Online Appendix to "Sales Effort Management Under All-or-Nothing Constraint"

Longyuan Du, Ming Hu, Jiahua Wu

We analyze how a seller can adjust the sales intensity to maximize her profit under an all-or-nothing constraint. This online appendix contains all proofs. The techniques we used to characterize the optimal policy, analyze the asymptotic bounds, and construct static and dynamic heuristics with provable performance bounds can be readily applied to the general stochastic point process control problem when the objective function is discontinuous. In particular:

1. We obtain rich structural properties of the optimal policies in Theorem 1 and Lemma A1 in this online appendix. We show the precise monotonicity of the optimal sales intensity. That is, when $b \le p$ the optimal sales intensity increases in t, whereas when $b > p$ the optimal sales intensity first increases then decreases in t. To the best of our knowledge, we are the first to show this "watershed" structure of the optimal sales intensity in the context of revenue management/sales effort management. The proof of Theorem 1 relies on constructing tight bounds of $\Delta_t(n)$ (for instance, when $b > p$ we prove that when t is sufficiently large, $\Delta_t(n) > p + \exp\left(-\int_0^t \lambda^*(s,n)\,\mathrm{d}s\right) F^{(n-1)}(t)$, where $F^{(n-1)}(t)$ is a polynomial function of order $n-1$ and the leading coefficient is positive), which, we believe, is novel and readily reusable in other settings.
2. We show that asymptotically optimal static heuristics require a markup on top of the optimal deterministic sales intensity, which is also unique due to the existence of the all-or-nothing constraint. The asymptotic analysis requires the bounding of the tail distribution of a Poisson variable. We use a tight bound in the proof of Proposition 5, which again can be applied in many other contexts.
3. Regarding resolving heuristics, we show that the standard resolving heuristic in revenue management that updates the sales intensity by periodically resolving the static problems is not asymptotically optimal. We propose a modified resolving heuristic with a carefully chosen switching time that can strike a balance between the loss in the probability of reaching the target and the cost of extra effort from the higher sales intensity. To the best of our knowledge, this two-stage resolving heuristic is novel in the literature, and the same technique can be used in many other settings when the objective function is discontinuous.

Proof of Proposition 1. When $n \ge 1$, for any $t > 0$, let us consider a small time period δ. We have

$$J^*_{t+\delta}(n) = \max_{\lambda}\ (1-\lambda\delta)\cdot J^*_t(n) + \lambda\delta \cdot J^*_t(n-1) - c(\lambda)\delta + o(\delta).$$

Rearranging the terms and letting $\delta \to 0$, we get Equation (2). The boundary conditions are derived from the following. At time 0, the optimal expected profit is just 0 if the threshold is not reached, i.e., $J_0^*(n) = 0$, $\forall n \geq 1$. When $n = 0$, the seller already reaches the threshold. As a result, at any time t, the optimal $\lambda^*(t,0)$ shall maximize the profit rate $\lambda p - c(\lambda)$. Thus, the profit rate when $n = 0$ is given by $\frac{\partial J_t^*(0)}{\partial t} = \max_{\lambda} \{\lambda p - c(\lambda)\}$. Given that $J_0^*(0) = b$, we thus obtain the announced result. □

Proof of Theorem 1. The monotonicity of $J_t^*(n)$ is obvious, and thus we omit the proof here. To prove parts (ii) and (iii) of the theorem, we prove the following lemma first.

LEMMA A1. *(i) $\forall n \geq 2$ and $0 < z \leq +\infty$, if $\frac{\partial \Delta_t(n-1)}{\partial t} \geq 0$ for any $t \in [0,z]$, then $\frac{\partial \Delta_t(n)}{\partial t} > 0$ for any $t \in [0,z]$;*

(ii) $\forall n \geq 1$, if $\left.\frac{\partial \Delta_t(n)}{\partial t}\right|_{t=z} \leq 0$, then $\frac{\partial \Delta_t(n)}{\partial t} \leq 0$ for any $t > z$;

(iii) $\forall n \geq 0$, $\lim_{t\to\infty} \lambda^*(t,n) = \lambda^*$.

Proof of Lemma A1. (i) For notational convenience, we let $J_t^*(-1) = J^*(0) + p$. First we show that

$$\lambda^*(t,n)\Big[\Delta_t(n-1) - \Delta_t(n)\Big] \leq \frac{\partial \Delta_t(n)}{\partial t} \leq \lambda^*(t,n-1)\Big[\Delta_t(n-1) - \Delta_t(n)\Big]. \tag{OA.1}$$

For any time t and a small time interval δ, we have

$$\begin{aligned} J_{t+\delta}^*(n) &= \max_{\lambda}\ (1-\lambda\delta)\cdot J_t^*(n) + \lambda\delta \cdot J_t^*(n-1) - c(\lambda)\delta + o(\delta) \\ &\geq (1-\lambda^*(t,n-1)\delta)\cdot J_t^*(n) + \lambda^*(t,n-1)\delta \cdot J_t^*(n-1) - c(\lambda^*(t,n-1))\delta + o(\delta). \end{aligned}$$

Rearranging the terms and letting $\delta \to 0$, we have $\frac{\partial J_t^*(n)}{\partial t} \geq \lambda^*(t,n-1)\Delta_t(n) - c(\lambda^*(t,n-1))$. Therefore,

$$\begin{aligned} \frac{\partial \Delta_t(n)}{\partial t} &= \frac{\partial J_t^*(n-1)}{\partial t} - \frac{\partial J_t^*(n)}{\partial t} \\ &\leq [\lambda^*(t,n-1)\Delta_t(n-1) - c(\lambda^*(t,n-1))] - [\lambda^*(t,n-1)\Delta_t(n) - c(\lambda^*(t,n-1))] \\ &= \lambda^*(t,n-1)\left[\Delta_t(n-1) - \Delta_t(n)\right]. \end{aligned}$$

Similarly, we can also show that $\frac{\partial \Delta_t(n)}{\partial t} \geq \lambda^*(t,n)\left[\Delta_t(n-1) - \Delta_t(n)\right]$.

Define

$$L_t(n) \equiv \int_0^t \lambda^*(s,n)\,\mathrm{d}s.$$

Since $\Delta_0(n) = 0$ for $n \geq 2$, applying Grönwall's inequality to Inequality (OA.1), we have:

$$\Delta_t(n) \leq \exp\left(-L_t(n-1)\right)\int_0^t \exp(L_s(n-1))\Delta_s(n-1)\lambda^*(s,n-1)\,\mathrm{d}s,$$

for any $t \leq z$. Since $\Delta_t(n-1)$ increases in t from the stipulation of Lemma A1(i), we have

$$\Delta_t(n) \leq \exp\left(-L_t(n-1)\right)\Delta_t(n-1)\int_0^t \exp(L_s(n-1))\lambda^*(s,n-1)\,\mathrm{d}s < \Delta_t(n-1).$$

Based on Inequality (OA.1), we have $\frac{\partial \Delta_t(n)}{\partial t} \geq \lambda^*(t,n)\Big[\Delta_t(n-1)-\Delta_t(n)\Big] > 0$.

(ii) We show this by induction. Consider first when $n=1$. Suppose the statement is not true, then there exists $t_2 > t_1 \geq z$ such that $\frac{\partial \Delta_t(1)}{\partial t}\Big|_{t=t_1} = 0$ and $\frac{\partial \Delta_t(1)}{\partial t} > 0$ for all $t \in (t_1,t_2]$. From Inequality (OA.1), we have $\Delta_{t_1}(0)-\Delta_{t_1}(1)=0$, and $\Delta_t(0)-\Delta_t(1)>0$ for all $t\in(t_1,t_2]$. Because $\Delta_s(0)=p$, $\forall s$ by construction, we have $\Delta_{t_1}(1)=p$ and $\Delta_t(1)<p$ for all $t\in(t_1,t_2]$. Because $\Delta_t(1)$ strictly increases between $[t_1,t_2]$, we have $\Delta_t(1)>p$ for any $t\in(t_1,t_2]$, which leads to contradiction. Thus, the statement is true for $n=1$.

