# Online Appendix – "Liking and Following and the Newsvendor: Operations and Marketing Policies under Social Influence"

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## OA.1. Proofs

*Proof of Lemma 1.* Recall that the probability that a consumer arriving at period $t+1$ will purchase product $i$, given $\boldsymbol{X}_t = \boldsymbol{x}$, is

$$f_i(\boldsymbol{x}) = P(u_i(\boldsymbol{x}) \geq u_{-i}(\boldsymbol{x}), u_i(\boldsymbol{x}) \geq 0) = \Pi_{[0,1]}\left[\min\left\{r - p + x_i^\alpha, \frac{x_i^\alpha - x_{-i}^\alpha + 1}{2}\right\}\right], \qquad i = A, B.$$

In order to show that the market is completely covered, it suffices to show that $r - p + x_i^\alpha \geq \frac{x_i^\alpha - x_{-i}^\alpha + 1}{2}$, $\forall\, x_i \in [0,1]$. As long as this inequality is satisfied, the non-purchase option never kicks in, and thus the market is completely covered. We have

$$x_i^\alpha + x_{-i}^\alpha \geq 0 \Leftrightarrow -x_{-i}^\alpha \leq x_i^\alpha \Leftrightarrow \frac{x_i^\alpha - x_{-i}^\alpha + 1}{2} \leq x_i^\alpha + \frac{1}{2} \Leftrightarrow \frac{x_i^\alpha - x_{-i}^\alpha + 1}{2} \leq r - p + x_i^\alpha,$$

where the first inequality is due to $x_i \in [0,1]$, and the last inequality is due to Assumption 1. Thus, we obtain the announced result. $\square$

*Proof of Lemma 3.* When $\alpha = 1$ or 2, the updating probability function is reduced to $f(x) = x$. It immediately follows from Zhu (2009, Theorem 2.1.4) that $X^*$ is uniformly distributed. When $\alpha \neq 1, 2$, it is easily verified that $f(x)$ has at least three fixed points within $[0,1]$, namely $0, \frac{1}{2}$ and 1. Next, we show that $f(x)$ does not have other fixed points. The second order derivative of (4) is given by $f''(x) = \frac{(1-\gamma)\alpha(\alpha-1)}{2}\left[x^{\alpha-2} - (1-x)^{\alpha-2}\right]$. When $1 < \alpha < 2$, $f(x)$ is convex when $x \in [0, \frac{1}{2}]$ and concave when $x \in [\frac{1}{2}, 1]$. Let $g(x) = f(x) - x$. As a linear combination of a convex function and a linear function, $g(x)$ is convex as well when $x \in [0, \frac{1}{2}]$. The maximum of $g(x)$ over the interval $[0, \frac{1}{2}]$ is reached at end points, i.e., either 0 or $\frac{1}{2}$, which is $g(0) = g(\frac{1}{2}) = 0$, leading to $f(x) < x$ when

$x \in (0, \frac{1}{2})$. Similarly, we have $f(x) > x$ when $x \in (\frac{1}{2}, 1)$. Consequently, when $1 < \alpha < 2$, $f(x)$ has no fixed points other than $0, \frac{1}{2}$ and 1. It is easy to verify that $f'(0) = f'(1) < 1$ and $f'(\frac{1}{2}) > 1$, and thus $X^*$ has the support of $\{0, 1\}$. As $X_2 = \frac{1}{2}$, by symmetry, the process converges to either 0 or 1 with the same probability $\frac{1}{2}$. Similarly, we can show that $\frac{1}{2}$ constitutes the unique stable equilibrium when $\alpha > 2$. $\square$

*Proof of Lemma 4.* We first show that the process converges to the unique point of 1, if $r - p \geq 1$, or $r - p < 1$ and $\alpha \geq \bar{\alpha}$. Recall that the probability that a random consumer arriving at period $t + 1$ will purchase product $A$, given that $X_t = x$, is $f(x) = \Pi_{[0,1]}[r - p + x^\alpha]$. When $r - p \geq 1$, $f(x)$ is reduced to $f(x) = 1$, $\forall x \in [0, 1]$. Based upon Lemma 2, we can verify that 1 constitutes the unique equilibrium.

