Online Appendix to "Global and Robust Stability in a General Price and Assortment Competition Model"

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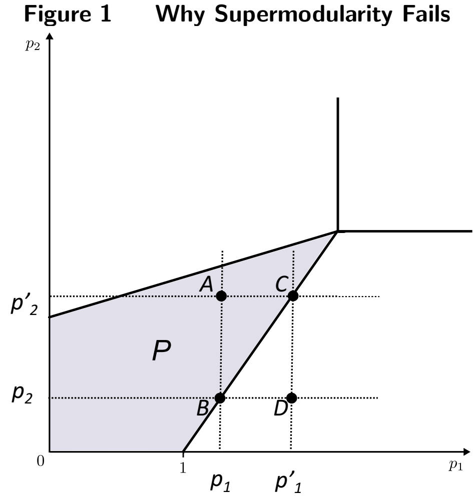
A. Analysis of Example 1

Federgruen and Hu (2015) investigate the same, in their Example 1; Muto (1993), Erkal (2005) and Zanchettin (2006) address this specific case, with the further restriction $\gamma_1 = \gamma_2$; for the case of a duopoly, the models in Ledvina and Sircar (2012) and Cumbul and Virág (2018) reduce to this case.

Figure 1 **Why Supermodularity Fails**



Note, the R-matrix is both row- and column-wise diagonally dominant. See Figure 1 for a pictorial representation of the polyhedron P when $\gamma_1 = 0.7$ and $\gamma_2 = 0.3$:

$$P = \left\{ p \geqslant 0 \;\middle|\; \begin{array}{l} 1 - p_1 + \gamma_1 p_2 \geqslant 0 \\ 1 - p_2 + \gamma_2 p_1 \geqslant 0 \end{array} \right\}.$$

Consider the four points A, B, C, D with A in the interior of P, $D \notin P$ and B and C on the boundary segment corresponding with the line $0 = q_1(p) = 1 - p_1 + \gamma_1 p_2$. By the regularity condition, the demand levels of point D are the same as those of point B. Thus, $d_1(p_1, p_2') > 0$, but $d_1(p_1, p_2) = d_1(p_1', p_2') = d_1(p_1', p_2) = 0$ and $d_1(p_1, p_2') - d_1(p_1, p_2) > 0 = d_1(p_1', p_2') - d_1(p_1', p_2)$ so that the demand function $d_1(\cdot)$ fails to have increasing differences and therefore fails to be supermodular. Moreover, assuming $p_1 > w_1$, we have $\pi_1(p_1, p_2') - \pi_1(p_1, p_2) > 0 = \pi_1(p_1', p_2') - \pi_1(p_1', p_2)$, where $\pi_i(p_1, p_2) = (p_i - w_i) d_i(p_1, p_2)$ is the profit function of firm i. Here, we see that the profit function $\pi_1(\cdot)$ fails to be supermodular, as well.

The demand and profit functions d_1 and π_1 also fail the weaker *single crossing property*, see Milgrom and Shannon (1994), Edlin and Shannon (1998), Topkis (1998) and Athey (2001): Note $d_1(p_1', p_2) \geqslant d_1(p_1, p_2)$ but $d_1(p_1', p_2') < d_1(p_1, p_2')$ and the same comparisons apply to the profit function $\pi_1(\cdot)$. Finally, with demand and profit values often equal to zero, the log-supermodularity property is irrelevant.

We now show that multiple equilibria may arise in P. Note first that

$$W = \left\{ w \geqslant 0 \;\middle|\; \begin{array}{l} 2 + \gamma_1 - (2 - \gamma_1\gamma_2)w_1 + \gamma_1 w_2 \geqslant 0 \\ 2 + \gamma_2 - (2 - \gamma_1\gamma_2)w_2 + \gamma_2 w_1 \geqslant 0 \end{array} \right\} \supseteq P. \tag{A.1}$$

Recall, W^o represents the set of wholesale prices under which a unique equilibrium prevails in the interior of P.

Consider the special case when $\gamma_1 = \gamma_2 = \gamma$. We construct an equilibrium at which firm 1 prices at cost, i.e., $\tilde{p}_1 = w_1$. Firm 2 sets the price $p_2 = \tilde{p}_2$ such that firm 1 is exactly priced out of the market, i.e., $0 = q_1 = 1 - \tilde{p}_1 + \gamma\tilde{p}_2$, i.e., $\tilde{p}_2 = \frac{w_1 - 1}{\gamma}$. We identify conditions under which the point $\tilde{p} = (\tilde{p}_1, \tilde{p}_2) = (w_1, \frac{w_1 - 1}{\gamma})$ is an equilibrium.

The point $\tilde{p} = (w_1, \frac{w_1 - 1}{\gamma})$ is on the boundary of P, denoted by ∂P, if $\tilde{p} \geqslant 0$ and $a - R\tilde{p} \geqslant 0$, which requires

$$1 \leqslant w_1 \leqslant \frac{1}{1 - \gamma}. \tag{A.2}$$

First, consider firm 1. The firm does not gain by pricing below $\tilde{p}_1 = w_1$, because that would result in a non-positive profit. Likewise, the firm does not gain by increasing its price above w_1: by the regularity condition (see Definition 1), its demand volume, and hence its profit value remains at zero.

Next, consider firm 2. If firm 2 decreases its price to some $p_2 < \tilde{p}_2$, the price vector $(\tilde{p}_1, p_2) \notin P$ and the associated demand volumes are obtained by applying the affine functions $q(\cdot)$ to its projection $\Omega(\tilde{p}_1, p_2)$. It is easily verified from the LCP conditions (2) and (3) that $\Omega_2(\tilde{p}_1, p_2) = p_2$, while $\bar{p}_1(p_2) \equiv \Omega_1(\tilde{p}_1, p_2) = 1 + \gamma p_2 \leqslant \tilde{p}_1$ since for this corrected price level for product 1, $q_1 = 0$. Thus

$$d_2(p_2) = q_2(\bar{p}_1(p_2), p_2) = 1 - p_2 + \gamma\bar{p}_1(p_2) = 1 - p_2 + \gamma(1 + \gamma p_2) = (1 + \gamma) - (1 - \gamma^2)p_2.$$