Now assume the statement is true for $n-1$ and let us consider n. Suppose the statement is not true for n, then there exists $t_2>t_1\geq z$ such that $\frac{\partial \Delta_t(n)}{\partial t}\Big|_{t=t_1}=0$ and $\frac{\partial \Delta_t(n)}{\partial t}>0$ for all $t\in(t_1,t_2]$. From Inequality (OA.1), we have $\Delta_{t_1}(n-1)-\Delta_{t_1}(n)=0$, and $\Delta_t(n-1)-\Delta_t(n)>0$ for all $t\in(t_1,t_2]$. First we know that, if $\frac{\partial \Delta_t(n-1)}{\partial t}\Big|_{t=z}>0$, $\frac{\partial \Delta_t(n-1)}{\partial t}>0$ for any $t\leq z$. (Otherwise, there exists a $t_1<z$ such that $\frac{\partial \Delta_t(n-1)}{\partial t}\Big|_{t=t_1}\leq 0$. Using the assumption for $n-1$, we have $\frac{\partial \Delta_t(n-1)}{\partial t}\leq 0$ for any $t\geq t_1$, which leads to contradiction.) According to Lemma A1(i), $\frac{\partial \Delta_t(n)}{\partial t}>0$ for any $t\leq z$, which in turn means $\frac{\partial \Delta_t(n)}{\partial t}\Big|_{t=z}>0$. This contradicts with our assumption that $\frac{\partial \Delta_t(n)}{\partial t}\Big|_{t=z}\leq 0$. Thus we must have $\frac{\partial \Delta_t(n-1)}{\partial t}\Big|_{t=z}\leq 0$. Using our assumption for $n-1$, this implies that $\frac{\partial \Delta_t(n-1)}{\partial t}\leq 0$ for any $t\geq z$. Because $\Delta_t(n-1)$ decreases in t and $\Delta_t(n)$ strictly increases in t for any $t\in[t_1,t_2]$, $\Delta_t(n-1)-\Delta_t(n)$ also strictly decreases in t for any $t\in[t_1,t_2]$. Thus $\Delta_t(n-1)-\Delta_t(n)<0$ for any $t\in(t_1,t_2]$, which leads to contradiction. We thus complete the proof.

(iii) In order to prove $\lim_{t\to\infty}\lambda^*(t,n)=\lambda^*$, it is sufficient to show that $\lim_{t\to\infty}\Delta_t(n)=p$ as $\lambda^*(t,n)=\arg\max_{\lambda\in[\underline{\lambda},\bar{\lambda}]}\{\lambda\Delta_t(n)-c(\lambda)\}$. We prove this by induction. For $n=0$, $\Delta_t(0)=p$ by construction. Now suppose $\lim_{t\to\infty}\Delta_t(n-1)=p$. Then for any $\epsilon>0$, there exists a z such that for any $t>z$, $p-\epsilon<\Delta_t(n-1)<p+\epsilon$. From Inequality (OA.1),

$$\begin{aligned}
\Delta_t(n) &\geq \Delta_z(n)\cdot\exp\left(L_z(n)-L_t(n)\right)+\exp\left(-L_t(n)\right)\int_z^t \exp\left(L_s(n)\right)\Delta_s(n-1)\lambda^*(s,n)\,\mathrm{d}s \\
&> \Delta_z(n)\cdot\exp(L_z(n)-L_t(n))+(p-\epsilon)\cdot\exp\left(-L_t(n)\right)\int_z^t\exp\left(L_s(n)\right)\lambda^*(s,n)\,\mathrm{d}s \\
&= \Delta_z(n)\cdot\exp(L_z(n)-L_t(n))+(p-\epsilon)[1-\exp\left(L_z(n)-L_t(n)\right)] \\
&= p-\epsilon-[p-\epsilon-\Delta_z(n)]\exp\left(L_z(n)-L_t(n)\right).
\end{aligned}$$

Since $\lim_{t\to\infty}L_t(n)=+\infty$, we can find a z_1 such that $\Delta_t(n)>p-2\epsilon$ for any $t>z_1$. Similarly, we can find a z_2 such that $\Delta_t(n)<p+2\epsilon$ for any $t>z_2$. Thus the statement is true for n, and we obtain the announced result. □

Now, we are ready to prove Theorem 1 parts (ii) and (iii). Recall that $\lambda^*(t,n) = \arg\max_{\lambda\in[\underline{\lambda},\bar{\lambda}]}\{\lambda\Delta_t(n) - c(\lambda)\} = \sup\{\lambda \le \bar{\lambda} : \Delta_t(n) \ge c'(\lambda)\} \vee \underline{\lambda}$. Since $c(\lambda)$ is convex, $c'(\lambda)$ increases in λ. Therefore the weak monotonicity in t and n of $\Delta_t(n)$ implies the weak monotonicity of $\lambda^*(t,n)$. We thus only need to show the monotonicity of $\Delta_t(n)$ in t and n. Furthermore, from Inequality (OA.1), we know that $\Delta_t(n)$ weakly increases (decreases) in t if and only if $\Delta_t(n-1) \ge \Delta_t(n)$ ($\Delta_t(n-1) \le \Delta_t(n)$). Therefore, it suffices to show the monotonicity of $\Delta_t(n)$ in t.

First, we prove Theorem 1(iii) when $b \le p$. Based on $\lambda^*(t,0) = \lambda^*$ and Inequality (OA.1), we have

$$\Delta_t(1) \le e^{-\lambda^* t}\left[\Delta_0(0) + \int_0^t \lambda^* e^{\lambda^* s}\Delta_s(0)\,\mathrm{d}s\right] = e^{-\lambda^* t}\left[b + p\int_0^t \lambda^* e^{\lambda^* s}\,\mathrm{d}s\right] = p - (p-b)e^{-\lambda^* t} \le p.$$

This further implies that $\Delta_t(1)$ increases in t based on Inequality (OA.1). From Lemma A1, we thus have $\Delta_t(n)$ strictly increases in t for any $n \ge 2$.

Now we prove Theorem 1(ii) when $b > p$. To that end, we prove a stronger statement that there exists an η_n such that for any $t \ge \eta_n$, $\Delta_t(n) \ge p + \exp(-L_t(n))\,F^{(n-1)}(t)$, where $F^{(n-1)}(t) = \theta^{(n-1)}t^{n-1} + \theta^{(n-2)}t^{n-2} + \cdots + \theta^{(0)}$ with $\theta^{(n-1)} > 0$. If this inequality holds, then we have $F^{(n-1)}(t) > 0$ when t is sufficiently large, which further implies that $\Delta_t(n) > p$ when t is sufficiently large. Note that $\lim_{t\to\infty}\Delta_t(n) = p$. This means $\Delta_t(n)$ must be approaching p from above. Combining with $\Delta_0(n) = 0 \le p$ for any $n \ge 2$ and Lemma A1(ii), we can then conclude that $\Delta_t(n)$ must first increase and then decrease in t.

We prove the inequality $\Delta_t(n) \ge p + \exp(-L_t(n))\,F^{(n-1)}(t)$ by induction. For $n = 1$,

$$\begin{aligned}
J_t^*(1) &= \exp(-L_t(1))\int_0^t \exp(L_s(1))\,[\lambda^*(s,1)\,(b + (\lambda^* p - c(\lambda^*))s) - c(\lambda^*(s,1))]\,\mathrm{d}s \\
&= \exp(-L_t(1))\int_0^t \exp(L_s(1))\cdot[\lambda^*(s,1)\,(b + (\lambda^* p - c(\lambda^*))s - p) + \lambda^*(s,1)p - c(\lambda^*(s,1))]\,\mathrm{d}s.
\end{aligned}$$

Because λ^* maximizes $\lambda p - c(\lambda)$, $\lambda^*(s,1)p - c(\lambda^*(s,1)) \le \lambda^* p - c(\lambda^*)$ for any $s \in [0,T]$. Thus,

$$\begin{aligned}
J_t^*(1) &\le \exp(-L_t(1))\int_0^t \exp(L_s(1))\cdot[\lambda^*(s,1)\,(b + (\lambda^* p - c(\lambda^*))s - p) + \lambda^* p - c(\lambda^*)]\,\mathrm{d}s \\
&= \exp(-L_t(1))\left[\int_0^t (b + (\lambda^* p - c(\lambda^*))s - p)\,\mathrm{d}\exp(L_s(1)) + \int_0^t \exp(L_s(1))\,(\lambda^* p - c(\lambda^*))\,\mathrm{d}s\right] \\
&= \exp(-L_t(1))\Bigg[(b + (\lambda^* p - c(\lambda^*))t - p)\exp(L_t(1)) - (b-p) - \int_0^t \exp(L_s(1))\,(\lambda^* p - c(\lambda^*))\,\mathrm{d}s + \\
&\qquad \int_0^t \exp(L_s(1))\,(\lambda^* p - c(\lambda^*))\,\mathrm{d}s\Bigg] \\
&= b + (\lambda^* p - c(\lambda^*))t - p - (b-p)\exp(-L_t(1)).
\end{aligned}$$

Consequently, we have $\Delta_t(1) = J_t^*(0) - J_t^*(1) \ge p + (b-p)\exp(-L_t(1))$.

Now suppose the statement is true for $n-1$. There exists an $\eta \geq 0$ such that $\Delta_t(n-1)$ (and thus also $\lambda^*(t,n-1)$) decreases in t when $t \geq \eta$. We first show that $L_t(n) - L_t(n-1)$ is bounded from below, i.e., $L_t(n) - L_t(n-1) \geq \mathscr{C}$, where $\mathscr{C}$ is a constant independent of t.

