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Technical Note—Global Robust Stability in a General Price and Assortment Competition Model

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Abstract. We analyze a general but parsimonious price competition model for an oligopoly in which each firm offers any number of products. The demand volumes are general piecewise affine functions of the full price vector, generated as the "regular" extension of a base set of affine functions. The model specifies a *product assortment*, along with their prices and demand volumes, in contrast to most commonly used demand models, such as the multinomial logit model or any of its variants. We show that a special equilibrium in this model has *global robust stability*. This means that, from any starting point, the market converges to this equilibrium when firms use a particular response mapping to dynamically adjust their own prices in response to their competitors' prices. The mapping involves each firm optimizing its own prices over a limited subset of possible prices and requires each firm to only know the demand function and cost structure for its own products (but not for other firms' products).

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1. Introduction and Summary

In the economics, marketing, and operations management literature, among others, oligopolistic price competition models are used to characterize and study various industries and interactions therein (see, e.g., Federgruen and Hu 2017). The development of such price competition models has a long history, starting with the seminal paper by Bertrand (1883). However, progress has been slow in developing a tractable model that represents many of the realities that are pertinent in most industries. The following is a list of desiderata for a general competition model:

i. the model should accommodate an arbitrary number of competing firms each offering an arbitrary potential assortment of products;

ii. the demand for any given product may depend on the prices charged for some or all of the products sold on the market, in accordance with general asymmetric customer preferences, allowing for general combinations of direct- and cross-price elasticities;

iii. the model should be parsimonious;

iv. the model specifies a *product assortment* sold on the market as a subset of the full collection of potential products, along with associated demand volumes;

v. the model has a guaranteed pure Nash equilibrium; in the presence of multiple equilibria, one should stand out in terms of predictability;

vi. under this special equilibrium, we should be able to attain a set of indirect equilibrium demand functions for upper-echelon competition models that are equally tractable and inherit properties (i)–(v) in a sequential oligopoly model.

In this paper, we address a general oligopolistic competition model. Each of I independent retailers, indexed by i, operating under linear cost structures potentially offers a distinct set of products $\mathcal{N}(i)$. The consumer demand for all products in $\mathcal{N} = \mathcal{N}(1) \cup \mathcal{N}(2) \cdots \cup \mathcal{N}(I)$, with $N = |\mathcal{N}|$, is specified by a set of (piecewise) affine demand functions. These demand functions are anchored on a set of purely affine functions. However, outside of a restricted polyhedron P in the complete price space $\mathbb{R}_+^N$ (i.e., the nonnegative orthant of the Euclidean space), these affine functions predict negative demand volumes for at least some of the products and must, therefore, be replaced by a suitable extension.

Shubik and Levitan (1980) stipulate that the extension of the affine demand functions, beyond the polyhedron P, must satisfy an intuitive *regularity* condition:

under any given price vector, when some product experiences zero demand (i.e., is priced *out of the market*), any *increase* of its own price keeps the product out of the market and has *no* impact on any of the demand volumes. This regularity condition specifies a *unique* set of demand functions, which are *piecewise* affine. The potential price space $\mathbb{R}_+^N$ is partitioned into 2^N polyhedra, each corresponding with a given subset of the full product line for which there is positive demand. Federgruen and Hu (2015) in this general model prove that a pure Nash equilibrium always exists, but depending on the choice of the cost rate vector, only part of the N products may "survive" the competition and be part of the equilibrium product assortment. The authors provide a sufficient condition for the equilibrium to be unique. However, multiple, even infinitely many equilibria may exist. (Federgruen and Hu (2015) report that, in the presence of multiple equilibria, all such equilibria are *equivalent* in the sense that all generate identical sales volumes for all products and identical profit levels for all competing firms. However, the result in Theorem 2(a) is incorrect. This is explained in detail in Section 3.)

Thus, Federgruen and Hu (2015) establish that the price competition model with extended affine demand functions satisfies the first four of the list of desiderata (i.e., (i)–(iv)). The objective of *this* paper is to show that the remaining two desiderata (v) and (vi) are satisfied as well. In particular, we identify a unique *special* equilibrium, with far stronger properties than merely being a Nash equilibrium. Specifically, we show that it is the *unique* equilibrium that has global robust stability. This means that, from any starting point, the market converges to this equilibrium when firms use a particular response mapping to dynamically adjust their own prices in response to their competitors' prices. The mapping involves each firm optimizing its own prices over a limited subset of possible prices; it requires each firm to only know the demand function and cost structure for its own products (but not for other firms' products).

Moreover, there is a simple representation for this special equilibrium—and to our knowledge, for no other: in an industry with N potential products, it can be computed with a limited number of multiplications and inversions of $N \times N$ matrices only, possibly in combination with the solution of a single linear program in N variables and constraints. Finally, under this special equilibrium, we obtain a set of *indirect* equilibrium demand functions for an upper-echelon competition model among suppliers. The supplier competition is equally tractable as the retailer competition model and the indirect equilibrium demand functions inherit all of the properties (i)–(v) (i.e., the last desideratum (vi) is also satisfied).

Establishing the existence of a pure Nash equilibrium in a noncooperative competition model greatly advances our understanding of the competitive dynamics in industry. Identification of Nash equilibria serves two potential objectives; see Holt and Roth (2004) for a more in-depth discussion. The first is *prescriptive*: any of the competing firms may be advised to adopt the strategy that is her part of the overall Nash equilibrium. Any other advice to the firm would, per definition, have the unappealing feature that it is dominated by a unilateral change toward a different strategy. The second objective associated with the identification of a Nash equilibrium is *predictive*: the equilibrium is used to predict the (steady-state) behavior in the market.

However, to support the *predictive* use of a Nash equilibrium, it is highly desirable to show that the market converges to the equilibrium from an *arbitrary starting position* as a result of a plausible iterative adjustment process. The most commonly investigated such adjustment process is the so-called tatônnement process, first introduced by Cournot (1838) in analyzing a duopoly where firms set sales quantities. In each iteration of the tatônnement process, each firm selects a best response to the current choices of its competitors. Vives (2001) calls a Nash equilibrium *globally* stable if this convergence property prevails regardless of the starting point of the process.[1]

Furthermore, an adjustment process is all the more plausible if the firms' (iterative) adjustments can be made with limited private information only (i.e., when each firm only needs to know the demand functions for its own products and its own cost structure). We specify such a best response, referred to as the robust best response, and say that an equilibrium has global *robust stability* if it arises as the limit of such an adjustment process with robust best responses, regardless of the market's starting point. This additional *robustness property* further supports the notion that the Nash equilibrium requires no more than limited private information or *bounded rationality*, even in the presence of full information. (We use the term "robust" analogously to its use in "distributionally *robust* stochastic optimization," an approach that requires only limited information about the full distributions of the random input factors.)

