Online Appendix to
"Revenue Volatility under Uncertain Network Effects"

A. Background Materials.

THEOREM OA.1 (POTENTIAL GAME; MONDERER AND SHAPLEY 1996). *Let Γ be a game in which the strategy sets are intervals of real numbers. Suppose the payoff functions are twice continuously differentiable. Then Γ is a potential game if and only if*

$$\frac{\partial^2 u_i}{\partial y_i \partial y_j} = \frac{\partial^2 u_j}{\partial y_i \partial y_j} \quad \text{for all } i,j \in N. \tag{OA.1}$$

Moreover, if the payoff functions satisfy (OA.1) and z is an arbitrary (but fixed) strategy profile in Y, then a potential for Γ is given by $P(y) = \sum_{i\in N}\int_0^1 \frac{\partial u_i}{\partial y_i}(x(t))(x^i)'(t)dt$, where $x:[0,1]\to Y$ is a piece-wise continuously differentiable path in Y that connects z to y (i.e., $x(0)=z$ and $x(1)=y$).

The spectral radius is closely related to the behavior of the convergence of the power sequence of a matrix; namely, we have the following.

THEOREM OA.2 (HORN AND JOHNSON 1990, THEOREM 5.6.12 AND COROLLARY 5.6.16). *Let A be an $n\times n$ matrix with spectral radius $\rho(A)$; then $\rho(A)<1$ if and only if $\lim_{k\to\infty} A^k = 0$. Furthermore, the Taylor series for $(I-A)^{-1}$ is convergent, i.e., $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$. Finally, if $\rho(A)>1$, $\left\|A^k\right\|_2$ is not bounded for increasing values of k.*

THEOREM OA.3 (BAI AND YIN 1988). *Let W_n be an $n\times n$ matrix. Suppose W_n is symmetric, entries on the diagonal are i.i.d., entries on the off-diagonal are i.i.d., and the diagonal and off-diagonal entries are also independent. Then, the necessary and sufficient conditions for $\lambda_{\max}\left(\frac{1}{\sqrt{n}}W_n\right)\to a$ a.s. are (1) $\mathbb{E}[[W_n]_{11}^{+}]\le\infty$, (2) $\mathbb{E}[[W_n]_{12}^{4}]<\infty$, (3) $\mathbb{E}[[W_n]_{12}^{2}]=\sigma^2$, $a=2\sigma$, (4) $\mathbb{E}[[W_n]_{12}]\le 0$. Here, $\lambda_{\max}\left(\frac{1}{\sqrt{n}}W_n\right)$ represents the maximum eigenvalue of $\frac{1}{\sqrt{n}}W_n$, and $[W_n]_{11}^{+}$ is 0 if $[W_n]_{11}<0$ and is $[W_n]_{11}$ otherwise.*

B. Other Proofs.

Proof of Proposition 1. This proposition directly follows from the results in Candogan et al. (2012). However, for the sake of completeness, we included a brief proof. For a given price vector $\mathbf{p}$, let

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} \left(-\frac{1}{2} x_i^2 + b_i x_i - p_i x_i \right) + \frac{1}{2} \sum_i \alpha \sum_{j \in \mathcal{V} \setminus \{i\}} W_{ij} x_i x_j .$$

It is easy to verify that $\Phi(\cdot)$ is the potential function of the second-stage game, and the consumption game $(W, \alpha, \mathbf{b}, \mathbf{p})$ is an exact potential game. An exact potential game has an essentially unique equilibrium, and myopic best responses converge to the equilibrium (see Monderer and Shapley 1996).

We next characterize the consumption equilibrium when $\mathbf{b} \geq \mathbf{p}$. The utility of each agent in her decision is concave and is characterized as $\mathbf{x} = \alpha W \mathbf{x} + (\mathbf{b} - \mathbf{p})$, where $\mathbf{x}$ represents the consumption vector in the equilibrium. By Assumption 1, $(I - \alpha W)$ is non-singular, and we have

$$\rho(\alpha W) \leq \alpha \|W\|_\infty = \alpha \max_i \sum_{j=1}^{n} W_{ij} < 1.$$

Therefore, by Theorem OA.2, the Taylor series of $(I - \alpha W)^{-1}$ is convergent, which implies that $(I - \alpha W)^{-1}$ is non-negative. Assuming $\mathbf{b} - \mathbf{p} \geq 0$ implies that $0 \leq (I - \alpha W)^{-1} (\mathbf{b} - \mathbf{p}) = \mathbf{x}$, completing the proof of the first part.