Applying Grönwall's Inequality on Inequality (OA.1) over $[\eta, t]$, we have

$$\begin{aligned}
\Delta_t(n) &\geq \Delta_\eta \exp\left(-\int_\eta^t \lambda^*(s,n)\,\mathrm{d}s\right) + \int_\eta^t \exp\left(-\int_s^t \lambda^*(s,n)\,\mathrm{d}s\right)\Delta_s(n-1)\,\mathrm{d}s \\
&= \Delta_\eta(n) \cdot \exp\left(L_\eta(n) - L_t(n)\right) + \exp\left(-L_t(n)\right)\int_\eta^t \exp\left(L_s(n)\right)\Delta_s(n-1)\lambda^*(s,n)\,\mathrm{d}s \\
&\geq \Delta_\eta(n) \cdot \exp\left(L_\eta(n) - L_t(n)\right) + \Delta_t(n-1) \cdot \exp\left(-L_t(n)\right)\int_\eta^t \exp\left(L_s(n)\right)\lambda^*(s,n)\,\mathrm{d}s \\
&= \Delta_\eta(n) \cdot \exp\left(L_\eta(n) - L_t(n)\right) + \Delta_t(n-1) \cdot [1 - \exp(L_\eta(n) - L_t(n))] \\
&= \Delta_t(n-1) + [\Delta_\eta(n) - \Delta_t(n-1)]\exp(L_\eta(n)) \cdot \exp(-L_t(n)),
\end{aligned}$$

where the second inequality is due to the stipulation that $\Delta_t(n-1)$ decreases in t for any $t \geq \eta$. Note that $\Delta_t(n-1)$ is bounded from above. Also η is a constant independent of t. We can thus find a constant $\mathscr{C}_1$ such that $[\Delta_\eta(n) - \Delta_t(n-1)]\exp(L_\eta(n)) \geq \mathscr{C}_1$ for any $t > \eta$, which implies $\Delta_t(n) - \Delta_t(n-1) \geq \mathscr{C}_1 \exp(-L_t(n))$.

Let $\tilde{\lambda}(t,n)$ be the unique solution of equation $\Delta_t(n) = c'(\lambda)$. Then for t's such that $\tilde{\lambda}(t,n) \leq \tilde{\lambda}(t,n-1)$, $\Delta_t(n) - \Delta_t(n-1) = c'(\tilde{\lambda}(t,n)) - c'(\tilde{\lambda}(t,n-1)) \leq \alpha(\tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1))$, where $\alpha = \min\limits_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} c''(\lambda) > 0$. Hence for any $t \geq \eta$, $\tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1) \geq \frac{\mathscr{C}_1}{\alpha}\exp(-L_t(n))$ if $\tilde{\lambda}(t,n) \leq \tilde{\lambda}(t,n-1)$. This further implies that $\tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1) \geq -\frac{|\mathscr{C}_1|}{\alpha}\exp(-L_t(n))$ for any $t \geq \eta$.

Note that $\lambda^*(t,n) = \arg\max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} \{\lambda\Delta_t(n) - c(\lambda)\} = \sup\{\lambda \leq \bar{\lambda} : \Delta_t(n) \geq c'(\lambda)\} \vee \underline{\lambda} = (\tilde{\lambda}(t,n) \wedge \bar{\lambda}) \vee \underline{\lambda}$. We show that there exists some constant $w \geq \eta$ such that for any $t \geq w$, $\lambda^*(t,n) - \lambda^*(t,n-1) \geq \tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1)$ if $\lambda^*(t,n) < \lambda^*(t,n-1)$. From Assumption 1(iii), $\lambda^* \in [\underline{\lambda}, \bar{\lambda}]$. Consider the following three cases:

(a) If $\lambda^* = \underline{\lambda}$, $\lim\limits_{t\to\infty} \lambda^*(t,n) = \lambda^* < \bar{\lambda}$ from Lemma A1(iii). Thus there exists some constant $w \geq \eta$ such that $\lambda^*(t,n) = \tilde{\lambda}(t,n) \vee \underline{\lambda}$ and $\lambda^*(t,n-1) = \tilde{\lambda}(t,n-1) \vee \underline{\lambda}$ for any $t \geq w$. If $\lambda^*(t,n) < \lambda^*(t,n-1)$, $\tilde{\lambda}(t,n-1) > \underline{\lambda}$ (otherwise $\lambda^*(t,n) < \lambda^*(t,n-1) = \tilde{\lambda}(t,n-1) \vee \underline{\lambda} = \underline{\lambda}$, which contradicts with $\lambda^*(t,n) \geq \underline{\lambda}$). This implies that $\lambda^*(t,n-1) = \tilde{\lambda}(t,n-1) \vee \underline{\lambda} = \tilde{\lambda}(t,n-1)$. Also note that $\lambda^*(t,n) \geq \tilde{\lambda}(t,n)$. Therefore, for any $t \geq w$, if $\lambda^*(t,n) < \lambda^*(t,n-1)$, $\lambda^*(t,n) - \lambda^*(t,n-1) = \lambda^*(t,n) - \tilde{\lambda}(t,n-1) \geq \tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1)$.

(b) If $\lambda^* = \bar{\lambda}$, $\lim\limits_{t\to\infty} \lambda^*(t,n) = \lambda^* > \underline{\lambda}$ from Lemma A1(iii). Thus there exists some constant $w \geq \eta$ such that $\lambda^*(t,n) = \tilde{\lambda}(t,n) \wedge \bar{\lambda}$ and $\lambda^*(t,n-1) = \tilde{\lambda}(t,n-1) \wedge \bar{\lambda}$ for any $t \geq w$. If $\lambda^*(t,n) < \lambda^*(t,n-1)$,

$\tilde{\lambda}(t,n) < \bar{\lambda}$ (otherwise $\lambda^*(t,n-1) > \lambda^*(t,n) = \tilde{\lambda}(t,n) \wedge \bar{\lambda} = \bar{\lambda}$, which contradicts with $\lambda^*(t,n-1) \leq \bar{\lambda}$). This implies that $\lambda^*(t,n) = \tilde{\lambda}(t,n) \wedge \bar{\lambda} = \tilde{\lambda}(t,n)$. Also note that $\lambda^*(t,n-1) \leq \tilde{\lambda}(t,n-1)$. Therefore, for any $t \geq w$, if $\lambda^*(t,n) < \lambda^*(t,n-1)$, $\lambda^*(t,n) - \lambda^*(t,n-1) = \tilde{\lambda}(t,n) - \lambda^*(t,n-1) \geq \tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1)$.

(c) If $\lambda^* \in (\underline{\lambda}, \bar{\lambda})$, $\lim_{t\to\infty} \lambda^*(t,n) = \lambda^* \in (\underline{\lambda}, \bar{\lambda})$ from Lemma A1(iii). Therefore $\lambda^*(t,n) = \tilde{\lambda}(t,n)$ and $\lambda^*(t,n-1) = \tilde{\lambda}(t,n-1)$ when t is sufficiently large. We can find some constant $w \geq \eta$ such that $\lambda^*(t,n) = \tilde{\lambda}(t,n)$ and $\lambda^*(t,n-1) = \tilde{\lambda}(t,n-1)$ for any $t \geq w$, which implies that $\lambda^*(t,n) - \lambda^*(t,n-1) = \tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1)$.

Recall our earlier result that $\tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1) \geq -\frac{|\mathscr{C}_1|}{\alpha} \exp(-L_t(n))$ for any $t \geq \eta$. Therefore there exists some constant $w \geq \eta$, such that for any $t \geq w$, if $\lambda^*(t,n) < \lambda^*(t,n-1)$, $\lambda^*(t,n) - \lambda^*(t,n-1) \geq \tilde{\lambda}(t,n) - \tilde{\lambda}(t,n-1) \geq -\frac{|\mathscr{C}_1|}{\alpha} \exp(-L_t(n))$. Thus,

$$
\begin{aligned}
L_t(n) - L_t(n-1) &= \int_0^t (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s \\
&\geq \int_0^w (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s + \int_w^t [\lambda^*(s,n) - \lambda^*(s,n-1)]^-\, \mathrm{d}s \\
&= \int_0^w (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s + \int_w^t (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathbf{1}\{\lambda^*(s,n) < \lambda^*(s,n-1)\}\, \mathrm{d}s \\
&\geq \int_0^w (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s - \int_w^t \frac{|\mathscr{C}_1|}{\alpha} \exp(-L_s(n)) \mathbf{1}\{\lambda^*(s,n) < \lambda^*(s,n-1)\}\, \mathrm{d}s \\
&\geq \int_0^w (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s - \int_0^t \frac{|\mathscr{C}_1|}{\alpha} e^{-\underline{\lambda} s}\, \mathrm{d}s,
\end{aligned}
$$

where $\mathbf{1}\{A\}$ is an indicator function that equals one if condition A holds and zero otherwise. Because w is a constant independent of t, $\int_0^w (\lambda^*(s,n) - \lambda^*(s,n-1))\, \mathrm{d}s$ is also a constant. Since $\int_0^t \frac{|\mathscr{C}_1|}{\alpha} e^{-\underline{\lambda} s}\, \mathrm{d}s$ is bounded from above, $L_t(n) - L_t(n-1)$ is bounded from below, i.e., we can find a constant $\mathscr{C}$ such that $L_t(n) - L_t(n-1) \geq \mathscr{C}$ for any t.