In this paper, we show that the *special* equilibrium has, in fact, *global robust stability*. The equilibrium arises as the limit of this simplest possible of tatônnement schemes where each firm acts on its own private information only. In particular, no firm needs to know the cost structure or demand functions of any of its competitors, often *private* information.

Global stability has, typically, only been demonstrated in *supermodular* games in which the *uniqueness* of the equilibrium can be demonstrated. In a *general* supermodular game, the set of equilibria is a bounded lattice with more than a single element; if so, none of the equilibria are *globally* stable, even though the component-wise smallest and largest equilibria enjoy a *restricted* stability region (i.e., the equilibrium arises as the limit of a tatônnement process that starts with an initial strategy tuple in this *restricted* stability region).

However, our competition model fails to be supermodular, as demonstrated in Section 3. As mentioned, it often has multiple equilibria. Interestingly, we obtain global robust stability of the special equilibrium, even in the presence of many alternative Nash equilibria. Thus, beyond the desiderata (i), (ii), (iii), and (iv), already highlighted, the global stability result establishes (v).

Finally, as to (vi), the equilibrium sales volumes *associated with the special equilibrium*, when viewed as functions of the products' variable cost rate vector, are (the unique) regular extension of a new set of affine functions. These have an easily computable intercept vector and price sensitivity matrix, similar to the structure of the retailers' consumer demand functions (see (6)). This characterization allows us to extend our equilibrium analysis to a *sequential* oligopoly in a *multiechelon* market: at each echelon of the supply process, an arbitrary number of firms compete, each offering one or multiple products to some or all of the firms in the next echelon, with firms at the most downstream echelon selling to the end consumer; see Federgruen and Hu (2016). We are not aware of any other oligopoly model that satisfies desideratum (vi). For example, even in the basic multinomial logit (MNL) model, the equilibrium demand functions result in an intractable upper-echelon competition game.

For any vector $x \in \mathbb{R}^N$ and product set $U \subseteq \mathcal{N}$, x_U (x_{-U}) denotes the subvector of x whose components belong to U ($\overline{U} = \mathcal{N} \setminus U$). Similarly, for any $N \times N$ matrix X and sets $U, V \subseteq \mathcal{N}$, $X_{U,V}$ denotes the subblock of the matrix whose row (column) indices constitute the set U (V).

The remainder of this paper is organized as follows. Section 2 gives a review of the literature. The model, notation, and preliminary results are part of Section 3. This section also characterizes the set of Nash equilibria. The global robust stability of the special price equilibrium is shown in Section 4. Section 5 ends the paper with concluding comments.

2. Literature Review

There is a vast literature on oligopolistic price competition models. See Topkis (1998) and Vives (2001) for surveys of the literature until the start of the current millennium. In the past 15 years, attention has focused on a few workhorse models that can be and have been used in empirical studies. This applies mostly to the MNL model and various of its generalizations, in particular the mixed multinomial logit (MMNL) model and the nested MNL model. However, in both the MMNL and nested MNL models, the existence of a pure Nash equilibrium can only be guaranteed under specific settings (see Aksoy-Pierson et al. 2013, Gallego and Wang 2014). Perhaps most importantly, under the MNL model or any of its variants, the *full set* of potential products arises as the product assortment in the market, regardless of how the product prices are selected. Such models are, therefore, intrinsically incapable of analyzing how various model primitives impact on the assortment that arises in equilibrium; see desideratum (iv). We refer to Federgruen and Hu (2015, 2017) for a survey of price competition models based on the MNL model or various generalizations thereof.

Several authors have analyzed price competition games based on the (extended) affine demand model of this paper or special cases thereof. Farahat and Perakis (2010) establish the existence of a *unique* Nash equilibrium in the special case where the price sensitivity matrix is symmetric and the cost rate vector is in the interior of the price polyhedron P. This is equivalent to assuming that there is positive demand for all potential products when the firms offer all of their products at cost. This assumption is rather restrictive: in many industries, one can expect that when even the most brand/feature attractive products in the market are offered at marginal cost, this is likely to push less attractive substitutes out of the market. The latter may only preserve a share in the market when offered at a low price advantage, thus appealing to the most price-sensitive customers. Moreover, under this assumption, all retailers in equilibrium always select a maximally available product assortment, which defies what we observe in practice.

Several papers in the economics literature have addressed a very special case of the competition model with (extended) affine demand functions. Although allowing for a *general cost rate vector*, these papers assume that (i) each firm has a unique product to offer rather than an arbitrary collection of such items and (ii) the price sensitivity matrix has identical diagonal and identical off-diagonal elements. (The recent paper by Thomassen et al. (2017) is an exception. Its estimated price sensitivity matrix has nonidentical diagonal and off-diagonal elements; however, these authors do not characterize the set of Nash equilibria.) See Ledvina and Sircar (2012) and Cumbul and Virág (2018a) and the references therein. Federgruen and Hu (2015) provide a characterization of the *set* of Nash equilibria in the general model, with an arbitrary vector of cost rates and a fully general, possibly asymmetric, price sensitivity matrix. In this general model, multiple, in fact infinitely many, pure Nash equilibria are common.

Several papers, starting with Singh and Vives (1984), analyze price competition models with a *purely affine* set of demand functions. As mentioned in Section 1, a purely affine set of demand functions can only prevail on a limited polyhedron P, rather than the full price space. An alternative approach to the problem is to confine the set of feasible price choices to the polyhedron P. A recent example is Gao et al. (2017) addressing a multiperiod model in which each firm offers a single product, with a limited supply for the full horizon. They show that the unique Nash equilibrium—under a diagonally dominant price sensitivity matrix—is globally stable among all price vectors in (the equivalent of) the polyhedron P.