Using the result from the first part, for a price vector $\mathbf{p}$, if Assumption 1 holds, the equilibrium consumption is $(I - \alpha W)^{-1} (\mathbf{b} - \mathbf{p})$ with revenue $(\mathbf{p} - c\mathbf{1})^\top (I - \alpha W)^{-1} (\mathbf{b} - \mathbf{p})$. To maximize the revenue, the seller chooses price vector $\mathbf{p} = \frac{1}{2}(\mathbf{b} + c\mathbf{1})$. Finally, according to Candogan et al. (2012, Lemma 7), the consumption levels in equilibrium are interior under Assumption 2 (i.e., $x_i > 0$ for all $i \in \mathcal{V}$), completing the proof. $\square$

Proof of Theorem 1. We break the proof into three parts.

Part 1, existence of an interior consumption equilibrium: We show that if (1) holds, then for all $\tilde{W} \in \mathcal{W}(W, \epsilon)$, $(I - \alpha \tilde{W})$ is non-singular and $(I - \alpha \tilde{W})^{-1}$ is non-negative. We first show that the spectral radius of $\alpha\tilde{W}$ is less than or equal to 1, i.e., $\rho(\alpha\tilde{W}) \leq 1$. Since $\tilde{W}$ and W are symmetric, we have $\rho(\tilde{W}) = \left\|\tilde{W}\right\|_2$ and $\rho(W) = \|W\|_2$ (the norm 2, $\|\cdot\|_2$, and spectral radius, $\rho(\cdot)$, of any symmetric matrix are equal). Therefore,

$$\begin{aligned}\rho(\alpha\tilde{W}) &= \left\|\alpha W + \alpha(\tilde{W} - W)\right\|_2 \leq \alpha\|W\|_2 + \alpha\left\|\tilde{W} - W\right\|_2 \leq \alpha\|W\|_2 + \alpha\epsilon\|W\|_2 \\ &= \|W\|_2\,\alpha(1+\epsilon) < \lambda_{\max}(W)\alpha(1 + \frac{1}{\alpha\lambda_{\max}(W)} - 1) = 1,\end{aligned}$$

where the second inequality follows from Definition 2 as $\tilde{W} \in \mathcal{W}(W,\epsilon)$ and the strict inequality follows from (1). As presented in the appendix (Theorem OA.2), if $\rho(M) < 1$, then $I - M$ is non-singular and the Taylor series for $(I-M)^{-1}$ is convergent. Therefore, $\rho(\alpha\tilde{W}) < 1$ implies that $(I - \alpha\tilde{W})^{-1}$ exists and is non-negative, which together with Proposition 1 completes the proof of the first part.

Part 2, lower bound on revenue volatility: We show that for a given W, there exists $\tilde{W} \in \mathcal{W}(W,\epsilon)$ and $\mathbf{b}$ for which $\left|R(W,\alpha,\mathbf{b}) - R\left(\tilde{W},\alpha,\mathbf{b}\right)\right| / R(W,\alpha,\mathbf{b})$ is equal to the right-hand side of Eq.(2). Recall that $\mathbf{v}_{\max}(W)$ denotes the eigenvector corresponding to the largest eigenvalue of W, $\lambda_{\max}(W)$. We let $\tilde{W} = (1+\epsilon)W$ (note that $\tilde{W} \in \mathcal{W}(W,\epsilon)$) and $\mathbf{b} = c\mathbf{1} + \mathbf{v}_{\max}(W)$. Note that $\mathbf{v}_{\max}(W)$ is also an eigenvector of $(I - \alpha W)$, $(I - \alpha\tilde{W})$ and their inverses, with the corresponding eigenvalues $(1 - \alpha\lambda_{\max}(W))$, $(1 - \alpha(1+\epsilon)\lambda_{\max}(W))$ and their reciprocals.