Our stipulation for $n-1$ says $\Delta_t(n-1) \geq p + \exp(-L_t(n-1)) F^{(n-2)}(t)$ for any $t \geq \eta_{n-1}$. Because the leading coefficient of $F^{(n-2)}(t)$ is positive and $\lim_{t\to\infty} \lambda^*(t,n) = \lambda^*$, we can find some $z \geq \eta_{n-1}$ such that $F^{(n-2)}(t) > 0$ and $\lambda^*(t,n) \geq \frac{\lambda^* + \underline{\lambda}}{2}$ for any $t > z$. Thus, by applying Grönwall's Inequality to Inequality (OA.1) over $[z,t]$, we have

$$
\begin{aligned}
\Delta_t(n) &\geq \Delta_z \exp\left(-\int_z^t \lambda^*(s,n)\, \mathrm{d}s\right) + \int_z^t \exp\left(-\int_s^t \lambda^*(s,n) \mathrm{d}s\right) \Delta_s(n-1)\, \mathrm{d}s \\
&= \Delta_z(n) \exp(L_z(n) - L_t(n)) + \exp(-L_t(n)) \int_z^t \exp(L_s(n))\, \lambda^*(s,n) \Delta_s(n-1)\, \mathrm{d}s \\
&= \exp(-L_t(n)) \left[\Delta_z(n) \exp(L_z(n)) + \int_z^t \exp(L_s(n))\, \lambda^*(s,n) \Delta_s(n-1)\, \mathrm{d}s\right] \\
&\geq \exp(-L_t(n)) \left[\Delta_z(n) \exp(L_z(n)) + \int_z^t \exp(L_s(n)) \left[p + \exp(-L_s(n-1)) F^{(n-2)}(s)\right] \lambda^*(s,n)\, \mathrm{d}s\right]
\end{aligned}
$$

$$= \exp\left(-L_t(n)\right)\left[\Delta_z(n)\exp(L_z(n)) + \int_z^t p\cdot\exp\left(L_s(n)\right)\lambda^*(s,n)\,\mathrm{d}s + \int_z^t \exp\left(L_s(n)-L_s(n-1)\right)F^{(n-2)}(s)\lambda^*(s,n)\,\mathrm{d}s\right]$$

$$= \exp\left(-L_t(n)\right)\left[\Delta_z(n)\exp(L_z(n)) + p\cdot\exp(L_t(n)) - p\cdot\exp(L_z(n)) + \int_z^t \exp\left(L_s(n)-L_s(n-1)\right)F^{(n-2)}(s)\lambda^*(s,n)\,\mathrm{d}s\right]$$

$$= p + \exp\left(-L_t(n)\right)\left[(\Delta_z(n)-p)\exp(L_z(n)) + \int_z^t \exp\left(L_s(n)-L_s(n-1)\right)F^{(n-2)}(s)\lambda^*(s,n)\,\mathrm{d}s\right].$$

Note that for any $s\in[z,t]$, $\lambda^*(s,n)\geq\frac{\lambda^*+\underline{\lambda}}{2}$, $L_s(n)-L_s(n-1)\geq\mathscr{C}$, and $F^{(n-2)}(s)\geq 0$. Therefore

$$\begin{aligned}\Delta_t(n) &\geq p + \exp\left(-L_t(n)\right)\left[(\Delta_z(n)-p)\exp(L_z(n)) + \int_z^t \exp(\mathscr{C})\cdot F^{(n-2)}(s)\cdot\frac{\lambda^*+\underline{\lambda}}{2}\,\mathrm{d}s\right]\\ &= p + \exp\left(-L_t(n)\right)\left[(\Delta_z(n)-p)\exp(L_z(n)) + \exp(\mathscr{C})\cdot\frac{\lambda^*+\underline{\lambda}}{2}\cdot\int_z^t F^{(n-2)}(s)\,\mathrm{d}s\right]\\ &= p + \exp\left(-L_t(n)\right)F^{(n-1)}(t),\end{aligned}$$

where we denote $F^{(n-1)}(t)\equiv(\Delta_z(n)-p)\exp(L_z(n))+\exp(\mathscr{C})\cdot\frac{\lambda^*+\underline{\lambda}}{2}\cdot\int_z^t F^{(n-2)}(s)\,\mathrm{d}s$. Since $F^{(n-2)}(t)$ is a polynomial function of t of order $n-2$ and the leading coefficient is positive, $\int_z^t F^{(n-2)}(s)\,\mathrm{d}s$ is a polynomial function of t of order $n-1$ and its leading coefficient is also positive. Because $(\Delta_z(n)-p)\exp(L_z(n))$ is a constant independent of t, $F^{(n-1)}(t)$ is a polynomial function of t of order $n-1$ and its leading coefficient is also positive. Therefore, the statement is also true for n.

The strict monotonicity of $\tau(n)$ is a direct result of Lemma A1(i), and thus we complete the proof of Theorem 1(ii).

$\lim_{t\to\infty}\lambda^*(t,n)=\lambda^*$, $\forall n\geq 0$ in Theorem 1(iv) is shown in Lemma A1(iii). $\lim_{t\to 0}\lambda^*(t,n)=\underline{\lambda}$, $\forall n>0$ is obvious, and thus we omit the proof here. □

Proof of Proposition 2. First, we solve the following maximization problem

$$\begin{aligned}\tilde{\Pi}_D = \max\nolimits_\Lambda\ \tilde{\pi}_D(\Lambda) &= b + p\left(\Lambda T - N\right) - c(\Lambda)T\\ \text{s.t.}\quad \Lambda T &\geq N.\end{aligned}$$

It is easy to verify that the optimal solution is given by $\Lambda^*=\lambda_D$ and $\tilde{\Pi}_D=\tilde{\pi}_D(\lambda_D)$.

Next we show that $\tilde{\Pi}_D$ is an upper bound for $\pi_D(\boldsymbol{\lambda})$. Based on Jensen's inequality, for any $\boldsymbol{\lambda}$, we have

$$\pi_D(\boldsymbol{\lambda}) = b + p\left(\int_0^T \lambda_t\,\mathrm{d}t - N\right) - \int_0^T c(\lambda_t)\,\mathrm{d}t$$

$$\leq b + p\left(\int_0^T \lambda_t \,\mathrm{d}t - N\right) - T\cdot c\left(\frac{1}{T}\int_0^T \lambda_t \,\mathrm{d}t\right)$$
$$= \tilde{\pi}_D\left(\frac{1}{T}\int_0^T \lambda_t \,\mathrm{d}t\right).$$

Therefore,

$$\Pi_D = \max\left\{\pi_D(\boldsymbol{\lambda}) : \int_0^T \lambda_t \,\mathrm{d}t \geq N\right\} \leq \max\left\{\tilde{\pi}_D\left(\frac{1}{T}\int_0^T \lambda_t \,\mathrm{d}t\right) : \int_0^T \lambda_t \,\mathrm{d}t \geq N\right\} = \tilde{\Pi}_D.$$

Because $\pi_D(\lambda_D) = \tilde{\Pi}_D$, we thus obtain the announced result. □

Proof of Proposition 3. Because $c(\lambda)$ is convex, we have $\int_0^\theta c(\lambda_s)\,\mathrm{d}s \geq \theta \cdot c\left(\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)$ based on Jensen's inequality. Therefore,

$$\Pi_u^{(\theta)} = \mathbb{E}_u\left(\theta b + p\left(\int_0^\theta \mathrm{d}D_s - \theta N\right)\middle|\int_0^\theta \mathrm{d}D_s \geq \theta N\right)\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - \mathbb{E}_u\int_0^\theta c(\lambda_s)\,\mathrm{d}s$$
$$\leq \theta\cdot\mathbb{E}_u\left(b + p\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - N\right) - c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\middle|\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s \geq N\right)\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right)$$
$$+\theta\cdot\left[\mathbb{E}_u\left(c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\middle|\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s \geq N\right)\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - \mathbb{E}_u c\left(\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)\right]$$