Recently, several papers have emphasized the fact that retailers compete in terms of their product *assortments* and not just in terms of their retail *prices*. To our knowledge, Besbes and Sauré (2016) is the first paper to address a joint price and assortment competition model. However, in their model, firms start by each making its own price and assortment decisions. The specific sales volumes for all selected products are then determined by an underlying pure or MMNL model. In other words, each firm is assumed to control its own product assortment, irrespective of the price-assortment decisions made by its competitors. The authors show that a unique equilibrium always exists, with the property that every firm selects an identical profit margin for all of its products. In our model, assortment choices are implied by price selections allowing for general firm- and product-dependent price sensitivities and explaining general profit margins.

The identification of globally stable equilibria has a long history as well. It seems to start with the seminal paper by Theocharis (1960) in the context of the classical Cournot competition model with homogeneous goods. al-Nowaihi and Levine (1985) derive a set of sufficient conditions for global stability of the Nash–Cournot equilibrium; these conditions imply that all firms' best-response functions are downward sloping. The authors show that these conditions are satisfied when the number of firms is less than or equal to five and point out difficulties in the earlier global stability results for the Cournot game in Hahn (1962) and Okuguchi (1964). For a more recent contribution, see Okuguchi and Yamazaki (2008) and the references therein.

3. Model and Equilibrium Characterization

Consider an oligopolistic market with I retailers, each having the option to bring one or several products to the market. For all $i = 1, \ldots, I$, let $\mathcal{N}(i)$ denote the set of (potential) products offered by retailer i, with (i,k) representing the kth product in this set. We use the following notation:

w_{ik} = the procurement (purchase and/or manufacturing) cost rate for product (i,k), which we refer to as the "wholesale price,"
p_{ik} = the retail price charged by retailer i for product k,
d_{ik} = the consumer demand for product (i,k).

We thus assume that the retailers' suppliers employ simple wholesale price contracts. Such contracts continue to be most popular, both in the theoretical literature and in practical supply chains (see, e.g., Hwang et al. 2015). The wholesale prices are exogenous inputs to the model.[2] The consumer demand for each product depends, in general, on the prices of *all* products (potentially) sold in the market. As in many theoretical and empirical studies, we assume that the foundation for the demand functions is provided by a set of affine functions, with the following vector specification:

$$q(p) = a - Rp. \tag{1}$$

Here, R is an $N \times N$ matrix, and $a \geq 0$, indicating that all products are relevant choices, with nonnegative demand, at least when they are offered for free.

Although the affine functions provide the foundation for the demand functions, they cannot be used per se. After all, for price vectors outside the polyhedron $P \equiv \{p \geq 0 \mid a - Rp \geq 0\}$, the affine functions $q(\cdot)$ in (1) predict negative demand volumes, for at least some of the products. Shubik and Levitan (1980) suggest that any extension of the demand functions beyond P should satisfy the following highly intuitive regularity property.

Definition 1 (Regularity). A demand function $D(p): \mathbb{R}_+^N \to \mathbb{R}_+^N$ is said to be regular if for any product l and any price vector p, $D_l(p) = 0$ implies that $D(p + \Delta \cdot e_l) = D(p)$ for any $\Delta > 0$, where e_l denotes the lth unit vector.

In other words, when under a given price vector p, a particular product $l = (i,k)$ attracts zero demand, any increase of the product's price has no impact on any of the demand volumes. Although seemingly innocuous, this intuitive regularity condition completely specifies the extension of the demand functions on $\mathbb{R}_+^N$: Soon et al. (2009) show that for any $p \in \mathbb{R}_+^N$, a set of price corrections t needs to be applied such that

$$d(p) = q(p - t) = a - R(p - t) \geq 0, \tag{2}$$
$$t^\top [a - R(p - t)] = 0, \quad \text{and} \quad t \geq 0, \tag{3}$$

where $^\top$ is the symbol for transpose. The problem of finding a vector t that satisfies (2) and (3) is a *linear complementarity problem* (LCP) (see, e.g., Cottle et al. 1992). The solution to the LCP is well defined and *unique*, under the following common assumption about the price sensitivity matrix R (see Cottle et al. 1992, theorem 3.1.6).

Assumption (P). *The matrix R is positive definite.*

In particular, the function $d(\cdot)$, defined via the LCP (2) and (3), is a true extension of the affine functions $q(\cdot)$: when $p \in P$, it is immediate that $t = 0$ solves the LCP, so that $d(p) = q(p)$; conversely, when $p \notin P$, a true set of positive price corrections $0 \neq t \geq 0$ is required. Thus, any price vector $p \in \mathbb{R}^N_+$ generates a unique vector of sales volumes $d(p)$.

We refer to the adjusted price vector $p' = p - t \leq p$ as the projection of the vector p onto the polyhedron P.

Definition 2 (Projection onto P). For any $p \in \mathbb{R}^N_+$, the projection $\Omega(p)$ of p onto P along the coordinate axes is defined as the vector $p' = p - t$ with t the unique solution to the LCP (2) and (3).

Beyond property (P), we assume that the products are *substitutes* (i.e., all cross-price elasticities are nonnegative). This is equivalent to assuming that the matrix R has nonpositive off-diagonal elements (i.e., satisfies property (Z)).

Assumption (Z). *The matrix R is a Z matrix (i.e., has nonpositive off-diagonal elements).*[3]

(Because R is positive definite, all diagonal elements are positive.) When the matrix R satisfies both properties (P) and (Z), the solution to LCP (2) and (3) may be determined by solving a linear program with N variables and constraints, more specifically by optimizing a linear objective over the polyhedron described by the inequalities (2).

Federgruen and Hu (2015) characterize the equilibrium behavior in the price competition game under a general *asymmetric* price sensitivity matrix R.[4] A minor regularity condition is, however, required, which the authors refer to as the NPW assumption. In this paper, we adopt a strong sufficient condition for the latter, imposing a limited type of symmetry on the matrix R (Federgruen and Hu 2015, proposition 3).

Assumption (IS). *The matrix R is intrafirm symmetric (IS; i.e., $R_{ik,ik'} = R_{ik',ik}$ for all $i = 1, \ldots, I$ and $k, k' \in \mathcal{N}(i)$).*

Assumption (IS) holds, of course, trivially in the important special case where each firm sells a single product. (Existing economics papers have confined themselves to this case.)

Under strictly affine demand functions without the extension of (2) and (3), it is well known and easily verified that the competition game is supermodular (as long as the matrix R is a Z matrix). However, the extended demand functions fail to be supermodular, in general. The following simple duopoly example in the classical paper by McGuire and Staelin (2008) provides a counterexample.