Next consider the case when $r - p < 1$. Let $\bar{x}_\alpha = [1 - (r - p)]^{1/\alpha}$, where $r - p + x^\alpha = 1$ when evaluated at $x = \bar{x}_\alpha$. Then, we can re-write $f(x)$ as

$$f(x) = \begin{cases} r - p + x^\alpha, & \text{if } 0 \leq x < \bar{x}_\alpha, \\ 1, & \text{if } x \geq \bar{x}_\alpha. \end{cases}$$

Due to $r - p < 1$ and Assumption 1, we know that $\bar{x}_\alpha < 1$, and thus 1 is always an equilibrium. To this end, we need to show that there exists no other equilibrium when $\alpha \leq \bar{\alpha}$. Let $g(x) = r - p + x^\alpha - x$, $\forall x \geq 0$. The first and second order derivatives of $g(x)$ are given by $g'(x) = \alpha x^{\alpha-1} - 1$, $g''(x) = \alpha(\alpha - 1)x^{\alpha-2}$. As $\alpha \geq 1$ and $x \geq 0$, $g(x)$ is concave in $[0, +\infty)$, and its minimum is realized at $x^*_\alpha = \alpha^{\frac{1}{1-\alpha}}$. If $x^*_\alpha \geq \bar{x}_\alpha$, the minimum of $g(x)$, within the interval $[0, \bar{x}_\alpha]$, is given by $g(\bar{x}_\alpha) = 1 - [1 - (r - p)]^{1/\alpha} > 0$. That is $g(x) > 0$, $\forall x \in [0, \bar{x}_\alpha]$, which is equivalent to $f(x) > x$, $\forall x \in [0, \bar{x}_\alpha]$. Thus there exists no other equilibrium. If $x^*_\alpha < \bar{x}_\alpha$, the minimum of $g(x)$, within the interval $[0, \bar{x}_\alpha]$ is given by $g(x^*_\alpha) = r - p + \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}$. We can verify that $g(x^*_\alpha)$ is decreasing in $\alpha$, when $\alpha \geq 1$, with $g(x^*_{\bar{\alpha}}) = 0$. Consequently, when $x^*_\alpha < \bar{x}_\alpha$ and $\alpha \leq \bar{\alpha}$, $g(x) \geq 0$, $\forall x \in [0, \bar{x}_\alpha]$, and thus there exists no other equilibrium. To sum up, when $r - p < 1$, we identify two conditions, i.e., either $x^*_\alpha \geq \bar{x}_\alpha$, or $x^*_\alpha < \bar{x}_\alpha$ and $\alpha \leq \bar{\alpha}$, under which there exists no other equilibrium. It is easy to verify that $x^*_\alpha \geq \bar{x}_\alpha$ is a sufficient condition for $\alpha \leq \bar{\alpha}$, thus the two conditions are reduced to one inequality $\alpha \leq \bar{\alpha}$.

Next consider the case when $r - p < 1$ and $\alpha > \bar{\alpha}$. As $g(x)$ is concave within the interval $[0, \bar{x}_\alpha]$, we know that there exist two points, say $x_1$ and $x_2$, such that $g(x_i) = 0$, $i = 1, 2$. Without loss of generality, we assume $x_1 < x_2$. As $g(0) = r - p > 0$, $g(\bar{x}_\alpha) = 1 - [1 - (r - p)]^{1/\alpha} > 0$ and the continuity of $g(x)$, $\forall x \in [0, \bar{x}_\alpha]$, we know that $g(x) > 0$, $\forall x \in [0, x_1) \cup (x_2, \bar{x}_\alpha]$, and $g(x) < 0$, $\forall x \in (x_1, x_2)$. Then we have $g'(x_1) < 0$ ($g'(x_2) > 0$), which is equivalent to $f'(x_1) < 1$ ($f'(x_2) > 1$). Thus, only the smaller root of $g(x) = 0$ constitutes another equilibrium, and we obtain the announced result. $\square$