Thus, when firm 2 decreases its price, its effective profit function is

$$\pi_2^-(p_2) = (p_2 - w_2)[(1 + \gamma) - (1 - \gamma^2)p_2].$$

Since this function is concave, we need:

$$\left. \frac{\mathrm{d}\pi_2^-(p_2)}{\mathrm{d}p_2} \right|_{p_2 = \frac{w_1 - 1}{\gamma}} \geqslant 0,$$

to ensure that firm 2 has no incentive to decrease its price. This results in the following condition:

$$\frac{2+\gamma-\gamma^2}{\gamma} - \frac{2(1-\gamma^2)}{\gamma} w_1 + (1-\gamma^2) w_2 \geqslant 0. \tag{A.3}$$

(The corresponding half plane is bounded by a line that contains an edge of W.) Given $\tilde{p}_1 = w_1$, if firm 2 *increases* its price marginally to some new level $p_2 > \tilde{p}_2$, the new price vector $(\tilde{p}_1, p_2)$ remains in P, so that

$$d_2(\tilde{p}_1 = w_1, p_2) = q_2(\tilde{p}_1 = w_1, p_2) = 1 - p_2 + \gamma \tilde{p}_1 = 1 - p_2 + \gamma w_1.$$

Thus, before firm 2 increases its price up to $(1+\gamma w_1)$, at which point it prices itself out of the market as well, the effective profit function is,

$$\pi_2^+(p_1 = w_1, p_2) = (p_2 - w_2)(1 - p_2 + \gamma w_1).$$

This function is again concave, so that

$$\left.\frac{\mathrm{d}\pi_2^+(p_1, p_2)}{\mathrm{d}p_2}\right|_{p_1 = w_1, p_2 = \frac{w_1 - 1}{\gamma}} \leqslant 0,$$

ensures that firm 2 has no incentive to increase its price to any level $\leqslant (1+\gamma w_1)$. This leads to the following condition:

$$\frac{2+\gamma}{\gamma} - \frac{2-\gamma^2}{\gamma} w_1 + w_2 \leqslant 0. \tag{A.4}$$

We conclude that the price vector $(w_1, \frac{w_1-1}{\gamma})$ is a Nash equilibrium when all three conditions (A.2)-(A.4) are satisfied, i.e., when the cost rate vector w lies in:

$$\widetilde{W} = \left\{ 1 \leqslant w_1 \leqslant \frac{1}{1-\gamma} \;\middle|\; \begin{array}{c} \frac{2+\gamma-\gamma^2}{\gamma} - \frac{2(1-\gamma^2)}{\gamma} w_1 + (1-\gamma^2) w_2 \geqslant 0 \\ \frac{2+\gamma}{\gamma} - \frac{2-\gamma^2}{\gamma} w_1 + w_2 \leqslant 0 \end{array} \right\}, \quad \text{the triangle } ACD,$$

which is contained in $W(\mathrm{II}) = \{w \geqslant 0 \mid [\Theta(w)]_1 \leqslant w_1, [\Theta(w)]_2 = w_2\}$, the half open area, bounded by the line segment AC, the horizontal axis and the horizontal line through the point $A = (\frac{1}{\gamma}, \frac{1}{\gamma})$. We display the polyhedron $\widetilde{W}$ for $\gamma = 0.7$ in Figure 2. It is easily verified that the polyhedron $\tilde{W}$ is non-empty: it contains the point $(2, 0.2)$, for example.

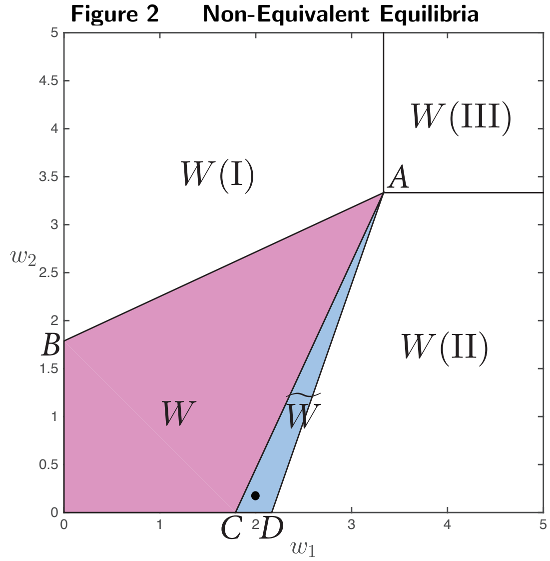
For this vector of cost rates $(2, 0.2)$, $(w_1, \frac{w_1-1}{\gamma}) = (2, 1.4286)$ is one Nash equilibrium.

Moreover, as shown in Proposition 1, $(p^*|w) = p^*(w') = (1.8808, 1.2583)$ is an alternative equilibrium, which we verify as follows.

If $w \in \mathrm{int}(W)$, the following price vector is the *unique* equilibrium in $\mathrm{int}(P)$, see (5), by Theorem 1(c):

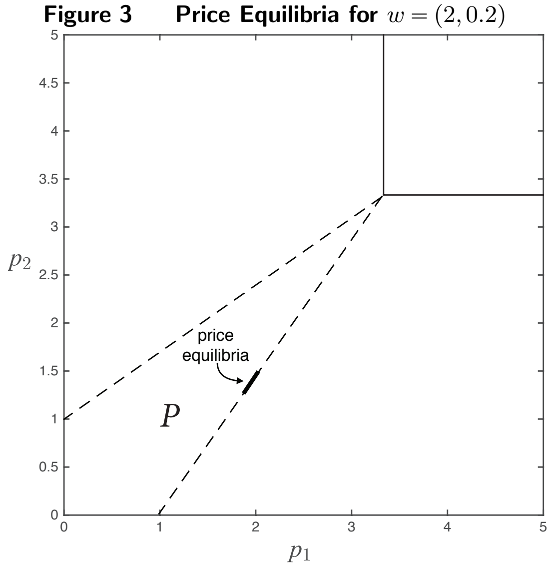
$$p_1^*(w) = \frac{2+\gamma_1}{4-\gamma_1\gamma_2} + \frac{2}{4-\gamma_1\gamma_2} w_1 + \frac{\gamma_1}{4-\gamma_1\gamma_2} w_2, \; p_2^*(w) = \frac{2+\gamma_2}{4-\gamma_1\gamma_2} + \frac{\gamma_2}{4-\gamma_1\gamma_2} w_1 + \frac{2}{4-\gamma_1\gamma_2} w_2. \tag{A.5}$$

Figure 2 Non-Equivalent Equilibria



Note. For $w \in \widetilde{W}$, there are at least two non-equivalent equilibria $p^*(\Theta(w))$ and $\tilde{p} = (w_1, \frac{w_1-1}{\gamma})$. The highlighted point in $\widetilde{W}$ is $w = (2, 0.2)$.