Example 1. Two retailers $i = 1, 2$ each offer a single product; the raw affine demand functions $q(\cdot)$ have an intercept vector $a = (1, 1)^\top$ and $R = \left(\begin{smallmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{smallmatrix}\right)$ with $\gamma_1, \gamma_2 \in (0, 1)$. See Online Appendix A for details.

We now characterize the equilibrium behavior in the price competition model, where each firm i's profit function $\pi_i(p)$ is given by $\pi_i(p) = \sum_{k \in \mathcal{N}(i)} (p_{ik} - w_{ik}) d_{ik}(p)$. Let

$$T(R) \equiv \begin{pmatrix} R^\top_{\mathcal{N}(1),\mathcal{N}(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^\top_{\mathcal{N}(I),\mathcal{N}(I)} \end{pmatrix} \quad \text{and}$$

$$\Psi(R) \equiv T(R)[R + T(R)]^{-1}.$$

Define the following polyhedron in the space of possible wholesale price vectors:

$$W \equiv \{w > 0 \mid \Psi(R)a - \Psi(R)Rw \geq 0\}^5 \text{ and}$$
$$\text{int}(W) \equiv \{w > 0 \mid \Psi(R)a - \Psi(R)Rw > 0\}. \tag{4}$$

Federgruen and Hu (2015, theorem 2(a)) show that if $w \in W$, a pure Nash equilibrium exists. Moreover, in case $w \in \mathbb{R}^N_{++} \setminus W$ (where $\mathbb{R}_{++} \equiv \{x \mid x > 0\}$), a Nash equilibrium is obtained by replacing w by $w' = \Theta(w)$, where $\Theta(\cdot)$ denotes the projection of w onto the polyhedron W along the coordinate axes. Federgruen and Hu (2015) show that under Assumptions (P), (Z), and (IS), the projection operator $\Theta(\cdot)$ is uniquely defined (again as the unique solution of an LCP, analogous to (2) and (3)). Moreover, for any $w \in \mathbb{R}^N_{++}$, $w' = \Theta(w) \geq 0$ because $\Psi(R)R$ is positive definite by Federgruen and Hu (2015, proposition 2(d)) and $\Psi(R)a \geq 0$ by Federgruen and Hu (2015, proposition 3).

Proposition 1. *The price competition model has a pure Nash equilibrium $(p^*|w)$ with the following expression:*

$$(p^*|w) = \begin{cases} p^*(w) = w + [R + T(R)]^{-1} q(w) \\ \qquad\qquad \text{if } w \in W, \\ p^*(w') = w' + [R + T(R)]^{-1} q(w'), \\ \quad \text{with } w' = \Theta(w) \quad \text{if } w \in \mathbb{R}^N_{++} \setminus W. \end{cases} \tag{5}$$

With some algebra, the associated equilibrium sales volumes are given by (see (4))

$$(d^*|w) = \begin{cases} \Psi(R)a - \Psi(R)Rw, & \text{if } w \in W, \\ \Psi(R)a - \Psi(R)Rw', & \text{with } w' = \Theta(w) \\ & \text{if } w \in \mathbb{R}^N_{++} \setminus W. \end{cases} \tag{6}$$

If $w \in W$, $w' = \Theta(w) = w$; the second expressions in (5) and (6) are therefore valid for all $w > 0$. Depending on the position of the cost rate vector w, the equilibrium $(p^*|w)$ may be the unique price equilibrium, or there may be alternative equilibria, possibly infinitely many equilibria. The main result of *this* paper is to show that $(p^*|w)$ is, however, the *unique* equilibrium that has global robust stability. This result is obtained in the *next* section. We complete *this* section with a characterization of the complete set of equilibria.

3.1. Characterization of the Set of Equilibria

Our characterization of the set of equilibria differs somewhat from the one provided in Federgruen and Hu (2015). The latter asserted that $(p^*|w)$ is the *unique* equilibrium in P (Federgruen and Hu 2015, theorem 1(a)). This implied that *all* equilibria are *equivalent* in the sense that they generate the same product assortment, the same sales volumes for all products, and the same profit levels for all firms. However, the proof for the uniqueness result for $(p^*|w)$ within P contains a subtle error. Indeed, multiple equilibria within P may arise even in the simplest of instances, as in the symmetric case of Example 1, where $\gamma_1 = \gamma_2 = \gamma$ (see Online Appendix A).

Theorem 1 characterizes the set of Nash equilibria.

Theorem 1.

a. *There exists at most one equilibrium in int*(P). *If an equilibrium exists in int*(P), *it is unique in* $\mathbb{R}_+^N$.

b. *When* $p^o \notin P$ *is an equilibrium, so is* $\Omega(p^o) \in P$. *Moreover,* $\Omega(p^o)$ *and* p^o *are equivalent equilibria.*

c. *If* $w \in int(W)$, $(p^*|w) = p^*(w) \in int(P)$ *is the unique equilibrium, and in equilibrium, all products are sold.*

d. *If* $w \in \mathbb{R}_{++}^N \setminus int(W)$, *the set of all equilibria is outside int*(P), *and* $(p^*|w) = p^*(w') \in \partial P$ *is one such equilibrium.*

e. $(p^*|w)_l \geq w_l$ *for any product* l *with positive equilibrium demand, under* $(p^*|w)$.

It follows from part (b) that if $(p^*|w)$ is the unique equilibrium in P, it is the component-wise smallest equilibrium; whether unique in P or not, if $(p^*|w)$ is the component-wise smallest equilibrium, it has the lowest associated profit levels: if $p^0 \leq p^1$ are two equilibria, $\pi_i(p^0) = \pi_i(p_i^0, p_{-i}^0) \leq \pi_i(p_i^1, p_{-i}^1) = \pi_i(p^1)$. The inequality follows from Federgruen and Hu (2016, theorem 4(g)). Cumbul and Virág (2018a) make the same observation in their special model (see Cumbul and Virág 2018a, footnote 4).

Part (e) shows that under the special equilibrium $(p^*|w)$, all products sold in the market have a nonnegative variable profit margin. However, in many industries, we find that some of the products are *sold* below cost, a practice often referred to as "loss leading." In the United Kingdom's retail industry, for example, the Competition Commission UK (2007, pp. 131–132) reports that "nearly all the main parties sold a small number of products at prices below the cost of purchase."[6] The phenomenon of "loss leading" can be explained by models that incorporate search costs or advertising strategies adopted to attract customers who are imperfectly informed of prices; see Chen and Rey (2012) and its references for recent examples of such models. Our model does not explain the phenomenon of "loss leading."