*Proof of Lemma 6.* Given the assumption that $X^*$ follows a continuous probability distribution $\Phi(\cdot)$ with density function $\phi(\cdot)$, the firm's profit function can be written as

$$\pi(q_A,q_B) = -c(q_A+q_B)-2K+\min\{q_A+q_B,1\}\cdot p\int_{\min\{q_A,1-q_B\}}^{\max\{q_A,1-q_B\}}\phi(\xi)d\xi$$
$$+p\int_{\max\{q_A,1-q_B\}}^{1}[q_A+(1-\xi)]\phi(\xi)d\xi+p\int_{0}^{\min\{q_A,1-q_B\}}(\xi+q_B)\phi(\xi)d\xi.$$

When $q_A+q_B\geq 1$, the profit function is reduced to

$$\pi(q_A,q_B)=-c(q_A+q_B)-2K+p\int_{1-q_B}^{q_A}\phi(\xi)d\xi+p\int_{q_A}^{1}[q_A+(1-\xi)]\phi(\xi)d\xi+p\int_{0}^{1-q_B}(\xi+q_B)\phi(\xi)d\xi.$$

Its first order conditions (FOCs) are

$$\frac{\partial\pi}{\partial q_A}=-c+p\int_{q_A}^{1}\phi(\xi)d\xi,\qquad \frac{\partial\pi}{\partial q_B}=-c+p\int_{0}^{1-q_B}\phi(\xi)d\xi,$$

and the Hessian matrix is given by

$$H(\pi)=\begin{bmatrix}-p\phi(q_A) & 0\\ 0 & -p\phi(1-q_B)\end{bmatrix}.$$

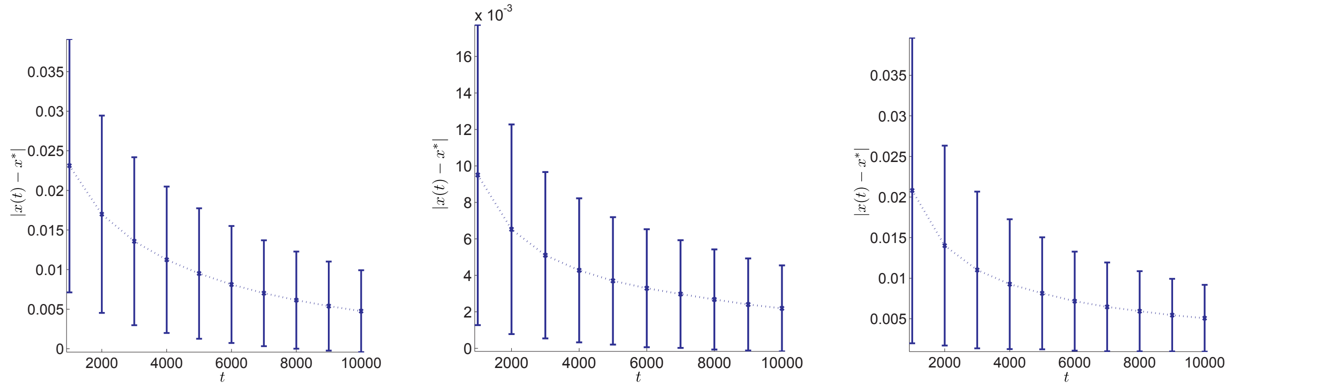
The Hessian matrix is negative definite, and the optimal solutions are given by FOCs, which is $q_A^*=\Phi^{-1}(1-c/p)$ and $q_B^*=1-\Phi^{-1}(c/p)$. It is easy to verify that when $q_A+q_B<1$, FOCs and the Hessian matrix are the same as those when $q_A+q_B\geq 1$. We thus obtain the desired result. □

## OA.2. Rate of Convergence

### OA.2.1. Numerical Results

We conduct extensive numerical experiments to see how accurate the approximation is. Figure OA.1 illustrates the result with 5,000 sample paths for each market size. We make the following observations. First, we observe that the approximation has an accuracy of 1-2% when the total market size is in thousands. This is consistent with the magnitude of production quantity in mass customization. The fast convergence to the limiting distribution indicates that the approximation becomes adequate when the market size is reasonably large. Second, interestingly, the Boundary Scenario has faster convergence rates than the other two scenarios. In other words, when consumers are more or less sensitive than the Boundary Scenario, the demand processes require more time to converge, although in general all converge reasonably fast. This provides another reason why we single out the Boundary Scenario as a special case. Lastly, the Impressionable Scenario tends to converge slightly slower than the Obstinate Scenario.