Figure 3 Price Equilibria for $w = (2, 0.2)$



Note. For $w = (2, 0.2)$, the set of price equilibria is given by the line segment between $p^*(\Theta(w)) = (1.8808, 1.2583)$ and $\tilde{p} = (w_1, \frac{w_1-1}{\gamma}) = (2, 1.4286)$ on the boundary of P.

If $w \in \mathbb{R}^N_{++} \backslash \text{int}(W)$, the special equilibrium $(p^*|w) \in \partial P$ is given by (A.5) with w replaced by $w' = \Theta(w)$:

$$p_1^*(w') = \frac{2+\gamma_1}{4-\gamma_1\gamma_2} + \frac{2}{4-\gamma_1\gamma_2} w_1' + \frac{\gamma_1}{4-\gamma_1\gamma_2} w_2', \; p_2^*(w') = \frac{2+\gamma_2}{4-\gamma_1\gamma_2} + \frac{\gamma_2}{4-\gamma_1\gamma_2} w_1' + \frac{2}{4-\gamma_1\gamma_2} w_2'. \quad \text{(A.6)}$$

Next, consider $w \in W(\text{II})$: In this area of the plane, $w_1' = w_1 - t_1$ and $w_2' = w_2$ (i.e., $t_2 = 0$), such that $0 = [Q(w-t)]_1 = [\Psi(R)a]_1 - [\Psi(R)R(w-t)]_1$, from which we get $w_1' = w_1 - t_1 = \frac{(2+\gamma_1)+\gamma_1 w_2}{2-\gamma_1\gamma_2}$.

Substituting into (A.6), we get

$$p_1^*(w') = \frac{2+\gamma_1}{2-\gamma_1\gamma_2} + \frac{\gamma_1}{2-\gamma_1\gamma_2}w_2, \quad p_2^*(w') = \frac{1+\gamma_2}{2-\gamma_1\gamma_2} + \frac{1}{2-\gamma_1\gamma_2}w_2. \tag{A.7}$$

For $w = (2, 0.2) \in W(\text{II})$ and $\gamma_1 = \gamma_2 = \gamma = 0.7$, we have $p^*(w') = \left(\frac{2+0.7}{2-0.49} + \frac{0.7}{2-0.49} \times 0.2, \frac{1+0.7}{2-0.49} + \frac{1}{2-0.49} \times 0.2\right) = (1.8808, 1.2583)$.

Moreover, the two equilibria fail to be equivalent. In both equilibria, product 1 does not survive the competition: Under the price equilibrium $(2, 1.4286)$, firm 2 has a sales volume $d_2 = 0.9714$ and a profit level $\pi_2 = 1.1935$. Under $(p^*|w) = (1.8808, 1.2583)$, firm 2 enjoys a larger sales volume $d_2 = 1.0583$, but a somewhat lower profit level $\pi_2 = 1.12$.

We show below that the full set of equilibria in P is the line segment connecting the two equilibria $(2, 1.4286)$ and $(1.8808, 1.2583)$, which lies on the boundary of P, see Figure 3. Since $w = (2, 0.2) \in \mathbb{R}^N_{++} \backslash \text{int}(W)$, by Theorem 1(d), we only need to consider ∂P, the boundary of P.

First, we verify that the boundary between P and $P(\text{I})$ does not contain an equilibrium. Consider a point p^o such that $d_1(p^o) > 0$ and $d_2(p^o) = 0$. Since $d_1(p^o) > 0$, it must be the case $p_1^o > w_1$. Then firm 2 can become strictly more profitable by reducing its price p_2 to $p_2 = p_1^o$, because now she still has a positive margin since $p_1^o > w_1 = 2 > 0.2 = w_2$ and reducing her price (moving from the boundary onto the diagonal within P) would boost her demand to a positive level.

Second, we verify that, on the boundary of P and $P(\text{II})$, any point with $p_2 < 1.2583$ is not an equilibrium. Consider a point $d_2(p^o) > 0$ and $d_1(p^o) = 0$ such that $p^o < (1.8808, 1.2583)$. Firm 2 has an incentive to (slightly) increase her price p_2. By doing so, the price point goes to P^o, the interior of P, in which the raw demand function applies without the need of modification. Then the profit function of firm 2 should be $(p_2 - w_2)(1 - p_2 + \gamma p_1)$. Note that

$$\begin{aligned}
\left.\frac{\mathrm{d}[(p_2 - w_2)(1 - p_2 + \gamma p_1)]}{\mathrm{d}p_2}\right|_{p=p^o} &= 1 - 2p_2^o + \gamma p_1^o + w_2 \\
&= 1 - 2p_2^o + \gamma(1 + \gamma p_2^o) + w_2 \\
&= 1 + \gamma + w_2 - (2 - \gamma^2)p_2^o \\
&> 0,
\end{aligned}$$

where the second equality is due to $0 = d_1(p^o) = 1 - p_1^o + \gamma p_2^o = 0$ and hence $p_1^o = 1 + \gamma p_2^o$, and the last inequality is due to $\gamma = 0.7$, $w_2 = 0.2$ and $p_2^o < 1.2583$. This verifies that firm 2 has an incentive to (slightly) increase her price p_2, at a point $p = p^o$ such that $d_2(p^o) > 0$, $d_1(p^o) = 0$ and $p_2^o < 1.2583$.

Third, we verify that, on the boundary of P and $P(\text{II})$, any point with $p_2 > 1.4286$ is not an equilibrium. Consider a point $d_2(p^o) > 0$ and $d_1(p^o) = 0$ such that $p^o > (2, 1.4286)$. Firm 1 has an incentive to slightly decrease her price p_1 but still make sure it is greater than 2. By doing so, the price point goes to P^o, the interior of P, in which the raw demand function applies without the need of modification. As a result, firm 1 has positive demand while still earns a positive profit margin, leading to a positive profit level. This verifies that firm 1 has an incentive to (slightly) decrease her price p_1, at a point $p = p^o$ such that $d_2(p^o) > 0$, $d_1(p^o) = 0$ and $p_2^o > 1.4286$.

Lastly, we can use the same way of verifying $(w_1, \frac{w_1 - 1}{\gamma}) = (2, 1.4286)$ as an equilibrium to show on the boundary of P and $P(\text{II})$, any point with $(1.8808, 1.2583) < p^o \leqslant (2, 1.4286)$ is an equilibrium.