4. Global Robust Stability

In this section, we show that the special equilibrium $(p^*|w)$ has global robust stability. To this end, define for each firm $i = 1, \ldots, I$ a robust best-response mapping as

$$\arg\max_{p_{\mathcal{N}(i)} \geq 0} \Big\{ \big(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)}\big)^\top \big[\big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}\big) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \big] : \big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}\big) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0 \Big\}. \tag{7}$$

To simplify the notation, note that the dependence of firm i's robust best response on the competitors' prices is fully determined by the $|\mathcal{N}(i)|$-dimensional vector: $\alpha \equiv a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)} \geq 0$. (Nonnegativity of α follows from $a \geq 0$, $p \geq 0$, and the fact that R is a Z matrix.) Thus, define the best-response mapping as

$$RB_i(w_{\mathcal{N}(i)}, \alpha) \equiv \underset{p_{\mathcal{N}(i)} \geq 0}{\arg\max} \Big\{ \big(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)}\big)^\top \cdot \big(\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\big) : \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0 \Big\}.$$

In other words, the $RB_i(\cdot)$ operator selects for any vector of prices $p_{-\mathcal{N}(i)}$ of the products offered by firm i's competitors, the price vector that maximizes firm i's profits among all vectors $p_{\mathcal{N}(i)}$ such that

$$p_{\mathcal{N}(i)} \in P_i(p_{-\mathcal{N}(i)}) \equiv \{ p_{\mathcal{N}(i)} \geq 0 \mid \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0 \}. \tag{8}$$

We shall demonstrate that, for any $p_{-\mathcal{N}(i)} \geq 0$, the optimization problem (7) has a *unique* optimizer indeed.

The robust best-response mapping $RB_i(\cdot)$ restricts firm i's choices to vectors $p_{\mathcal{N}(i)}$ such that $p_{\mathcal{N}(i)} \in P_i(p_{-\mathcal{N}(i)})$. Firm i, of course, has the option to select a price vector $p_{\mathcal{N}(i)}$ that falls outside of $P_i(p_{-\mathcal{N}(i)})$, and such a *fully best* selection may result in additional profit enhancements. However, as shown in Section 3, to assess the resulting sales volumes requires the solution of an LCP, defined by (2) and (3), which requires knowledge of the structure of all affine demand functions $q_{-\mathcal{N}(i)}$ for all of the competitors' products. Such information may not be available to firm i.

In contrast, application of the robust best-response mapping $RB_i(\cdot)$ merely requires robust information of the foundational affine functions $q_{\mathcal{N}(i)}(\cdot)$ pertaining to firm i's *own* products, as well as its own cost vector $w_{\mathcal{N}(i)}$. In this section, we show that even if firms restrict possible price adjustments to price vectors in $P_i(p_{-\mathcal{N}(i)})$, the adjustment process converges, in fact geometrically fast, to the overall equilibrium $(p^*|w)$; thus, $(p^*|w)$ is globally stable under the robust best-response operator. In contrast, the convergence properties under fully best responses are more complex; see Federgruen and Hu (2019).

Our analysis uses various matrix norms. A matrix norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ assigns to any $m\times n$ matrix A a number $\|A\|$ with the following properties: (i) $\|\lambda A\| = |\lambda|\,\|A\|$, for all $\lambda\in\mathbb{R}$; (ii) $\|A+B\|\le\|A\|+\|B\|$, for any $m\times n$ matrix B; and (iii) $\|A\|\ge 0$ and $\|A\|=0$ if and only if $A=0_{m,n}$. A matrix norm is *induced by a vector norm* $\|\cdot\|_v$ on $\mathbb{R}^n$ if $\|A\|=\sup\{\|Ax\|_v/\|x\|_v : x\in\mathbb{R}^n \text{ with } x\neq 0\}$. A vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is *absolute* if $\|x\| = \||x|\|$, where $|x|\equiv(|x_1|,|x_2|,\ldots,|x_n|)$. It is well known (see, e.g., Johnson and Nylen 1991) that any absolute vector norm is *monotone* (i.e., $\|x\|\le\|y\|$) if $|x|\le|y|$. In fact, monotonicity and absoluteness are equivalent properties for vector norms; see Johnson and Nylen (1991).

We first verify that the "robust best-response" optimization problem (7) has a unique optimizer.

Lemma 1. *Fix $i=1,\ldots,I$ and $p^o_{-\mathcal{N}(i)}\in\mathbb{R}_+^{N-|\mathcal{N}(i)|}$. The robust best-response mapping $RB_i(\alpha;w_{\mathcal{N}(i)})$ is well defined because the optimization problem (7) has a unique maximizer.*

Proof of Lemma 1. Because R is positive definite, $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is positive definite. Then, the optimization problem (7) is a concave program with linear constraints, which has a unique solution. ■

Let $RB:\mathbb{R}^N\to\mathbb{R}^N$ denote the complete robust best-response mapping (i.e., $RB(p^o)\in\mathbb{R}^N$ is the price vector that arises when each firm $i=1,\ldots,I$ selects its robust best-response price vector $RB_i(p^o_{-\mathcal{N}(i)})$: $RB(p) = (RB_1(p_{-\mathcal{N}(1)}), RB_2(p_{-\mathcal{N}(2)}),\ldots,RB_I(p_{-\mathcal{N}(I)})))$. Let $RB^{(n)}(\cdot)$ be its n-fold application. We first prove that $(p^*|w)$ is a fixed point of the robust best-response mapping $RB(\cdot)$. We then show that it is the unique such fixed point.

Theorem 2. *Fix $w\in\mathbb{R}^N_{++}$. The equilibrium $(p^*|w)$ is a fixed point of the robust best-response mapping $RB(\cdot)$.*

Proof of Theorem 2. Recall that

$$(p^*|w) = \begin{cases} p^*(w) & \text{if } w\in W,\\ p^*(w') & \text{if } w\in\mathbb{R}^N_{++}\setminus W,\end{cases}$$

and for any i,

$$p^*_{\mathcal{N}(i)}(w) = \arg\max_{p_{\mathcal{N}(i)}}\Big\{\left(p_{\mathcal{N}(i)}-w_{\mathcal{N}(i)}\right)^\top \cdot\Big[\Big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}(w)\Big) - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}\Big]\Big\} \quad (9)$$

because $p^*(w)$ is the unique solution to the set of first-order conditions when each retailer solves an *unconstrained* maximization problem of a strictly concave profit function (Federgruen and Hu 2015, proposition 2(a)).