Applying Proposition 1 for $\mathbf{b} = c\mathbf{1} + \mathbf{v}_{\max}(W)$ leads to

$$\begin{aligned}\frac{|R(W,\alpha,\mathbf{b}) - R(\tilde{W},\alpha,\mathbf{b})|}{R(W,\alpha,\mathbf{b})} &= \frac{\left|(\mathbf{b}-c\mathbf{1})^\top(I-\alpha W)^{-1}(\mathbf{b}-c\mathbf{1}) - (\mathbf{b}-c\mathbf{1})^\top(I-\alpha\tilde{W})^{-1}(\mathbf{b}-c\mathbf{1})\right|}{(\mathbf{b}-c\mathbf{1})^\top(I-\alpha W)^{-1}(\mathbf{b}-c\mathbf{1})} \\ &= \frac{\left|\mathbf{v}_{\max}^\top(W)(I-\alpha W)^{-1}\mathbf{v}_{\max}(W) - \mathbf{v}_{\max}^\top(W)(I-\alpha\tilde{W})^{-1}\mathbf{v}_{\max}(W)\right|}{\mathbf{v}_{\max}^\top(W)(I-\alpha W)^{-1}\mathbf{v}_{\max}(W)} \\ &= \frac{\left|\frac{1}{1-\alpha\lambda_{\max}(W)} - \frac{1}{1-\alpha\lambda_{\max}(W)(1+\epsilon)}\right|}{\frac{1}{1-\alpha\lambda_{\max}(W)}} = \frac{\alpha\lambda_{\max}(W)\epsilon}{1-\alpha(1+\epsilon)\lambda_{\max}(W)}.\end{aligned}$$

Part 3, upper bound on revenue volatility: We prove that for any $\tilde{W} \in \mathcal{W}(W,\epsilon)$, the revenue volatility cannot be larger than the right-hand side of (2). This establishes that $\mathrm{RV}(W,\epsilon) =$

$\max_{\tilde{W}\in\mathcal{W}(W,\epsilon)} \mathrm{RV}(W,\tilde{W})$ is also upper bounded by the right-hand side of (2). For ease of notation, we define $\mathbf{f}=\mathbf{b}-c\mathbf{1}$. Therefore, there is a one-to-one mapping between $\mathbf{f}$ and $\mathbf{b}$, which leads to

$$\mathrm{RV}(W,\tilde{W}) = \max_{\mathbf{f}} \left| \frac{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f} - \mathbf{f}^\top (I-\alpha \tilde{W})^{-1}\mathbf{f}}{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}} \right|. \tag{OA.2}$$

We break the rest of the proof of part 3 into four steps.

Step 1: We first show that without loss of generality, we can assume $\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}=1$, i.e.,

$$\begin{aligned} \mathrm{RV}(W,\tilde{W}) = \max_{\mathbf{f}} & \left| \mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f} - \mathbf{f}^\top (I-\alpha \tilde{W})^{-1}\mathbf{f} \right| \\ \text{s.t. } & \mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}=1. \end{aligned} \tag{OA.3}$$

Note that for any $\mathbf{f}$ such that $\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}=1$, the objective of the optimization problem (OA.2) becomes equal to the objective of the optimization problem (OA.3). As (OA.3) is a more constrained problem, its right-hand side is smaller than (or equal to) the right-hand side of (OA.2). We next show the other direction of this inequality. Suppose $\mathbf{f}$ achieves the maximum of (OA.2). We show that plugging $\mathbf{f}' = \frac{1}{\left(\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}\right)^{1/2}}\mathbf{f}$ into the objective of (OA.3) achieves the same value as the right-hand side of (OA.2) (using Assumptions 1 and 2, $\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}\geq 0$, and therefore, $\mathbf{f}'$ is in $\mathbb{R}^n$). First, note that

$$\mathbf{f}'^\top (I-\alpha W)^{-1}\mathbf{f}' = \frac{\mathbf{f}^\top}{\left(\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}\right)^{1/2}} (I-\alpha W)^{-1} \frac{\mathbf{f}}{\left(\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}\right)^{1/2}} = 1.$$