In general, the set of equilibria in P may be more complex. See Example 1 in Federgruen and Hu (2015) for other types of equilibria in $\mathbb{R}_+^N$ and how these equilibria vary with w.

B. Auxiliary Lemmas

We use some properties of square matrices of special structure.

Definition B.1 (Z-matrix). A square matrix whose off-diagonal entries are non-positive is called a Z-matrix.

Definition B.2 (P-matrix). A square matrix whose principal minors are all positive (non-negative) is called a P-matrix (P_0-matrix).

Definition B.3 (ZP-matrix). A matrix that is both a Z-matrix and a P-matrix is called a ZP-matrix.

It is well known that all positive definite matrices are P-matrices, see, e.g., Cottle et al. (1992, Chapter 3). However, the class of P-matrices is significantly broader because it accommodates asymmetric matrices. It is well known that a *symmetric* matrix is positive definite if and only if it is a P-matrix.

Lemma B.1. *Let A be a ZP-matrix and B be a Z-matrix such that $A \leqslant B$, i.e., $B - A \geqslant 0$. Then*

(a) A^{-1} exists and $A^{-1} \geqslant 0$.

(b) B is a ZP-matrix and $B^{-1} \leqslant A^{-1}$.

(c) AB^{-1} and $B^{-1}A$ are ZP-matrices.

Let $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ represent a block decomposition of the matrix Y, with A and D square matrices. The Schur complement of A in Y is defined as $(D - CA^{-1}B)$.

(d) If Y is a ZP-matrix, so are A and its Schur complement $(D - CA^{-1}B)$.

(e) If A is non-singular and Y is the inverse *of a ZP-matrix, then the Schur complement of A, $(D - CA^{-1}B)$ is the inverse of a ZP-matrix, itself, and hence $(D - CA^{-1}B) \geqslant 0$.*

(f) Let X be the inverse of a symmetric ZP-matrix $Y = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$, with A a square $|\overline{S}| \times |\overline{S}|$ matrix and D a square $|S| \times |S|$ matrix. Thus, $X_{\overline{S},S} X_{S,S}^{-1} X_{S,\overline{S}} \geqslant 0$.

(g) If $A = I - Q$ is a ZP-matrix, with $Q \geqslant 0$, then $\rho(Q) < 1$, with $\rho(Q)$ the spectral radius of Q.

Proof of Lemma B.1. Parts (a)-(d) can be found in Horn and Johnson (1991, Section 2.5). (Horn and Johnson refer to ZP-matrices as M-matrices.) Part (e) follows from Imam (1984).

Part (f). Since Y is symmetric, $X = Y^{-1}$ is symmetric. With $Y = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}$. Then

$$X = Y^{-1} = \begin{pmatrix} (A - BD^{-1}B^\top)^{-1} & -A^{-1}B(D - B^\top A^{-1}B)^{-1} \\ -(D - B^\top A^{-1}B)^{-1}B^\top A^{-1} & (D - B^\top A^{-1}B)^{-1} \end{pmatrix} \geqslant 0,$$

see Horn and Johnson (1985, Section 0.7.3), and hence,

$$\begin{aligned} X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}} &= A^{-1}B(D - B^\top A^{-1}B)^{-1}(D - B^\top A^{-1}B)(D - B^\top A^{-1}B)^{-1}B^\top A^{-1} \\ &= A^{-1}B(D - B^\top A^{-1}B)^{-1}B^\top A^{-1} \geqslant 0, \end{aligned}$$

where the inequality is due to Y being a ZP-matrix, hence $B \leqslant 0$, $A^{-1} \geqslant 0$ and $(D - B^\top A^{-1}B)^{-1} \geqslant 0$ (since A and its Schur complement $D - B^\top A^{-1}B$ are ZP-matrices, by part (d)).

Part (g) follows from Horn and Johnson (1991, Lemma 2.5.2.1). ■

Lemma B.2. *For any $p, p', w \in \mathbb{R}^N$,*

$$\|\max(p, w) - \max(p', w)\|_\infty \leqslant \|p - p'\|_\infty .$$

Proof. From the definition of the matrix-norm $\|\cdot\|_\infty$, it suffices to show that $|\max(p, w) - \max(p', w)| \leqslant |p - p'|$, i.e., for all $l = 1, \ldots, N$, $|\max(p_l, w_l) - \max(p'_l, w_l)| \leqslant |p_l - p'_l|$. The latter is easily verified, assuming, without loss of generality, that $p_l \leqslant p'_l$ and considering all three possible rankings: (i) $p_l \leqslant p'_l \leqslant w_l$, (ii) $p_l \leqslant w_l < p'_l$ and (iii) $w_l < p_l \leqslant p'_l$. ■

C. Characterization of the Set of Equilibria in P: Proof of Theorem 1

In this Online Appendix, we provide a characterization of the equilibria in P. To this end, we consider, as in Federgruen and Hu (2015), in conjunction with the *full* competition game in which each retailer is able to select an arbitrary price vector, a *restricted* game in which the industry-wide price vector p must be selected within the polyhedron P. This is a *generalized* Nash game with coupled constraints, a term coined by Rosen (1965), i.e., even the feasible price range for any retailer i depends on the price choices made by the competitors; see also Topkis (1998) for a treatment of such generalized games.

While the structure of the feasible strategy space is more complex in this restricted game, it has the advantage that the profit functions are simple quadratic functions, because for $p \in P$, $d(p) = q(p) = a - Rp$ is affine. Note that any equilibrium $p^* \in P$ in the actual competition game is an equilibrium in the restricted game, but, the converse may fail to be true. For any equilibrium p^o in the restricted game and any retailer i, $p^o_{\mathcal{N}(i)}$ must solve the quadratic program:

$$\begin{aligned} \max\nolimits_{p_{\mathcal{N}(i)}} \;& (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^o_{-\mathcal{N}(i)}) \\ \text{s.t.} \quad & a - R \begin{pmatrix} p_{\mathcal{N}(i)} \\ p^o_{-\mathcal{N}(i)} \end{pmatrix} \geqslant 0 \quad \text{and} \quad p_{\mathcal{N}(i)} \geqslant 0. \end{aligned}$$