Now, we verify that $(p^*|w)$ is indeed a fixed point of $p=RB(p)$.

Case 1. $w\in W$. First, because $w\in W$, $w'=w$ and $(p^*|w) = p^*(w)\in P$ (see Federgruen and Hu (2015, proposition 2(b)). Hence, $p_{\mathcal{N}(i)} = p^*_{\mathcal{N}(i)}(w)$ for any i satisfies the constraint set:

$$\Big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}(w)\Big) - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}\ge 0. \quad (10)$$

Because $p^*_{\mathcal{N}(i)}(w)$ is an unconstrained maximizer of problem (9) and it satisfies the constraint set (10), it must be a maximizer of the following constrained problem:

$$\begin{aligned}\max_{p_{\mathcal{N}(i)}}\quad & \left(p_{\mathcal{N}(i)}-w_{\mathcal{N}(i)}\right)^\top\Big[\Big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}(w)\Big) - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}\Big]\\ \text{s.t.}\quad & \Big(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}(w)\Big) - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}\ge 0,\\ & p_{\mathcal{N}(i)}\ge 0,\end{aligned} \quad (11)$$

which is exactly the robust best-response problem, given the competitors' price choices $p^*_{-\mathcal{N}(i)}(w)$. This shows that $(p^*|w) = p^*(w)$ is a fixed point of $p=RB(p)$, if $w\in W$.

Case 2. $w\in\mathbb{R}^N_{++}\setminus W$.

Note that $w'=\Theta(w)$ is the projection of w, along the coordinate axes, onto the polyhedron W, described by (4):

$$w'\le w, \quad (12)$$
$$Q(w')^\top(w-w')=0, \quad (13)$$
$$Q(w') = \Psi(R)a - \Psi(R)Rw'\ge 0. \quad (14)$$

The LCP (12)–(14) has a unique solution (see Federgruen and Hu 2015, (8)) because the matrix $\Psi(R)R$ is positive definite by Federgruen and Hu (2015, proposition 2(d)). Moreover, in view of Assumption (IS), property (NPW) is satisfied (Federgruen and Hu 2015, proposition 3). The property (NPW) guarantees that $w'=\Theta(w)\ge 0$. Combined with (14), this implies that

$$w'\in W \quad (15)$$

(i.e., $p^*(w') \in P$ by the definition of W). Now, consider a setting where the cost rate vector w is replaced by $w' \in W$. By Case 1, $p^*_{\mathcal{N}(i)}(w')$ is a maximizer of the following constrained problem (see (11)):

$$\max_{p_{\mathcal{N}(i)}} \left(p_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)}\right)^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\right]$$
$$\text{s.t.} \quad \left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0,$$
$$p_{\mathcal{N}(i)} \geq 0.$$

For any firm i,

$$\begin{aligned}
&\left[p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}\right]^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)}(w')\right] \\
&= \left[p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}\right]^\top Q_{\mathcal{N}(i)}(w') \\
&= \left[p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)} + \left(w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)}\right)\right]^\top Q_{\mathcal{N}(i)}(w') \\
&= \left[p^*_{\mathcal{N}(i)}(w') - w'_{\mathcal{N}(i)}\right]^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)}(w')\right]. \quad (16)
\end{aligned}$$

The first equality follows from Federgruen and Hu (2015, proposition 2(b)), showing that if $w' \in W$, $q(p^*(w')) = Q(w')$. The second equality follows from $(w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})^\top Q_{\mathcal{N}(i)}(w') = 0$ (see (13)). The last equality again uses $Q(w') = q(p^*(w'))$. For any retailer i,

$$\begin{aligned}
&\left[p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}\right]^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)}(w')\right] \\
&\leq \max_{q(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w')) \geq 0, p_{\mathcal{N}(i)} \geq 0} \left[p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)}\right]^\top \cdot \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\right] \\
&\leq \max_{q(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w')) \geq 0, p_{\mathcal{N}(i)} \geq 0} \left[p_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)}\right]^\top \cdot \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\right] \\
&= \left[p^*_{\mathcal{N}(i)}(w') - w'_{\mathcal{N}(i)}\right]^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p^*_{\mathcal{N}(i)}(w')\right]. \quad (17)
\end{aligned}$$

The first inequality is because of the fact that $p_{\mathcal{N}(i)} = p^*_{\mathcal{N}(i)}(w')$ satisfies the constraints $q(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w')) \geq 0$ and $p_{\mathcal{N}(i)} \geq 0$ because $p^*(w') \in P$ by Theorem 1(d). The second inequality is because of $0 \leq w' \leq w$ (see (12)), so that the objective function of the maximization problem to the left of the inequality is dominated by the objective function to its right. The last equality is because of the fact that $p_{\mathcal{N}(i)} = p^*_{\mathcal{N}(i)}(w')$ is the *unconstrained* optimizer of the optimization problem to its left. Moreover, it indeed satisfies the constraints $q(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w')) \geq 0$ and $p_{\mathcal{N}(i)} \geq 0$ because $w' \in W$ (see (15)). By Equation (16), all inequalities in (17) hold as equalities, and in particular, $p^*_{\mathcal{N}(i)}(w')$ is the maximizer of the following constrained problem:

$$\max_{p_{\mathcal{N}(i)}} \left(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)}\right)^\top \left[\left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\right]$$
$$\text{s.t.} \quad \left(a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)}(w')\right) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0,$$
$$p_{\mathcal{N}(i)} \geq 0.$$

This is exactly the robust best-response problem, given the competitors' price choices $p^*_{-\mathcal{N}(i)}(w')$. It shows that $p^*(w')$ is a fixed point of $p = RB(p)$, if $w \in \mathbb{R}^N_{++} \setminus W$. ■

We now show that the robust best-response mapping is a *contraction mapping*. Together with Theorem 2, this implies that $(p^*|w)$ is its (unique) fixed point. A further implication is that, regardless of the market's initial price vector p^0, it converges to $(p^*|w)$ through a series of dynamic adjustments as prescribed by the robust best-response mapping $RB(\cdot)$. Moreover, convergence is *geometrically* fast, implying that the number of iterations required to approach $(p^*|w)$ within an arbitrary ϵ ball is a *logarithmic* function of the original distance $\|p^0 - (p^*|w)\|$ (i.e., the number of iterations is $O(\log(\|p^0 - (p^*|w)\|))$).