Using the definition of $\Pi_D^{(\theta)}$, $\theta\cdot\mathbb{E}_u\left(b + p\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - N\right) - c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\middle|\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s \geq N\right) \leq \Pi_D^{(\theta)}$. Therefore,

$$\Pi_u^{(\theta)} \leq \Pi_D^{(\theta)}\cdot\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right)$$
$$+\theta\cdot\left[\mathbb{E}_u\left(c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\middle|\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s \geq N\right)\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - \mathbb{E}_u c\left(\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)\right]$$
$$< \Pi_D^{(\theta)}\cdot\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) + \theta\cdot\mathbb{E}_u\left[c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right) - c\left(\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)\right].$$

Because $\left(\int_0^\theta \mathrm{d}D_s - \int_0^\theta \lambda_s \,\mathrm{d}s\right)^2 - \int_0^\theta \lambda_s \,\mathrm{d}s$ is also a martingale, $\mathrm{Var}\left\{\int_0^\theta \mathrm{d}D_s - \int_0^\theta \lambda_s \,\mathrm{d}s\right\} = O(\theta)$. Therefore

$$\mathbb{E}_u\left[c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right) - c\left(\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)\right] = \mathbb{E}_u\left[c'(\xi)\cdot\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - \frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right)\right]$$
$$\leq c'(\bar{\lambda})\cdot\mathbb{E}_u\left|\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - \frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s\right| \leq c'(\bar{\lambda})\cdot\frac{1}{\theta}\left[\mathbb{E}_u\left(\int_0^\theta \mathrm{d}D_s - \int_0^\theta \lambda_s \,\mathrm{d}s\right)^2\right]^{1/2} = O(1/\sqrt{\theta}),$$

where ξ is between $\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s$ and $\frac{1}{\theta}\int_0^\theta \lambda_s \,\mathrm{d}s$. Therefore,

$$\Pi_u^{(\theta)} < \Pi_D^{(\theta)}\cdot\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) + O(\sqrt{\theta}).$$

Since $\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) \leq 1$ and $\Pi_D^{(\theta)} = \Theta(\theta)$, $\limsup_{\theta\to\infty}\frac{\Pi_u^{(\theta)}}{\Pi_D^{(\theta)}} \leq 1$.

Now for policies that have $\lim_{\theta\to\infty}\frac{\mathbb{E}_u\int_0^\theta \lambda_s\,\mathrm{d}s}{\theta N}<1$. Using Markov's inequality, we have

$$\mathbb{P}_u\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) \leq \frac{\mathbb{E}_u\int_0^\theta \mathrm{d}D_s}{\theta N} = \frac{\mathbb{E}_u\int_0^\theta \lambda_s\,\mathrm{d}s}{\theta N}.$$

Therefore, $\lim_{\theta\to\infty}\frac{\Pi_u^{(\theta)}}{\Pi_u^{(\theta)}} \leq \lim_{\theta\to\infty}\frac{\mathbb{E}_u\int_0^\theta \lambda_s\,\mathrm{d}s}{\theta N}<1$. □

Proof of Proposition 4. Our proof uses the following result.

LEMMA A2. *Let X be a random variable with Poisson distribution with rate λ. For any $0<x<\lambda$,*

$$\mathbb{P}\left(X\leq \lambda - x\right) \leq \exp\left(-\frac{x^2}{2\lambda}\right).$$

Lemma A2 provides a bound for the tail of a Poisson distribution. We refer interested readers to Canonne (2017) and the remark on page 13 of Pollard (2015) for the proof.

Without loss of generality, we assume that $T=1$. When $\lambda^*>N$, we know that $\Pi_D^{(\theta)} = \theta\left[b+p(\lambda^*-N)-c(\lambda^*)\right]$. For the stochastic problem with the static heuristic, the sales process follows a homogeneous Poisson process with rate $\lambda^*>N$, and thus the total sales $\int_0^\theta dD_s$ has a Poisson distribution with rate $\lambda^*\theta$. Based on Lemma A2, we have

$$\mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) = 1-\mathbb{P}\left(\int_0^\theta \mathrm{d}D_s < \lambda^*\theta - \theta(\lambda^*-N)\right) \geq 1-\exp\left[-\frac{(\lambda^*-N)^2}{2\lambda^*}\theta\right].$$

Therefore,

$$\begin{aligned}
\Pi_{SH}^{(\theta)} &= \mathbb{E}\left(\theta b + p\left(\int_0^\theta \mathrm{d}D_s - \theta N\right)\middle|\int_0^\theta \mathrm{d}D_s\geq\theta N\right)\mathbb{P}\left(\int_0^\theta \mathrm{d}D_s\geq\theta N\right) - \int_0^\theta c(\lambda_s)\,\mathrm{d}s\\
&\geq \mathbb{E}\left(\theta b + p\left(\int_0^\theta \mathrm{d}D_s - \theta N\right)\right)\left(1-\exp\left[-\frac{(\lambda^*-N)^2}{2\lambda^*}\theta\right]\right) - c(\lambda^*)\theta\\
&= \Pi_D^{(\theta)} - \theta[b+p(\lambda^*-N)]\exp\left[-\frac{(\lambda^*-N)^2}{2\lambda^*}\theta\right].
\end{aligned}$$

Because $\Pi_D^{(\theta)}\geq\Pi_{SH}^{(\theta)}$ when θ is sufficiently large, we thus conclude that $\lim_{\theta\to\infty}\left(\Pi_D^{(\theta)}-\Pi_{SH}^{(\theta)}\right)=0$. □

Proof of Proposition 5. Without loss of generality, we assume that $T=1$. When $\lambda^*\leq N$, the optimal profit for the deterministic problem is given by $\Pi_D^{(\theta)}=\theta\left[b-c(N)\right]$. Next consider the stochastic problem. Given that $\lambda_{SH}^{(\theta)}=\lambda_D+f(\theta)$, the total sales $\int_0^\theta dD_s$ follows a Poisson distribution with mean $\theta N+\theta f(\theta)$. It is easy to verify that the static heuristic is not asymptotically optimal when $\lim_{\theta\to\infty}|f(\theta)|>0$. Thus, we focus solely on heuristics such that $\lim_{\theta\to\infty}f(\theta)=0$.

First, consider the case when $\lim_{\theta\to\infty}\sqrt{\theta}f(\theta)<\infty$. Let $Y_\theta=\frac{\int_0^\theta \mathrm{d}D_s-(\theta N+\theta f(\theta))}{\sqrt{\theta N+\theta f(\theta)}}$. Thus, $\int_0^\theta \mathrm{d}D_s\geq\theta N$ is equivalent to $Y_\theta\geq-\frac{\theta f(\theta)}{\sqrt{\theta N+\theta f(\theta)}}$. Because $\lim_{\theta\to\infty}\sqrt{\theta}f(\theta)<\infty$, there exists $-\infty\leq\mathscr{B}<+\infty$, such that $\lim_{\theta\to\infty}\frac{\theta f(\theta)}{\sqrt{\theta N+\theta f(\theta)}}=\mathscr{B}$. Based on Central Limit Theorem, we have

$$\mathbb{P}\left(Y_\theta\geq-\frac{\theta f(\theta)}{\sqrt{\theta N+\theta f(\theta)}}\right)=\Phi\left(\mathscr{B}\right)+o(1),$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution. From the proof of Proposition 3, we know that

$$\Pi_{SH}^{(\theta)} < \Pi_D^{(\theta)} \cdot \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) + O(\sqrt{\theta}) = \Phi(\mathscr{B}) \cdot \Pi_D^{(\theta)} + o(\theta).$$

Since $\Phi(\mathscr{B}) < 1$, $\lim_{\theta\to\infty} \frac{\Pi_{SH}^{(\theta)}}{\Pi_D^{(\theta)}} < 1$.