**Figure OA.1 The absolute deviations of $x(t)$ from its corresponding convergence point $x^*$ at various time periods $t$.**




(a) Impressionable Scenario ($\alpha = 1.5$)

(b) Boundary Scenario ($\alpha = 2$)

(c) Obstinate Scenario ($\alpha = 5$)

*Note.* The error bar is one standard deviation unit in length above and below the average absolute deviation of $x(t)$ from its corresponding convergence point $x^*$.

### OA.2.2. Theoretical Results with Firm-recruited Influencers

Here we study how fast the sales process converges with influencers. In practice, because of the finite market size, the firm can only recruit a finite number of influencers by running promotions. The rate of convergence lends credibility to the accuracy of our approximation. Again for tractability, we focus on the cases when a sufficient condition in Proposition 2 is satisfied for a given scenario. The definition below formalizes the concept of the convergence rate utilized in this paper.

Definition OA.1 (Converge Faster). Let $\{Y_t\}$ and $\{Z_t\}$ be two discrete-time stochastic processes. Assume they converge to unique points, and let $y^*$ and $z^*$ be their unique convergence points, respectively. Suppose, for any discrete-time series $\{\beta_t\}$, $\limsup_{t\to\infty}\beta_t|Y_t - y^*| = \mu_y$ and $\limsup_{t\to\infty}\beta_t|Z_t - z^*| = \mu_z$. If $\mu_y < \mu_z$ ($\mu_y > \mu_z$), then we say that $\{Y_t\}$ converges at a faster (slower) rate than $\{Z_t\}$.

We can interpret this definition in the following sense. The limit superiors on $\{Y_t\}$ and $\{Z_t\}$ indicate that, for any $\epsilon > 0$, there exists $T$, such that $|Y_t - y^*| < (\mu_y + \epsilon)/\beta_t$ and $|Z_t - z^*| < (\mu_z + \epsilon)/\beta_t$ for all $t > T$. That is, after a finite number of observations, the process $\{Y_t\}$ is constrained within a narrower band around its convergence point than that of $\{Z_t\}$, if $\mu_y < \mu_z$. As a result, we say that $\{Y_t\}$ converges at a faster rate than $\{Z_t\}$.

Given the definition above, we compare the convergence rates under the policy of recruiting influencers to the case where all consumers' decisions are solely contingent on their own intrinsic preferences, that is, without social influence. Recall that the probability that any influencer chooses product $A$ without social influence is $\Pi_{[0,1]}\left[\frac{1+(1-\beta)p}{2}\right]$, and the probability that any follower chooses product $A$ without social influence is $1/2$. This process with independent decision making converges to $(1-\gamma)\frac{1}{2}+\gamma\Pi_{[0,1]}\left[\frac{1+(1-\beta)p}{2}\right]$ at an *exponential* rate given by the law of large numbers (Baum et al. 1962). Against this benchmark, we compare the case when consumers are susceptible to others' decisions. The result is summarized as follows.

Lemma OA.1 (Comparison of Convergence Rates). *Assume the firm recruits influencers such that the process converges to a unique stable equilibrium $x^*$. The process with social influence converges faster than the process without if and only if $\frac{x^*(1-x^*)}{f(1/2)(1-f(1/2))}<1-2f'(x^*)$ and $f'(x^*)<1/2$.*

*Proof of Lemma OA.1.* When consumers are susceptible to others' decisions, the sales process is characterized by (6). The convergence rate of the process can be derived from Arthur et al. (1987, Theorems 2 and 3), i.e.,

$$\limsup_{t\to\infty}\sqrt{\frac{t}{\ln\ln t}}|X_t-x^*|=\begin{cases}\sqrt{\frac{2x^*(1-x^*)}{1-2f'(x^*)}} & \text{if } f'(x^*)<\frac{1}{2},\\ +\infty & \text{if } f'(x^*)\geq\frac{1}{2}.\end{cases}\tag{OA.1}$$