Thus, consider an arbitrary pair of price vectors $\widehat{p}, \widetilde{p} \in \mathbb{R}^N_+$, and let $\check{p} = \min\{\widehat{p}, \widetilde{p}\}$. Let $\widehat{\alpha} = a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} \widehat{p}_{-\mathcal{N}(i)}$, $\widetilde{\alpha} = a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} \widetilde{p}_{-\mathcal{N}(i)}$, and $\check{\alpha} = \min\{\widehat{\alpha}, \widetilde{\alpha}\} = a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} \check{p}_{-\mathcal{N}(i)}$ because $R_{\mathcal{N}(i),-\mathcal{N}(i)} \leq 0$. We first need the following lemma, the proof of which is relegated to Online Appendix D. Let $\widehat{p}^* = RB_i(\widehat{p}_{-\mathcal{N}(i)})$, $\widetilde{p}^* = RB_i(\widetilde{p}_{-\mathcal{N}(i)})$, and $\check{p}^* = RB_i(\check{p}_{-\mathcal{N}(i)})$.

Lemma 2. *Fix* $\widehat{p}, \widetilde{p} \in \mathbb{R}^N_+$. *Fix* $i = 1, \ldots, I$,

$$0 \leq \widehat{p}^* - \check{p}^* \leq R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widehat{\alpha} - \check{\alpha}),$$
$$0 \leq \widetilde{p}^* - \check{p}^* \leq R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widetilde{\alpha} - \check{\alpha}). \quad (18)$$

Now, we are ready to present our main result.

Theorem 3.

a. $RB(\cdot)$ *is a contraction mapping: that is, there exists a norm* $\|\cdot\|$ *on* $\mathbb{R}^N$ *and a constant* $0 < \gamma < 1$ *such that*

$$\|RB(\widehat{p}) - RB(\widetilde{p})\| \leq \gamma \|\widehat{p} - \widetilde{p}\| \quad \textit{for all } \widehat{p}, \widetilde{p} \in \mathbb{R}^N_+. \quad (19)$$

b. $\|RB^{(n)}(p) - (p^*|w)\| \leq \gamma^n \|p - (p^*|w)\|$ *and* $\lim_{n\to\infty} RB^{(n)}(p) = (p^*|w)$ *for all* $p \in \mathbb{R}^N_+$. *In other words,* $(p^*|w)$ *is globally stable under* $RB(\cdot)$.

Proof of Theorem 3. It suffices to prove part (a). By Theorem 2, $(p^*|w)$ is a fixed point of the $RB(\cdot)$ operator. Part (b) then follows by setting $\widetilde{p} = (p^*|w)$ in (19) and iterating inequality (19) n times.

For part (a), fix $\widehat{p}, \widetilde{p} \in \mathbb{R}_+^N$. Then, fix $i = 1, \ldots, I$. By (18),

$$-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widetilde{\alpha} - \check{\alpha}) \le (\widehat{p}^* - \check{p}^*) - (\widetilde{p}^* - \check{p}^*) \le R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widehat{\alpha} - \check{\alpha}). \quad (20)$$

Let $U = \{l \in \mathcal{N}(i) \mid \widehat{\alpha}_l < \widetilde{\alpha}_l\}$. Because $\check{\alpha} = \min\{\widehat{\alpha}, \widetilde{\alpha}\}$, $\check{\alpha}_U = \widehat{\alpha}_U$ and $\check{\alpha}_{\overline{U}} = \widetilde{\alpha}_{\overline{U}}$. Recall that $|x| = (|x_i|)$ for any vector x. Then, because $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)} \ge 0$ (by Online Appendix B, Lemma B.1(a)):

$$\begin{aligned}
R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widehat{\alpha} - \check{\alpha}) &= R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\widehat{\alpha} - R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\check{\alpha} \\
&= R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\left[\begin{pmatrix}\widehat{\alpha}_U \\ \widehat{\alpha}_{\overline{U}}\end{pmatrix} - \begin{pmatrix}\widehat{\alpha}_U \\ \widetilde{\alpha}_{\overline{U}}\end{pmatrix}\right] \\
&= R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\begin{pmatrix}0 \\ \widehat{\alpha}_{\overline{U}} - \widetilde{\alpha}_{\overline{U}}\end{pmatrix} \\
&\le R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\begin{pmatrix}|\widehat{\alpha}_U - \widetilde{\alpha}_U| \\ \widehat{\alpha}_{\overline{U}} - \widetilde{\alpha}_{\overline{U}}\end{pmatrix} \\
&= R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}|\widehat{\alpha} - \widetilde{\alpha}| \\
&= R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\left|R_{\mathcal{N}(i),-\mathcal{N}(i)}(\widetilde{p}_{-\mathcal{N}(i)} - \widehat{p}_{-\mathcal{N}(i)})\right| \\
&\le -R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\left|\widetilde{p}_{-\mathcal{N}(i)} - \widehat{p}_{-\mathcal{N}(i)}\right| \\
&= -R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\left|\widehat{p}_{-\mathcal{N}(i)} - \widetilde{p}_{-\mathcal{N}(i)}\right|.
\end{aligned} \quad (21)$$

By symmetry,

$$\begin{aligned}
&R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}(\widetilde{\alpha} - \check{\alpha}) \\
&\quad \le -R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\left|\widetilde{p}_{-\mathcal{N}(i)} - \widehat{p}_{-\mathcal{N}(i)}\right| \\
&\quad = -R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\left|\widehat{p}_{-\mathcal{N}(i)} - \widetilde{p}_{-\mathcal{N}(i)}\right|.
\end{aligned} \quad (22)$$

Combining (20), (21), and (22),

$$\begin{aligned}
|\widehat{p}^* - \widetilde{p}^*| &= \left|RB_i(\widehat{p}_{-\mathcal{N}(i)}) - RB_i(\widetilde{p}_{-\mathcal{N}(i)})\right| \\
&\le -R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}\left|\widehat{p}_{-\mathcal{N}(i)} - \widetilde{p}_{-\mathcal{N}(i)}\right|.
\end{aligned}$$