Next consider the case when $\lim_{\theta\to\infty} \sqrt{\theta} f(\theta) = \infty$. From Lemma A2, we have

$$\begin{aligned}
\mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) &= \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq (\theta N + \theta f(\theta)) - \theta f(\theta)\right) \\
&\geq 1 - \exp\left[-\frac{(\theta f(\theta))^2}{2(\theta N + \theta f(\theta))}\right] \\
&> 1 - \exp\left(-\frac{\theta f^2(\theta)}{2(N+1)}\right) (\because f(\theta) < 1).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{1}{\theta}\Pi_{SH}^{(\theta)} &= \left[b - pN + p\frac{1}{\theta}\mathbb{E}\left(\int_0^\theta \mathrm{d}D_s \middle| \int_0^\theta \mathrm{d}D_s \geq \theta N\right)\right] \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - \frac{1}{\theta}\int_0^\theta c(\lambda_s)\,\mathrm{d}s \\
&\geq \left[b - pN + p\frac{1}{\theta}\mathbb{E}\left(\int_0^\theta \mathrm{d}D_s\right)\right] \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - c(\lambda_D + f(\theta)) \\
&= [b + pf(\theta)] \cdot \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) - c(\lambda_D + f(\theta)) \\
&> [b + pf(\theta)] \cdot \left[1 - \exp\left(-\frac{\theta f^2(\theta)}{2(N+1)}\right)\right] - c(\lambda_D + f(\theta)) \\
&= \frac{1}{\theta}\Pi_D(\theta) - b\exp\left(-\frac{\theta f^2(\theta)}{2(N+1)}\right) - (c'(\lambda_D) - p)f(\theta) - \frac{c''(\lambda_D)}{2}[f(\theta)]^2 + o\left([f(\theta)]^2\right)).
\end{aligned}$$

Since $\theta f^2(\theta) \to \infty$, $\frac{1}{\theta}\Pi_{SH}^{(\theta)} \geq \frac{1}{\theta}\Pi_D^{(\theta)} + o(1)$. Coupling with the result that $\lim_{\theta\to\infty} \frac{1}{\theta}\Pi_D^{(\theta)} \geq \lim_{\theta\to\infty} \frac{1}{\theta}\Pi_{opt}^{(\theta)} \geq \lim_{\theta\to\infty} \frac{1}{\theta}\Pi_{SH}^{(\theta)}$, we conclude that $\lim_{\theta\to\infty} \frac{1}{\theta}\Pi_{SH}^{(\theta)} = \lim_{\theta\to\infty} \frac{1}{\theta}\Pi_D^{(\theta)}$. That is, the static heuristic is asymptotically optimal when $\lim_{\theta\to\infty} \sqrt{\theta} f(\theta) = \infty$. In particular, if $\lambda^* = N$, we have $c'(\lambda_D) = p$. Then for any $\epsilon > 0$, we can let $f(\theta) = \theta^{-0.5+\epsilon/2}$, and the performance loss of the corresponding static heuristic is bounded by $\Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} = O(\theta^\epsilon)$. On the other hand, if $\lambda^* < N$, then for any $\epsilon > 0$, we can let $f(\theta) = \theta^{-0.5+\epsilon}$, and the performance loss of the corresponding static heuristic is bounded by $\Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} = O(\theta^{0.5+\epsilon})$.

Lastly, we show that the performance gap increases at a rate greater than $\sqrt{\theta}$ when $\lambda^* < N$. Notice that

$$\begin{aligned}
\Pi_{SH}^{(\theta)} &\leq \theta \cdot \mathbb{E}\left(b + p\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - N\right) - c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right) \middle| \int_0^\theta \mathrm{d}D_s \geq \theta N\right) \mathbb{P}\left(\int_0^\theta \mathrm{d}D_s \geq \theta N\right) + O(\sqrt{\theta}) \\
&< \theta \cdot \mathbb{E}\left(b + p\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s - N\right) - c\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\right) + O(\sqrt{\theta}) \\
&\leq \theta\left[b + p \cdot \left(\mathbb{E}\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right) - N\right) - c\left(\mathbb{E}\left(\frac{1}{\theta}\int_0^\theta \mathrm{d}D_s\right)\right)\right] + O(\sqrt{\theta})
\end{aligned}$$

$$= \theta\left[b + pf(\theta) - c(\lambda_D + f(\theta))\right] + O(\sqrt{\theta}),$$

where the last inequality is due to Jensen's inequality. With Taylor's expansion, we know that $c(\lambda_D + f(\theta)) = c(\lambda_D) + c'(\lambda_D)f(\theta) + o(f(\theta))$. Thus, we have

$$\Pi_{SH}^{(\theta)} \leq \theta\left(b - c(\lambda_D)\right) + \theta f(\theta)\left[p - c'(\lambda_D)\right] + O(\sqrt{\theta}) = \Pi_D^{(\theta)} + f(\theta)\left[p - c'(\lambda_D)\right] + O(\sqrt{\theta}).$$

Because $\lambda^* < \lambda_D$, we have $p - c'(\lambda_D) < 0$, and thus $\frac{1}{\theta}\left(\Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)}\right) = \Omega(f(\theta))$. Because $\lim_{\theta\to\infty} \sqrt{\theta} f(\theta) = \infty$, we conclude that $\frac{1}{\sqrt{\theta}}\left(\Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)}\right) = \infty$. $\square$

Next, we show two auxiliary results, which will be used in the proofs of Proposition 6 and Theorem 2.

Lemma A3. *Denote $\hat{D}_t$ as the realized demand between time-to-go t and $t-1$, and let $\delta_t = \hat{D}_t - \mathbb{E}\hat{D}_t$. If $\sum_{s=l+1}^{T} \frac{\delta_s}{s-1} < \lambda_D - \lambda^*$ for any $l \geq t$, then*

$$\hat{\lambda}_t = \lambda_D - \sum_{s=t+1}^{T} \frac{\delta_s}{s-1}. \tag{OA.2}$$

Proof of Lemma A3. We prove the lemma by induction. At $t = T$, we have $\hat{\lambda}_T = \lambda_D = N/T$ and $\hat{n}_T = N$. Because $\hat{D}_T$ is a Poisson random variable with mean $\hat{\lambda}_T$, $\mathbb{E}\hat{D}_T = N/T$. Thus, we can update the threshold as follows

$$\hat{n}_{T-1} = N - \hat{D}_T = N - \delta_T - \mathbb{E}\hat{D}_T = \frac{T-1}{T}N - \delta_T.$$

If $\frac{\delta_T}{T-1} < \lambda_D - \lambda^* = N/T - \lambda^*$, then we have $\hat{n}_{T-1} > (T-1)\lambda^*$. Thus,

$$\hat{\lambda}_{T-1} = \max\left\{\lambda^*, \frac{\hat{n}_{T-1}}{T-1}\right\} = \frac{\hat{n}_{T-1}}{T-1} = \frac{\frac{T-1}{T}N - \delta_T}{T-1} = \lambda_D - \frac{\delta_T}{T-1}.$$

Now suppose that Equation (OA.2) holds for t. That is, $\hat{\lambda}_t = \lambda_D - \sum_{s=t+1}^{T} \frac{\delta_s}{s-1}$, which implies that $\hat{\lambda}_t > \lambda^*$ and $\hat{n}_t = \hat{\lambda}_t t$. Because $\hat{D}_t$ is a Poisson random variable with mean $\hat{\lambda}_t$, we have $\mathbb{E}\hat{D}_t = \hat{\lambda}_t$. Thus, the updated threshold is given by

$$\hat{n}_{t-1} = \hat{n}_t - \hat{D}_t = \hat{n}_t - \delta_t - \mathbb{E}\hat{D}_t = \hat{\lambda}_t(t-1) - \delta_t.$$

Therefore,

$$\frac{n_{t-1}}{t-1} = \frac{\hat{\lambda}_t(t-1) - \delta_t}{t-1} = \hat{\lambda}_t - \frac{\delta_t}{t-1} = \lambda_D - \sum_{s=t}^{T} \frac{\delta_s}{s-1}.$$

When $\sum_{s=t}^{T} \frac{\delta_s}{s-1} < \lambda_D - \lambda^*$, we have $\frac{n_{t-1}}{t-1} > \lambda^*$. Thus, $\hat{\lambda}_{t-1} = \max\left\{\lambda^*, \frac{\hat{n}_{t-1}}{t-1}\right\} = \frac{\hat{n}_{t-1}}{t-1} = \lambda_D - \sum_{s=t}^{T} \frac{\delta_s}{s-1}$, and we obtain the announced result. $\square$

Lemma A4. *For any* $0 < x < \min\left\{\bar{\lambda} - \lambda_D, \lambda_D - \lambda^*\right\}$, *let* $\tau(x)$ *be the first time-to-go such that* $|\hat{\lambda}_t - \lambda_D| \geq x$. *There exists* $\Psi(x) > 0$ *(independent of* t*), such that, for any* $1 \leq t \leq T - 2$,

$$\mathbb{P}\left(\tau(x) > t\right) < \frac{\Psi(x)}{t}.$$

Proof of Lemma A4. Based on Lemma A3, we know that

$$\mathbb{P}\left(\tau(x) > t\right) = \mathbb{P}\left(\max_{t+1 \leq l \leq T-1} \left|\sum_{s=l+1}^{T} \frac{\delta_s}{s-1}\right| \geq x\right).$$

Notice that $\sum_{s=t}^{T} \frac{\delta_s}{s-1}$ is a backwards martingale w.r.t. filtration $\{\mathscr{H}_t\}$, where $\mathscr{H}_t$ is the observed history up to time-to-go t. Based on Doob's maximal inequality, we thus have

$$\mathbb{P}\left(\tau(x) > t\right) = \mathbb{P}\left(\max_{t+1 \leq l \leq T-1} \left|\sum_{s=l+1}^{T} \frac{\delta_s}{s-1}\right| \geq x\right) \leq \frac{1}{x^2}\mathbb{E}\left(\sum_{s=t+2}^{T} \frac{\delta_s}{s-1}\right)^2.$$