Compare it with the benchmark case where all consumers' purchase decisions are solely based upon their own preferences. In this case, any random consumer purchases product $A$ with probability $y^*\equiv(1-\gamma)/2+\gamma\Pi_{[0,1]}\left[\frac{1+(1-\beta)p}{2}\right]$, and purchase product $B$ with probability $1-y^*$. Denote, by $Y_t$, the fraction of consumers who purchase product $A$ at the beginning of period $t$. The convergence rate of the process $\{Y_t\}$ is given by (Durrett 2010, Section 2.5.1 and Theorem 8.8.2),

$$\limsup_{t\to\infty}\sqrt{\frac{t}{\ln\ln t}}|Y_t-y^*|=\sqrt{2y^*(1-y^*)}.\tag{OA.2}$$

Notice that $f(\frac{1}{2})=y^*$, and thus we obtain the desired result by comparing the right-hand-sides of Equations (OA.1) and (OA.2). □

Lemma OA.1 shows that social influence facilitates a marketing campaign in shaping a collective decision. This is because the convergence point is being pulled closer towards 1 with social influence than without social influence.[1] Notice that the rate of convergence is affected by the convergence point: ceteris paribus, the process converges in the slowest pace when the convergent point is $1/2$, and the rate of convergence increases as the equilibrium deviates from $1/2$.[2] When the unique

[1] Notice that the convergence point without social influence is $f(1/2)>1/2$. Because of $f'(x)>0,\forall x\in[0,1]$, we have $f(f(1/2))>f(1/2)$. Thus the convergence point with social influence, $x^*$, which is given by $f(x)=x$, must be greater than $f(1/2)$.

[2] To see this, consider an independent and identically distributed sequence of Bernoulli random variables with parameter $p\in[0,1]$. When $p=0$ or 1, the average process of the sequence reaches its convergence point at either 0 or 1 at the first step and stays there forever. However when $p=1/2$, the average process fluctuates, but eventually it converges to $1/2$ by the law of large numbers.

stable equilibrium $x^*$ is closer to 1, the condition $\frac{x^*(1-x^*)}{f(1/2)(1-f(1/2))} < 1 - 2f'(x^*)$ is more likely to hold, and consequently, we are more confident in using the limit as an approximation to our finite market-size model because of its fast convergence.

## OA.3. Optimal Production Time under Production Postponement

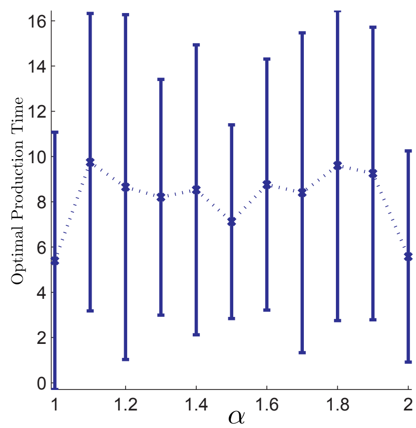
In this section, we allow the firm to choose when to produce, and investigate how the optimal production time depends on consumers' sensitivity to others' decisions. As postponing production means a shorter time to deliver products to market, the cost of production inevitably increases in the time delay for production. Essentially, the firm faces a trade-off between the reduced uncertainty in future demand and the increased production cost. The complexity of the demand distributions prevents us from identifying the structure of the optimal time to produce. However, we show in the example below that the optimal production time can be highly sensitive to consumers' sensitivity to others' decisions, i.e., $\alpha$.

Example OA.1 (Sensitivity of Optimal Production Time). Let us denote the optimal production time as $T^*_\alpha$, with a given $\alpha$. For ease of presentation, we assume that only the fixed cost is increasing in the production delay, say $K_t = \tau_t K$, where $\tau_t$ is greater than or equal to 1, and non-decreasing in $t$. The expected profit when producing at time $T$, with production quantities $q_A$ and $q_B$, is given by $\pi_T(q_A, q_B) = pE[\min\{q_A, X^*\} + \min\{q_B, 1 - X^*\}] - [c \cdot (q_A + q_B) + 2\tau_T K]$.