Combining each individual retailer's best response to get the best-response mapping, we have for any absolute vector norm (and its associated matrix norm) $\|\cdot\|$,

$$\begin{aligned}
\|RB(\widehat{p}) - RB(\widetilde{p})\| &\le \|\Lambda(R) \cdot |\widehat{p} - \widetilde{p}|\| \\
&\le \|\Lambda(R)\| \cdot \|\widehat{p} - \widetilde{p}\|,
\end{aligned} \quad (23)$$

where

$$\Lambda(R) = \begin{pmatrix}
0 & -R^{-1}_{\mathcal{N}(1),\mathcal{N}(1)}R_{\mathcal{N}(1),\mathcal{N}(2)} & \cdots & -R^{-1}_{\mathcal{N}(1),\mathcal{N}(1)}R_{\mathcal{N}(1),\mathcal{N}(I)} \\
-R^{-1}_{\mathcal{N}(2),\mathcal{N}(2)}R_{\mathcal{N}(2),\mathcal{N}(1)} & 0 & \cdots & -R^{-1}_{\mathcal{N}(2),\mathcal{N}(2)}R_{\mathcal{N}(2),\mathcal{N}(I)} \\
\vdots & \vdots & \ddots & \vdots \\
-R^{-1}_{\mathcal{N}(I),\mathcal{N}(I)}R_{\mathcal{N}(I),\mathcal{N}(1)} & -R^{-1}_{\mathcal{N}(I),\mathcal{N}(I)}R_{\mathcal{N}(I),\mathcal{N}(2)} & \cdots & 0
\end{pmatrix} \ge 0.$$

Note that $\Lambda(R) = I - [T(R)]^{-1}R$. Because $R \le T(R)$ due to that $T(R)$ is symmetric (see Assumption (IS)), $[T(R)]^{-1}R = I - \Lambda(R)$ is a ZP matrix, by Online Appendix B, Lemma B.1(c). Because $\Lambda(R) \ge 0$, it follows from Online Appendix B, Lemma B.1(g) that $\rho(\Lambda(R)) < 1$, with $\rho(\cdot)$ denoting the spectral radius.

By Horn and Johnson (1985, lemma 5.6.10), for any $\epsilon > 0$, there is a matrix norm $\|\cdot\|$ induced by an absolute vector norm such that $\rho(\Lambda(R)) \le \|\Lambda(R)\| \le \rho(\Lambda(R)) + \epsilon$. The absoluteness of the norm is because the constructed norm in the proof of Horn and Johnson (1985, lemma 5.6.10) is based on the $\|\cdot\|_1$-matrix norm, which is absolute. Because $\rho(\Lambda(R)) < 1$, there exists an absolute norm $|||\cdot|||$ and a constant γ such that $|||\Lambda(R)||| = \gamma < 1$. Therefore, by (23), which holds for any absolute norm, we have $|||RB(\widehat{p}) - RB(\widetilde{p})||| \le |||\Lambda(R)||| \cdot |||\widehat{p} - \widetilde{p}||| = \gamma|||\widehat{p} - \widetilde{p}|||$, proving (19) and hence, part (a) of the theorem. ■

5. Conclusion

We have analyzed a general but parsimonious price competition model for an oligopoly in which each firm offers any number of products. The demand volumes are general piecewise affine functions of the full price vector, generated as the "regular" extension of a base set of affine functions. The model specifies a *product assortment*, along with their prices and demand volumes, in contrast to most commonly used demand models. Depending on the choice of the cost rate vector, the model may have a unique Nash equilibrium or multiple such equilibria. Our first objective was to provide a full characterization of the set of Nash equilibria.

Regardless of whether there is a unique equilibrium or not, there exists an equilibrium $(p^*|w)$ that can be computed with a limited number of matrix multiplications and inversions of $N \times N$ matrices, sometimes combined with the solution of a single linear program with N variables and constraints. Moreover, the induced equilibrium sales functions have the same structure as the original retail demand functions, thus enabling a similar analysis of the competition among suppliers at higher echelons.

As our main result, we have shown that the special equilibrium $(p^*|w)$ has *global robust stability* (i.e., regardless of its starting point, the market converges geometrically fast to this equilibrium, if firms iteratively adjust their prices as *robust best responses* to the competitors' prices). To determine a robust best response, a firm only needs to know its own demand functions and cost structure and selects a best response among a limited choice set. These stability results show that the special Nash equilibrium $(p^*|w)$ satisfies a *strong* equilibrium property, adding to its predictive power.

We have introduced the concept of *robust* stability and hope that this concept will be pursued in future work for other competition games, in which the determination of a *fully* best response for any given firm requires more than just its own private information. Nevertheless, its remains of great interest to study the market dynamics, when firms adjust prices repeatedly based on *fully best* responses, rather than the robustly best responses. This topic has been addressed in Federgruen and Hu (2019).

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Endnotes

[1] He adds: "Although [the tatônnement process] can (and has) been criticized for being ad hoc, it can also be interpreted as a crude way of expressing the *bounded rationality* of agents. Indeed, several learning mechanisms for agents to play a Nash equilibrium can be understood as refinements of the basic tatônnement" (Vives 2001, p. 49).

[2] See, however, Federgruen and Hu (2016), where these prices are endogenized in a sequential oligopoly model.

[3] However, in Federgruen and Hu (2017, section 6.7), we have relaxed Assumption (Z), allowing for certain types of complementarity among the products.

[4] Ledvina and Sircar (2012) and Cumbul and Virág (2018a, b) and the references therein had confined themselves to the special case where R^{-1} has diagonal elements λ and off-diagonal elements $\lambda\theta$, $\theta \in (0,1)$ (i.e., $R^{-1}_{ii} = \lambda$ for all i and $R^{-1}_{ij} = \lambda\theta$ for all $i \neq j$). This corresponds with the matrix R where $R_{ii} = \frac{1+\theta(N-2)}{\lambda(1-\theta)(1+\theta(N-1))}$ for all i and $R_{ij} = -\frac{\theta}{\lambda(1-\theta)(1+\theta(N-1))}$ for all $i \neq j$.

[5] This definition of W is slightly different from that in Federgruen and Hu (2015), confining the set to positive cost rate vectors $w > 0$ only.

[6] Different European countries and different states within the United States have opposite views on whether loss leading should be prohibited as a predatory practice; some of them have general sales-below-cost laws on the books. At the same time, the Organization for Economic Co-operation and Development (2007) argued that "rules against loss leading are likely to protect inefficient competitors and harm consumers."

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