For any $s < t$, we know that $\mathbb{E}[\delta_s \delta_t] = \mathbb{E}\left[\delta_t \mathbb{E}\left(\delta_s | \delta_t\right)\right] = 0$. Therefore,

$$\mathbb{E}\left(\sum_{s=t+2}^{T} \frac{\delta_s}{s-1}\right)^2 = \sum_{s=t+2}^{T} \frac{\mathbb{E}\delta_s^2}{(s-1)^2} = \sum_{s=t+2}^{T} \frac{\mathrm{Var}(\hat{D}_s)}{(s-1)^2} < \sum_{s=t+2}^{T} \frac{\bar{\lambda}}{(s-1)^2} < \sum_{s=t+2}^{T} \frac{\bar{\lambda}}{(s-1)(s-2)} < \frac{\bar{\lambda}}{t}.$$

Let $\Psi(x) = \frac{\bar{\lambda}}{x^2}$. We can then conclude that $\mathbb{P}\left(\tau(x) > t\right) < \frac{\Psi(x)}{t}$. $\square$

Proof of Proposition 6. Without loss of generality, we assume that $T = 1$. Thus,

$$\begin{aligned}
\Pi_{RH}^{(\theta)} &= \left[\theta b + p \cdot \mathbb{E}\left(\sum_{t=1}^{\theta} \hat{D}_t - \theta N \middle| \sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)\right] \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) - \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) \\
&= \theta\left(b - pN\right)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) + p\mathbb{E}\left(\sum_{t=1}^{\theta} \hat{D}_t \middle| \sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) - \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) \\
&< \theta\left(b - pN\right) - \theta\left(b - pN\right)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t < \theta N\right) + p\mathbb{E}\left(\sum_{t=1}^{\theta} \hat{D}_t\right) - \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t).
\end{aligned}$$

Using Jensen's inequality, we have

$$\mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) \geq \theta \cdot c\left(\frac{1}{\theta}\mathbb{E}\sum_{t=1}^{\theta} \hat{\lambda}_t\right) = \theta \cdot c\left(\frac{1}{\theta}\mathbb{E}\sum_{t=1}^{\theta} \hat{D}_t\right).$$

Therefore,

$$\Pi_{RH}^{(\theta)} < \theta(b - pN) + p\mathbb{E}\sum_{t=1}^{\theta} \hat{D}_t - \theta \cdot c\left(\frac{1}{\theta}\mathbb{E}\sum_{t=1}^{\theta} \hat{D}_t\right) - \theta(b - pN)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t < \theta N\right).$$

Notice that $\mathbb{E}\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N$ due to $\hat{\lambda}_t = \max\left\{\lambda^*, \frac{n_t}{t}\right\}$. Coupling with the result that $\lambda p - c(\lambda)$ decreases in λ for any $\lambda \geq N$ (this is due to $\lambda^* < N$), we have

$$\Pi_{RH}^{(\theta)} < \theta(b - pN) + \theta pN - \theta c(N) - \theta(b - pN)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) = \Pi_{D}^{(\theta)} - \theta(b - pN)\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t < \theta N\right).$$

So to complete the proof, we only need to show that $\lim_{\theta\to\infty} \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t < \theta N\right) > 0$. Note that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{s=t}^{\theta} \hat{D}_t < \theta N\right) = 1 - \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) > 1 - \mathbb{P}\left(\hat{\lambda}_t = \lambda^*\right) = \mathbb{P}\left(\hat{\lambda}_t > \lambda^*\right) \\
&> \mathbb{P}\left(\forall s \geq t : \left|\hat{\lambda}_s - \lambda_D\right| \leq x\right) = \mathbb{P}\left(\tau(x) < t\right) = 1 - \mathbb{P}\left(\tau(x) \geq t\right),
\end{aligned}$$

where $0 < x \leq \lambda_D - \lambda^*$ and $\tau(x)$ is the first tim-to-go that $|\hat{\lambda}_t - \lambda_D| \geq x$ as defined in Lemma A4. Let $x = \frac{\lambda_D - \lambda^*}{2}$ and $t = 2\Psi(x) = \frac{8\bar{\lambda}}{(\lambda_D - \lambda^*)^2}$. Based on Lemma A4, we have $\mathbb{P}[\tau(x) > 2\Psi(x)] < \frac{1}{2}$. That is, when $t = \frac{8\bar{\lambda}}{(\lambda_D - \lambda^*)^2}$, the probability that $\hat{\lambda}_t \geq \frac{\lambda^* + \lambda_D}{2}$ is greater than 1/2, which also implies that the probability of the updated threshold $\hat{n}_{2\Psi(x)}$ being greater than zero is greater than 1/2. Because $2\Psi(x) = \frac{8\bar{\lambda}}{(\lambda_D - \lambda^*)^2}$ is finite and does not depend on θ, we can conclude that the probability the threshold being reached when time expires must be strictly less than 1 in the limit. $\square$

Proof of Theorem 2. Without loss of generality, we assume that $T = 1$. Let $x = min\left\{\frac{\bar{\lambda} - \lambda_D}{2}, \lambda_D - \lambda^*\right\}$. From Lemma A4, we have

$$\mathbb{E}\tau^{(\theta)}(x) = \sum_{t=1}^{\theta-1} \mathbb{P}\left(\tau^{(\theta)}(x) \geq t\right) < 1 + \sum_{t=1}^{\theta-2} \mathbb{P}\left(\tau^{(\theta)}(x) > t\right) < 1 + \Psi(x) \sum_{t=1}^{\theta-2} \frac{1}{t} = O(\log\theta).$$

Therefore, there exists an $M > 0$, which is independent of θ, such that $\mathbb{E}\tau^{\theta}(x) \leq M\log\theta$. Denote $\hat{\tau}^{\theta} = \max\{\tau^{\theta}(x), M\log\theta\}$. Thus,

$$\mathbb{E}\hat{\tau}^{\theta} = \mathbb{E}\max\left\{\tau^{\theta}(x), M\log\theta\right\} \leq \mathbb{E}\left(\tau^{\theta}(x) + M\log\theta\right) \leq 2M\log\theta,$$

which implies that $M\log\theta \leq \mathbb{E}\hat{\tau}^{\theta} \leq 2M\log\theta$.

The expected profit of the modified resolving heuristic is given by

$$\begin{aligned}
\Pi_{MRH}^{(\theta)} &= \left[\theta(b - pN) + p \cdot \mathbb{E}\left(\sum_{t=1}^{\theta} \hat{D}_t \middle| \sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)\right] \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) - \mathbb{E}\left(\sum_{t=1}^{\theta} c(\hat{\lambda}_t)\right) \\
&\geq \left[\theta(b - pN) + p \cdot \mathbb{E}\left(\sum_{t=\hat{\tau}^{\theta}+1}^{\theta} \hat{D}_t\right)\right] \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) - \mathbb{E}\left(\sum_{t=\hat{\tau}^{\theta}+1}^{\theta} c(\hat{\lambda}_t)\right) - O(\log\theta) \\
&= \left[\theta(b - pN) + \mathbb{E}\left(\sum_{t=\hat{\tau}^{\theta}+1}^{\theta} (p\hat{D}_t - c(\hat{\lambda}_t))\right)\right] \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right) - \\
&\quad \left(1 - \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)\right) \mathbb{E}\left(\sum_{t=\hat{\tau}^{\theta}+1}^{\theta} c(\hat{\lambda}_t)\right) - O(\log\theta).
\end{aligned}$$

Next, we provide bounds for the two terms $\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)$ and $\mathbb{E}\left(\sum_{t=\hat{\tau}^{\theta}+1}^{\theta} (p\hat{D}_t - c(\hat{\lambda}_t))\right)$. First, consider $\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N\right)$. Recall that $\hat{\tau}^{\theta}$ is the first time such that $\left|\hat{\lambda}_t - \lambda_D\right| \geq$