This quadratic program can be rewritten as

$$\begin{aligned} \min\nolimits_{p_{\mathcal{N}(i)}} \;& -(w^\top_{\mathcal{N}(i)} R_{\mathcal{N}(i),\mathcal{N}(i)} + a^\top_{\mathcal{N}(i)} - (p^o_{-\mathcal{N}(i)})^\top R^\top_{\mathcal{N}(i),-\mathcal{N}(i)}) p_{\mathcal{N}(i)} + \frac{1}{2} p^\top_{\mathcal{N}(i)} (2R_{\mathcal{N}(i),\mathcal{N}(i)}) p_{\mathcal{N}(i)} \\ & + w^\top_{\mathcal{N}(i)} (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^o_{-\mathcal{N}(i)}) \\ \text{s.t.} \quad & -R \begin{pmatrix} p_{\mathcal{N}(i)} \\ p^o_{-\mathcal{N}(i)} \end{pmatrix} \geqslant -a, & \text{(C.1)} \\ & p_{\mathcal{N}(i)} \geqslant 0. & \text{(C.2)} \end{aligned}$$

Since R is positive definite, $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is positive definite, as well. Let $y^i \geqslant 0$ and $s_{\mathcal{N}(i)} \geqslant 0$ denote the Lagrange multipliers associated with the constraint sets (C.1) and (C.2), respectively. Also let $t^i = (t^i_{\mathcal{N}(i)}, t^i_{-\mathcal{N}(i)}) \geqslant 0$ denote the surplus variables of constraint set (C.1). Since $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is positive definite, the optimal solution to this quadratic program is the unique solution to the complementarity conditions:

$$\begin{pmatrix} s_{\mathcal{N}(i)} \\ t^i_{\mathcal{N}(i)} \\ t^i_{-\mathcal{N}(i)} \end{pmatrix} - \begin{pmatrix} R_{\mathcal{N}(i),\mathcal{N}(i)} + R^\top_{\mathcal{N}(i),\mathcal{N}(i)} & R_{\mathcal{N}(i),-\mathcal{N}(i)} & R^\top_{\mathcal{N}(i),\mathcal{N}(i)} & R^\top_{-\mathcal{N}(i),\mathcal{N}(i)} \\ -R_{\mathcal{N}(i),\mathcal{N}(i)} & -R_{\mathcal{N}(i),-\mathcal{N}(i)} & 0 & 0 \\ -R_{-\mathcal{N}(i),\mathcal{N}(i)} & -R_{-\mathcal{N}(i),-\mathcal{N}(i)} & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{\mathcal{N}(i)} \\ p^o_{-\mathcal{N}(i)} \\ y^i_{\mathcal{N}(i)} \\ y^i_{-\mathcal{N}(i)} \end{pmatrix} = \begin{pmatrix} -(R^\top_{\mathcal{N}(i),\mathcal{N}(i)} w_{\mathcal{N}(i)} + a_{\mathcal{N}(i)}) \\ a_{\mathcal{N}(i)} \\ a_{-\mathcal{N}(i)} \end{pmatrix}, \quad \text{(C.3)}$$

and

$$\begin{aligned} s_{\mathcal{N}(i)} \geqslant 0, \qquad & p_{\mathcal{N}(i)} \geqslant 0, \quad s^\top_{\mathcal{N}(i)} p_{\mathcal{N}(i)} = 0, \\ t^i = (t^i_{\mathcal{N}(i)}, t^i_{-\mathcal{N}(i)}) \geqslant 0, \qquad & y^i = (y^i_{\mathcal{N}(i)}, y^i_{-\mathcal{N}(i)}) \geqslant 0, \quad (t^i)^\top y^i = 0. \end{aligned} \quad \text{(C.4)}$$

This implies that a price vector p is a generalized Nash equilibrium if and only if vectors $s, y^i, t^i \in \mathbb{R}^N_+$ can be found, for all i, such that (C.3) and (C.4) are satisfied for all i, *simultaneously*. In other words, the price vector p is a generalized Nash equilibrium if and only if the extended vector $(p, y^1, y^2, \ldots, y^I) \in \mathbb{R}^{N(I+1)}_+$ is a solution to a specific *master* LCP that takes the following form:

$$\big(s, t^1, \ldots, t^I\big)^\top = \big(-T(R)w - a, a, \ldots, a\big)^\top + \tilde{R}\,\big(p, y^1, \ldots, y^I\big)^\top, \quad \text{(C.5)}$$

$$\big(s, t^1, \ldots, t^I\big) \geqslant 0, \quad \big(p, y^1, \ldots, y^I\big) \geqslant 0, \quad \text{(C.6)}$$

$$\left(s^\top, (t^1)^\top, \ldots, (t^I)^\top\right) \begin{pmatrix} p \\ y^1 \\ \vdots \\ y^I \end{pmatrix} = 0, \tag{C.7}$$

where

$$\tilde{R} \equiv \begin{pmatrix} R + T(R) & \mathring{R}_{\mathcal{N}(1)} & \cdots & \mathring{R}_{\mathcal{N}(I)} \\ -R & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N(I+1)\times N(I+1)}, \quad \mathring{R}_{\mathcal{N}(i)} \equiv \begin{pmatrix} 0 \\ \vdots \\ R^\top_{\mathcal{N},\mathcal{N}(i)} \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

We first prove the following result regarding the master LCP:

Lemma C.1. *For any $w \in \mathbb{R}^N_{++}$, the LCP (C.5)-(C.7) has at most one solution whose price component lies in int(P).*

Proof of Lemma C.1. For a solution p^o to the LCP (C.5)-(C.7) to be in int(P), $t^i = a - Rp^o > 0$ and by complementarity, $y^i = 0$. Hence, the LCP (C.5)-(C.7) reduces to $s = -T(R)w - a + [R + T(R)]p \geqslant 0$, $p \geqslant 0$ and $s^\top p = 0$. Since $R + T(R)$ is *positive definite*, by Cottle et al. (1992, Theorem 3.1.6), this reduced LCP has a *unique* solution, but the solution may not necessarily be in int(P). Hence, the LCP (C.5)-(C.7) has at most one solution whose price component lies in int(P). ■

Now we characterize the set of Nash equilibria in the original game.

Proof of Theorem 1. Part (a). In the narrative proceeding Proposition 1, we showed that $p^*(w)$ is the only possible equilibrium in int(P).

Part (b). Follows from the proof of Theorem 1 in Federgruen and Hu (2015), see the Online Appendix thereof.

Part (c). We showed, in Proposition 1, that when $w \in$ int(W), $p^*(w)$ is an equilibrium, indeed. By part (a), we already know that no other equilibrium in int(P) can exist. To show that there are no alternative equilibria, *anywhere*, we first establish that there are no alternative equilibria in the restricted game. This is equivalent to showing that the master LCP corresponding with this restricted game, i.e., the LCP (C.5)-(C.7), has a unique solution.