Note that under the Obstinate Scenario, the firm knows that demand will be split evenly between the two products even without any pre-order information. Thus, the firm does not benefit from postponing production, and the optimal production time under the Obstinate Scenario is to start production at the beginning of the sales season, i.e., $T^*_\alpha = 0$ for all $\alpha > 2$. Under the Boundary Scenario, given that $N_T = n$, we know that $X^*$ follows a $Beta(n, T - n)$ distribution. The expected profit when the firm produces at any time $T \geq 0$ is bounded by $-2\tau_T K \leq \pi_T(q_A, q_B) \leq p - c - 2\tau_T K \leq p - c - 2K$, where the lower bound is derived when $q_A = q_B = 0$. The second inequality is shown in the proof of Proposition 1, and the third inequality is due to $\tau_t \geq 1$. As $T$ goes to infinity, we know from Lemma 5 that the variance of $X^*$ converges to 0, and the distribution degenerates to a single constant. Thus if marginal cost increase from postponing production, $\tau_{T+1} - \tau_T$, is sufficiently small, the optimal production time can be very large and the optimal profit is close to $p - c - 2K$. That is, under the Boundary Scenario, i.e., $\alpha = 2$, the benefit of extra sales information can outweigh the marginal increase in production cost for a long time, leading to a large optimal production time. However, when $\alpha$ is slightly greater than 2 as under the Obstinate Scenario, the optimal decision is always to produce without postponement. Consequently, when the cost of postponing production increases slowly over time, $T_\alpha^*$ can vary significantly for a small change in $\alpha$ around 2.

On another note, the benefit from postponing production is bounded by $p-c$, because $-2\tau_T K \le \pi_T(q_A,q_B) \le p-c-2\tau_T K$ for all $T \ge 0$ and $\tau_T$ is non-decreasing in $T$. When $\tau_T$ increases quickly, say $\tau_{T+1}-\tau_T \ge (p-c)/K$ for some $\tilde{T}$, it will never be optimal for the firm to produce after period $\tilde{T}$. As a direct consequence, the firm produces without postponement when the preceding inequality is satisfied for $\tilde{T}=0$. The sensitivity of the optimal production time with respect to the consumers' responsiveness to others' decisions can be moderated when the marginal cost increase from production postponement is substantial.

**Figure OA.2 The optimal production time $T_\alpha^*$ as $\alpha$ varies.**




*Note.* The parameters are specified as follows. $\delta$ follows a uniform distribution with support $[0,1]$, and $p=10$, $c=3$, $K=1$, $\tau_t=0.04$. We run the simulation 1,000 times for each value of $\alpha$. The error bar is one standard deviation unit in length above and below the average optimal production time $T_\alpha^*$.

To further understand how the optimal production time depends on $\alpha$, we conduct comprehensive numerical analysis under the Boundary and Impressionable Scenario, and the results are plotted in Figure OA.2. Our numerical results indicate that: (1) the optimal production time is generally smaller under the Boundary Scenario, i.e., $\alpha=1$ or 2, than that under the Impressionable Scenario, i.e., $1<\alpha<2$; (2) the optimal production time first tend to decrease, and then tend to increase as $\alpha$ varies within $(1,2)$. Notice that consumers are the most sensitive to others' decisions when $\alpha$ is around 1.5. In that neighborhood, a small difference between the market shares of two products may influence the decisions of later arrivals substantially, and thus the firm can afford to produce as soon as a small lead in the market share by one product is established. However, as consumers become more independent, the trend is more likely to be reversed, and thus the firm needs to wait longer for more sales information before determining production quantities.

## References

Arthur, W. B., Y. M. Ermoliev, Y. M. Kaniovski. 1987. Limit theorems for proportions of balls in a generalized urn scheme. Working Paper, International Institute for Applied Systems Analysis, Austria.

Baum, L. E., M. Katz, R. R. Read. 1962. Exponential convergence rates for the law of large numbers. *Transactions of the American Mathematical Society* **102**(2) 187–199.

Durrett, R. 2010. *Probability: Theory and Examples*. 4th ed. Cambridge University Press, UK.

Zhu, T. 2009. Nonlinear Pólya urn models and self-organizing processes. Dissertation, University of Pennsylvania, PA.