$\min\left\{\frac{1}{2}\left(\bar{\lambda}-\lambda_D\right),\lambda_D-\lambda^*\right\}$. Thus, $\hat{\lambda}_t=\frac{\hat{n}_t}{t}\geq\frac{1}{2}\left(\bar{\lambda}+\lambda_D\right)$ for any $t\geq\hat{\tau}^\theta+1$. Coupling with the result that $\hat{n}_t=\theta N-\sum_{s=t+1}^{\theta}\hat{D}_s$, we have $\theta N-\sum_{t=\hat{\tau}^\theta+2}^{\theta}\hat{D}_t\leq\frac{\bar{\lambda}+\lambda_D}{2}(\hat{\tau}^\theta+1)$. Thus,

$$
\begin{aligned}
\mathbb{P}\left(\sum_{t=1}^{\theta}\hat{D}_t\geq\theta N\right)&=\mathbb{P}\left(\sum_{t=1}^{\hat{\tau}^\theta+1}\hat{D}_t\geq\theta N-\sum_{t=\hat{\tau}^\theta+2}^{\theta}\hat{D}_t\right)\\
&\geq\mathbb{P}\left(\sum_{t=1}^{\hat{\tau}^\theta}\hat{D}_t\geq\frac{\bar{\lambda}+\lambda_D}{2}(\hat{\tau}^\theta+1)\right)\\
&=\mathbb{E}\left[\mathbb{P}\left(\sum_{t=1}^{\hat{\tau}^\theta}\hat{D}_t\geq\frac{\bar{\lambda}+\lambda_D}{2}(\hat{\tau}^\theta+1)\middle|\hat{\tau}^\theta\right)\right].
\end{aligned}
$$

Conditional on $\hat{\tau}^\theta$, $\sum_{t=1}^{\hat{\tau}^\theta}\hat{D}_t$ follows a Poisson distribution with mean $\bar{\lambda}\tau^\theta$. Thus, based on Lemma A2, we have

$$
\begin{aligned}
\mathbb{P}\left(\sum_{t=1}^{\theta}\hat{D}_t\geq\theta N\right)&\geq1-\mathbb{E}\left[\mathbb{P}\left(\sum_{t=1}^{\hat{\tau}^\theta}\hat{D}_t<\bar{\lambda}\hat{\tau}^\theta-\frac{\bar{\lambda}-\lambda_D}{2}\tau^\theta+\frac{\bar{\lambda}+\lambda_D}{2}\middle|\hat{\tau}^\theta\right)\right]\\
&\geq1-\mathbb{E}\left[\exp\left(-\frac{\left(\frac{\bar{\lambda}-\lambda_D}{2}\hat{\tau}^\theta-\frac{\bar{\lambda}+\lambda_D}{2}\right)^2}{2\bar{\lambda}\hat{\tau}^\theta}\right)\right]\\
&=1-\mathbb{E}\left[\exp\left(\frac{\bar{\lambda}^2-\lambda_D^2}{4\bar{\lambda}}\right)\cdot\exp\left(-\frac{(\bar{\lambda}-\lambda_D)^2}{8\bar{\lambda}}\hat{\tau}^\theta\right)\cdot\exp\left(-\frac{(\bar{\lambda}+\lambda_D)^2}{8\bar{\lambda}\hat{\tau}^\theta}\right)\right]\\
&>1-\exp\left(\frac{\bar{\lambda}^2-\lambda_D^2}{4\bar{\lambda}}\right)\cdot\mathbb{E}\left[\exp\left(-\frac{(\bar{\lambda}-\lambda_D)^2}{8\bar{\lambda}}\hat{\tau}^\theta\right)\right]\\
&\geq1-\exp\left(\frac{\bar{\lambda}^2-\lambda_D^2}{4\bar{\lambda}}\right)\cdot\mathbb{E}\left[\exp\left(-\frac{(\bar{\lambda}-\lambda^D)^2}{8\bar{\lambda}}M\log\theta\right)\right].
\end{aligned}
$$

The last inequality is due to $\hat{\tau}^\theta\geq M\log\theta$. Let $M\geq\frac{8\bar{\lambda}}{(\bar{\lambda}-\lambda_D)^2}$, and thus we can conclude that there exists a Γ such that $\mathbb{P}\left(\sum_{t=1}^{\theta}\hat{D}_t\geq\theta N\right)\geq1-\frac{\Gamma}{\theta}$.

Next, consider $\mathbb{E}\left(\sum_{t=\hat{\tau}^\theta+1}^{\theta}(p\hat{D}_t-c(\hat{\lambda}_t))\right)$. Denote $\epsilon_t=\sum_{s=t}^{\theta}\frac{\delta_s}{s-1}$, where $\delta_t=\hat{D}_t-\mathbb{E}\hat{D}_t$. Based on Taylor's expansion, we have

$$
\begin{aligned}
\sum_{t=\hat{\tau}^\theta+1}^{\theta}(p\hat{D}_t-c(\hat{\lambda}_t))&=\sum_{t=\hat{\tau}^\theta+1}^{\theta}\left[p\left(\lambda_D-\epsilon_{t+1}+\delta_t\right)-c(\lambda_D)+c'(\lambda_D)\epsilon_{t+1}-\frac{1}{2}c''(z_t)\epsilon_{t+1}^2\right]\\
&=\sum_{t=\hat{\tau}^\theta+1}^{\theta}[p\lambda_D-c(\lambda_D)]-\sum_{t=\hat{\tau}^\theta+1}^{\theta}(p-c'(\lambda_D))\epsilon_{t+1}-\sum_{t=\hat{\tau}^\theta+1}^{\theta}\frac{1}{2}c''(z_t)\epsilon_{t+1}^2+p\sum_{t=\hat{\tau}^\theta+1}^{\theta}\delta_t.
\end{aligned}
$$

The existence of $z_t\in[\hat{\lambda}_t,\lambda_D]$ is guaranteed by mean value theorem. Note that $\sum_{s=t}^{\theta}\delta_s$ and $\sum_{s=t}^{\theta}\epsilon_s$ are backwards martingales. Because $\mathbb{E}\hat{\tau}^\theta > 0$, we have $\mathbb{E}\left(\sum_{t=\hat{\tau}^\theta+1}^{\theta} \delta_t\right) = \mathbb{E}\left(\sum_{t=\hat{\tau}^\theta+1}^{\theta} \epsilon_{t+1}\right) = 0$ based on the optimal stopping time theorem. Moreover,

$$
\begin{aligned}
\mathbb{E}\sum_{t=\hat{\tau}^\theta}^{\theta} c''(z_t)\epsilon_t^2 &= \mathbb{E}\sum_{t=1}^{\hat{\tau}^\theta}\sum_{1\le s,v\le t} c''(z_t)\frac{\delta_s\delta_v}{(\theta-s)(\theta-v)} \\
&= \mathbb{E}\sum_{t=1}^{\hat{\tau}^\theta}\sum_{s=1}^{t} c''(z_t)\frac{\delta_s^2}{(\theta-s)^2} \\
&\le \mathbb{E}\sum_{t=1}^{\theta}\sum_{s=1}^{t} c''(z_t)\frac{\delta_s^2}{(\theta-s)^2} \\
&= O(\log\theta).
\end{aligned}
$$

Therefore, we have

$$
\mathbb{E}\left[\sum_{t=\hat{\tau}^\theta+1}^{\theta}\left(p\hat{D}_t - c(\hat{\lambda}_t)\right)\right] = \mathbb{E}\left[\sum_{t=\hat{\tau}^\theta+1}^{\theta}[p\lambda_D - c(\lambda_D)]\right] - O(\log\theta) = [p\lambda_D - c(\lambda_D)]\left(\theta - \mathbb{E}\hat{\tau}^\theta\right) - O(\log\theta).
$$

Recall that $\mathbb{E}\hat{\tau}^\theta \le 2M\log\theta$ and $p\lambda_D - c(\lambda_D) = pN - c(N)$. Thus,

$$
\mathbb{E}\left[\sum_{t=\hat{\tau}^\theta+1}^{\theta}\left(p\hat{D}_t - c(\hat{\lambda}_t)\right)\right] = [pN - c(N)]\,\theta - O(\log\theta).
$$

Finally, we assemble all pieces together, and obtain

$$
\begin{aligned}
\Pi_{MRH}^{(\theta)} \ge& \left[\theta(b-pN) + \mathbb{E}\left(\sum_{t=\hat{\tau}^\theta+1}^{\theta}(p\hat{D}_t - c(\hat{\lambda}_t))\right)\right]\mathbb{P}\left(\sum_{t=1}^{\theta}\hat{D}_t \ge \theta N\right) - \\
&\left(1 - \mathbb{P}\left(\sum_{t=1}^{\theta}\hat{D}_t \ge \theta N\right)\right)\mathbb{E}\sum_{t=\hat{\tau}^\theta+1}^{\theta} c(\hat{\lambda}_t) - O(\log\theta) \\
\ge& \,[\theta(b-pN) + (pN - c(N))\,\theta - O(\log\theta)]\cdot\left(1 - \frac{\Gamma}{\theta}\right) - \frac{\Gamma}{\theta}\cdot O(\theta) - O(\log\theta) \\
=& \,\theta b - c(N)\theta - O(\log\theta) \\
=& \,\Pi_D^{(\theta)} - O(\log\theta).
\end{aligned}
$$

Thus, we obtain the announced results. □

References

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