$\tilde{R}$ is a P_0-matrix, a fact shown as part of the proof of Theorem 1 in Federgruen and Hu (2015), see the Online Appendix thereof. Were this matrix a P-matrix, this would immediately imply that the LCP (C.5)-(C.7) has a *unique* solution, see Theorem 3.1.6. in Cottle et al. (1992). Since $\tilde{R}$ is only a P_0-matrix, we need an additional property to arrive at the same conclusion. However, Corollary 5 in Jones and Gowda (1998) shows that the LCP has a unique solution if it has a *locally* *unique* solution; indeed we have shown that $p^*(w)$ is a locally unique solution since no other point in $\text{int}(P)$ is an equilibrium.

Therefore, when $w \in \text{int}(W)$, no price vector $\tilde{p} \in P$, with $\tilde{p} \neq p^*(w)$ is an equilibrium of the restricted game; and, a fortiori, of the full price competition game. Finally, to exclude the existence of equilibria in $\mathbb{R}^N_+ \backslash P$, we invoke part (b): if such an equilibrium $\tilde{p} \notin P$ were to exist, then $\Omega(\tilde{p})$, on the boundary of P, is an equilibrium as well, and we have just proven that no vector on the boundary of P is an equilibrium.

Finally, if $w \in \text{int}(W)$, $q(p^*(w)) > 0$, i.e., all products have a positive market share, or $\mathcal{N}^+(p^*|w) = \mathcal{N}$.

Part (d): When $w \in \mathbb{R}^N_{++} \backslash \text{int}(W)$, the proof of Proposition 1 showed that there is no equilibrium in $\text{int}(P)$: after all, only $p^*(w)$ is a candidate equilibrium in $\text{int}(P)$; however, when $w \in \mathbb{R}^N_{++} \backslash \text{int}(W)$, $p^*(w) \notin \text{int}(P)$. This proves the first statement. The fact that $p^*(w') \in \partial P$ is an equilibrium was shown in Proposition 1(b).

Part (e): For any product $l \in \mathcal{N}(i)$ for some $i = 1, \ldots, I$ with $d(p^*|w)_l > 0$, we must have that $(p^*|w)_l \geqslant w_l$. This is because if $(p^*|w)_l < w_l$, firm i can strictly improve its profit by increasing its price $(p^*|w)_l$ to w_l. The argument is as follows. Without loss of generality, we assume $(p^*|w)_{l'} \geqslant w_{l'}$, for $l' \neq l$; otherwise, we simultaneously raise prices of those product that are below its cost to its cost, and the same argument as follows holds. As the price of product l increases, its demand volume d_l decreases, while the demand volumes of all other products d_{-l} increase, see Farahat and Perakis (2010, Corollary 1), i.e., $d(w_l; (p^*|w)_{-l})_{l'} \geqslant d(p^*|w)_{l'}$ for $l' \neq l$. If the demand volume $d(w_l; (p^*|w)_{-l})_l$ for product l at price w_l is still nonnegative, the profit level of firm i strictly improves because firm i now eliminates a negative profit for product l while improving profitability for all of its other products. That is,

$$
\begin{aligned}
\pi_i(w_l; (p^*|w)_{-l}) &= (w_l - w_l) d(w_l; (p^*|w)_{-l})_l + \sum_{l, l' \in \mathcal{N}(i)} ((p^*|w)_{l'} - w_{l'}) d(w_l; (p^*|w)_{-l})_{l'} \\
&\geqslant ((p^*|w)_l - w_l) d(p^*|w)_l + \sum_{l, l' \in \mathcal{N}(i)} ((p^*|w)_{l'} - w_{l'}) d(p^*|w)_{l'}, \qquad \text{(C.8)}
\end{aligned}
$$

where the inequality is due to $(p^*|w)_l < w_l$, $d(p^*|w)_l \geqslant 0$, $(p^*|w)_{l'} \geqslant w_{l'}$ for $l' \neq l$, and $d(w_l; (p^*|w)_{-l})_{l'} \geqslant d(p^*|w)_{l'}$ for $l' \neq l$. Otherwise, in the process of increasing the price of product l from $(p^*|w)_l$ to w_l, at some price point, its demand volume hits zero. By the regularity condition, a further price increase leaves all demand volumes unaltered. Both scenarios eliminate a negative profit value for product l while the profits for any other products sold by firm i are (weakly) improved, i.e., inequality (C.8) holds for both cases. This contradicts the fact that $(p^*|w)$ is an equilibrium. ■

D. Proposition 2 and Proof of Lemma 2

Proposition 2. *For any $w \in \mathbb{R}^N_{++}$, $p^*(w')$, coupled with Lagrange multipliers $y^i_{\mathcal{N}(i)} = [w - w']_{\mathcal{N}(i)}$ and $y^i_{-\mathcal{N}(i)} = 0$ for all i, constitutes a solution to the LCP (C.5)-(C.7).*

Proof of Proposition 2. It suffices to verify that $\tilde{p} = [R + T(R)]^{-1}[a + T(R)w'] = p^*(w')$, $\tilde{y}^i_l = \begin{cases} w_l - w'_l & \text{if } l \in \mathcal{N}(i), \\ 0 & \text{otherwise,} \end{cases}$ for all i, $\tilde{s} = 0$, $\tilde{t}^i = a - R\tilde{p}$, for all i, satisfy the LCP (C.5)-(C.7).

The nonnegative conditions (C.6) can be easily verified.

