rounds of updates is the most effective when the remaining time is neither too long or too short.