To verify (C.5), we have

$$\begin{aligned}
&[R + T(R)]\tilde{p} - [a + T(R)w] + \begin{pmatrix} R^\top_{\mathcal{N},\mathcal{N}(1)}\tilde{y}^1 \\ \vdots \\ R^\top_{\mathcal{N},\mathcal{N}(I)}\tilde{y}^I \end{pmatrix} \\
= {} & [R + T(R)]\tilde{p} - [a + T(R)w] + \begin{pmatrix} R^\top_{\mathcal{N}(1),\mathcal{N}(1)}[w - w']_{\mathcal{N}(1)} \\ \vdots \\ R^\top_{\mathcal{N}(\mathcal{I}),\mathcal{N}(I)}[w - w']_{\mathcal{N}(I)} \end{pmatrix} \\
= {} & [R + T(R)]\tilde{p} - [a + T(R)w] + T(R)(w - w') \\
= {} & [R + T(R)]\tilde{p} - [a + T(R)w'] = 0,
\end{aligned}$$

Lastly, we verify the complementarity conditions (C.7). Since $\tilde{s} = 0$, $\tilde{s}^\top \tilde{p} = 0$. Note that

$$\tilde{t}^i = a - R\tilde{p} = q(\tilde{p}) = Q(w') \geqslant 0 \text{ for all } i,$$

where the last inequality is due to Federgruen and Hu (2015, Proposition 2(b)). By the definition of the projection, $(w - w')^\top Q(w') = 0$. Thus $\tilde{y}^i_l \tilde{t}^i_l = (w_l - w'_l)[Q(w')]_l = 0$ for $l \in \mathcal{N}(i)$. Moreover, $\tilde{y}^i_l \tilde{t}^i_l = 0$ for $l \notin \mathcal{N}(i)$ because $\tilde{y}^i_l = 0$. ■

Proof of Lemma 2. Consider, for any $i = 1, \ldots, I$, the single player price game, with firm i as the "monopolist" and demand functions given by the (unique) *regular* extension on $\mathbb{R}^{|\mathcal{N}(i)|}_+$ of the affine functions

$$\widehat{q}_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}) = \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)},$$

where $\alpha = a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} \geqslant 0$. The regular extension of these affine functions on $\mathbb{R}^{|\mathcal{N}(i)|}_+$ is well defined, since $R_{\mathcal{N}(i),\mathcal{N}(i)}$, as a major principal of the full matrix R, is both a Z-matrix and positive definite, and is thus given by

$$\widehat{d}_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}) = \widehat{q}_{\mathcal{N}(i)}(p_{\mathcal{N}(i)} - t_{\mathcal{N}(i)}) = \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)} - t_{\mathcal{N}(i)})$$

with the vector $t_{\mathcal{N}(i)}$ the *unique* solution to the LCP:

$$\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)} - t_{\mathcal{N}(i)}) \geqslant 0,$$

$$t_{\mathcal{N}(i)}^{\top}[\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)} - t_{\mathcal{N}(i)})] = 0, \quad \text{and} \quad t_{\mathcal{N}(i)} \geqslant 0.$$

Now fix a vector $p^o_{-\mathcal{N}(i)} \geqslant 0$. Recall that $RB_i(p^o_{-\mathcal{N}(i)})$ is the price vector which optimizes firm i's profits in the above single player robust competition game. In this game the matrix R is to be replaced by $R_{\mathcal{N}(i),\mathcal{N}(i)}$ and $T(R_{\mathcal{N}(i),\mathcal{N}(i)}) = R_{\mathcal{N}(i),\mathcal{N}(i)}$ by property (IS). Applying the expression in (5) to this single firm game, we get the following expression for $RB_i(w_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)})$:

$$\begin{aligned} RB_i(w_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)}) &= w'_{\mathcal{N}(i)} + [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} w'_{\mathcal{N}(i)}) \\ &= [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}\alpha + \frac{1}{2}[w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)})], \end{aligned} \tag{D.1}$$

where $w'_{\mathcal{N}(i)} = w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)})$ is the projection, along the coordinates, of $w_{\mathcal{N}(i)}$ onto

$$W_i \equiv \{w_{\mathcal{N}(i)} > 0 \mid \Psi(R_{\mathcal{N}(i),\mathcal{N}(i)})[\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} w_{\mathcal{N}(i)}] \geqslant 0\} = P_i(p^o_{-\mathcal{N}(i)}),$$

where the last equality is due to $\Psi(R_{\mathcal{N}(i),\mathcal{N}(i)}) = T(R_{\mathcal{N}(i),\mathcal{N}(i)})[R_{\mathcal{N}(i),\mathcal{N}(i)} + T(R_{\mathcal{N}(i),\mathcal{N}(i)})]^{-1} = \frac{1}{2}$ and (8).

Let

$$d_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \alpha) = \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)}(w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, \alpha)),$$

and

$$B_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \alpha) = \alpha + R_{\mathcal{N}(i),\mathcal{N}(i)}(w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, \alpha)).$$

Then in view of (D.1), we can write

$$RB_i(w_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)}) = [2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1} B_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \alpha). \tag{D.2}$$

Since the projection of $w_{\mathcal{N}(i)}$ onto W_i along the coordinates is defined by a linear complementarity problem, for any $l \in \mathcal{N}(i)$, by complementarity,

if $[t(w_{\mathcal{N}(i)}, \alpha)]_l > 0$, $[d_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \alpha)]_l = 0$, so that $\alpha_l = [R_{\mathcal{N}(i),\mathcal{N}(i)}(w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, \alpha))]_l$ and hence, $[B(w_{\mathcal{N}(i)}, \alpha)]_l = 2\alpha_l$;

if $[t(w_{\mathcal{N}(i)}, \alpha)]_l = 0$, $[d_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \alpha)]_l > 0$, so that $\alpha_l > [R_{\mathcal{N}(i),\mathcal{N}(i)}(w_{\mathcal{N}(i)} - t(w_{\mathcal{N}(i)}, \alpha))]_l$ and hence, $[B(w_{\mathcal{N}(i)}, \alpha)]_l < 2\alpha_l$.

Let $\widehat{t} = t(w_{\mathcal{N}(i)}, \widehat{p}_{-\mathcal{N}(i)})$, $\widetilde{t} = t(w_{\mathcal{N}(i)}, \widetilde{p}_{-\mathcal{N}(i)})$, $\breve{t} = t(w_{\mathcal{N}(i)}, \breve{p}_{-\mathcal{N}(i)})$, $\widehat{B} = B_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widehat{\alpha})$, $\widetilde{B} = B_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \widetilde{\alpha})$, $\breve{B} = B_{\mathcal{N}(i)}(w_{\mathcal{N}(i)}, \breve{\alpha})$.

Since $\breve{\alpha} \leqslant \widehat{\alpha}$, $\breve{W}_i \subseteq \widehat{W}_i$. It follows from Lemma B.2(d) in the Online Appendix of Federgruen and Hu (2016) that the projection of the vector $w_{\mathcal{N}(i)}$ onto $\widehat{W}_i$ is component-wise smaller than that onto $\breve{W}_i \subseteq \widehat{W}_i$, i.e., $\widehat{t} \leqslant \breve{t}$. As a consequence, only the following three cases need to be considered.

(i) If $\breve{t}_l > 0$ and $\widehat{t}_l = 0$, then $\breve{B}_l = 2\breve{\alpha}_l$ and $\widehat{B}_l < 2\widehat{\alpha}_l$ and hence,

$$[\widehat{B} - \breve{B}]_l < [2(\widehat{\alpha} - \breve{\alpha})]_l.$$

(ii) If $\breve{t}_l > 0$ and $\widehat{t}_l > 0$, then $\breve{B}_l = 2\breve{\alpha}_l$ and $\widehat{B}_l = 2\widehat{\alpha}_l$ and hence,

$$[\widehat{B} - \breve{B}]_l = [2(\widehat{\alpha} - \breve{\alpha})]_l.$$

(iii) If $\breve{t}_l = 0$ and $\widehat{t}_l = 0$, by (D.1),

$$\begin{aligned}[\widehat{p}^* - \breve{p}^*]_l &= \{[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}\widehat{\alpha}\}_l + \frac{1}{2}w_l - \{[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}\breve{\alpha}\}_l - \frac{1}{2}w_l \\ &= \{[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(\widehat{\alpha} - \breve{\alpha})\}_l.\end{aligned}$$

Thus, let $S = \{l \in \mathcal{N}(i) \mid \breve{t}_l = \widehat{t}_l = 0\}$. Then

$$[\widehat{p}^* - \breve{p}^*]_S = \{[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1}(\widehat{\alpha} - \breve{\alpha})\}_S. \tag{D.3}$$

For notational simplicity, let

$$[2R_{\mathcal{N}(i),\mathcal{N}(i)}]^{-1} = X, \quad \widehat{B} - \breve{B} = y, \quad \widehat{\alpha} - \breve{\alpha} = z \geqslant 0.$$

Since R is a positive-definite Z-matrix, $R_{\mathcal{N}(i),\mathcal{N}(i)}$ has the same properties. It follows from Lemma B.1(a) that $X \geqslant 0$.

By (D.2), we expand the two sides of (D.3): $X_{S,S}y_S + X_{S,\overline{S}}y_{\overline{S}} = X_{S,S}z_S + X_{S,\overline{S}}z_{\overline{S}}$. Then

$$y_S = z_S + X_{S,S}^{-1}X_{S,\overline{S}}(z_{\overline{S}} - y_{\overline{S}}). \tag{D.4}$$

Therefore,

$$\begin{aligned}[\widehat{p}^* - \breve{p}^*]_{\overline{S}} &= X_{\overline{S},S}y_S + X_{\overline{S},\overline{S}}y_{\overline{S}} = X_{\overline{S},S}[z_S + X_{S,S}^{-1}X_{S,\overline{S}}(z_{\overline{S}} - y_{\overline{S}})] + X_{\overline{S},\overline{S}}y_{\overline{S}} \\ &= X_{\overline{S},S}z_S + X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}}z_{\overline{S}} + [X_{\overline{S},\overline{S}} - X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}}]y_{\overline{S}} \\ &\leqslant X_{\overline{S},S}z_S + X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}}z_{\overline{S}} + [X_{\overline{S},\overline{S}} - X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}}](2z_{\overline{S}}) \\ &= X_{\overline{S},S}z_S + [2X_{\overline{S},\overline{S}} - X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}}]z_{\overline{S}} \\ &\leqslant 2X_{\overline{S},S}z_S + 2X_{\overline{S},\overline{S}}z_{\overline{S}} \\ &= 2[Xz]_{\overline{S}} = [R_{\mathcal{N}(i),\mathcal{N}(i)}^{-1}(\widehat{\alpha} - \breve{\alpha})]_{\overline{S}},\end{aligned} \tag{D.5}$$

where the second equality is due to (D.4). The first inequality follows from $y_{\overline{S}} \leqslant 2z_{\overline{S}}$ (combining cases (i) and (ii)) while $X_{\overline{S},\overline{S}} - X_{\overline{S},S}X_{S,S}^{-1}X_{S,\overline{S}} \geqslant 0$, as the Schur complement of $X_{\overline{S}}$ in X, the inverse of a ZP-matrix, is the inverse of a ZP-matrix, itself, see Lemma B.1(e), and the last inequality is due to $X, z \geqslant 0$ and $X_{\overline{S},S} X_{S,S}^{-1} X_{S,\overline{S}} \geqslant 0$. The latter follows from Lemma B.1(f) with $Y = 2R_{\mathcal{N}(i),\mathcal{N}(i)}$, which is symmetric under Assumption (IS).

We thus have the first part of (18), where the first inequality is due to $\widehat{\alpha} \geqslant \breve{\alpha}$ and the monotonicity of equilibrium prices with respect to the intercept vector α (see Federgruen and Hu 2016, Theorem 4(b)), and the second inequality is due to the combination of (D.3) and (D.5). By the same argument, we have the second part of (18). ■

References

Athey, S. 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica* **69**(4) 861–889.

Cottle, RW, JS Pang, RE Stone. 1992. *The Linear Complementarity Problem*. Academic Press, Boston.

Cumbul, E, G Virág. 2018. Multilateral limit pricing in price-setting games. *Games and Economic Behavior* **111** 250–273.

Edlin, AS, C Shannon. 1998. Strict single crossing and the strict Spence-Mirrlees condition: A comment on monotone comparative statics. *Econometrica* **66**(6) 1417–1425.

Erkal, N. 2005. Optimal licensing policy in differentiated industries. *Economic Record* **81**(252) 51–60.

Farahat, A, G Perakis. 2010. A nonnegative extension of the affine demand function and equilibrium analysis for multiproduct price competition. *Oper. Res. Lett.* **38**(4) 280–286.

Federgruen, A, M Hu. 2015. Multi-product price and assortment competition. *Oper. Res.* **63**(3) 572–584.

Federgruen, A, M Hu. 2016. Sequential multiproduct price competition in supply chain networks. *Oper. Res.* **64**(1) 135–149.

Horn, RA, CR Johnson. 1985. *Matrix Analysis*. Cambridge University Press, Cambridge, UK.

Horn, RA, CR Johnson. 1991. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, UK.

Imam, IN. 1984. The Schur complement and the inverse M-matrix problem. *Linear Algebra Its Appl.* **62** 235–240.

Jones, C, MS Gowda. 1998. On the connectedness of solution sets in linear complementarity problems. *Linear Algebra Its Appl.* **272**(1-3) 33–44.

Ledvina, A, R Sircar. 2012. Oligopoly games under asymmetric costs and an application to energy production. *Mathematics and Financial Economics* **6**(4) 261–293.