Second, by plugging $\mathbf{f}'$ into the right-hand side of (OA.3), we obtain

$$\left| \frac{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}}{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}} - \frac{\mathbf{f}^\top (I-\alpha \tilde{W})^{-1}\mathbf{f}}{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}} \right| = \left| \frac{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f} - \mathbf{f}^\top (I-\alpha \tilde{W})^{-1}\mathbf{f}}{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}} \right|.$$

Step 2: For any $\tilde{W}\in\mathcal{W}(W,\epsilon)$ and $\mathbf{f}$ such that $\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}=1$, we can rewrite

$$\begin{aligned} \left| \frac{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f} - \mathbf{f}^\top (I-\alpha \tilde{W})^{-1}\mathbf{f}}{\mathbf{f}^\top (I-\alpha W)^{-1}\mathbf{f}} \right| &= \left| \mathbf{f}^\top \left((I-\alpha W)^{-1} - \left(I-\alpha\tilde{W})^{-1}\right) \right) \mathbf{f} \right| \\ &= \left| \mathbf{f}^\top (I-\alpha W)^{-1} \left(I - (I-\alpha W)\left(I-\alpha\tilde{W}\right)^{-1} \right) \mathbf{f} \right| \\ &= \left| \mathbf{f}^\top L^{-1} M \mathbf{f} \right|, \end{aligned}$$

where L denotes $I-\alpha W$ and M denotes $I-(I-\alpha W)\left(I-\alpha\tilde{W}\right)^{-1}$. Note that since L is symmetric, the matrix L^{-1} is also symmetric. Moreover,

$$L^{-1}M=(I-\alpha W)^{-1}\left(I-(I-\alpha W)\left(I-\alpha\tilde{W}\right)^{-1}\right)=(I-\alpha W)^{-1}-(I-\alpha\tilde{W})^{-1},$$

which is the difference of two symmetric matrices and hence is symmetric.

Step 3: In this step, we show a lemma implying that the maximum of $|\mathbf{f}^\top L^{-1}M\mathbf{f}|$ is either zero or the absolute value of one of the eigenvalues of M.

Lemma OA.1. *Given $L=I-\alpha W$ and $M=I-(I-\alpha W)\left(I-\alpha\tilde{W}\right)^{-1}$, if $\mathbf{f}$ is the argument of the maximum (or, similarly, argument of the minimum) of $\mathbf{f}^\top L^{-1}M\mathbf{f}$ over all vectors $\mathbf{f}$ such that $\mathbf{f}^\top L^{-1}\mathbf{f}=1$, then $M\mathbf{f}=\mu\mathbf{f}$ for some $\mu\in\mathbb{R}$.*

Proof of Lemma OA.1. Throughout this proof, we use the fact that L^{-1} and $L^{-1}M$ are symmetric, which we just established in the proof of step 2 in part 3 of the proof of Theorem 1. We show the lemma for maximization and note that a similar proof holds for minimization. Suppose f is the solution of the following optimization problem:

$$\begin{aligned}\max_{\mathbf{f}}\quad & \mathbf{f}^\top L^{-1}M\mathbf{f}\\ \text{s.t.}\quad & \mathbf{f}^\top L^{-1}\mathbf{f}=1.\end{aligned}$$

Using Bertsekas (1999, Proposition 3.1.1), if $\mathbf{f}$ is the optimal solution, then we either have $\nabla_{\mathbf{f}}\mathbf{f}^\top L^{-1}\mathbf{f}=0$ or

$$\nabla_{\mathbf{f}}\mathcal{L}(\mathbf{f},\mu)=0,$$

for some unique μ, where $\mathcal{L}(\mathbf{f},\mu)$ is the Lagrangian defined as $\mathcal{L}(\mathbf{f},\mu)=\mathbf{f}^\top L^{-1}M\mathbf{f}-\mu\left(\mathbf{f}^\top L^{-1}\mathbf{f}-1\right)$. If $\nabla_{\mathbf{f}}\mathbf{f}^\top L^{-1}\mathbf{f}=2L^{-1}\mathbf{f}=0$, then $\mathbf{f}=0$ and $M\mathbf{f}=\mu\mathbf{f}$ hold for any μ. Otherwise, we obtain

$$\nabla_{\mathbf{f}}\mathcal{L}(\mathbf{f},\mu)=2L^{-1}M\mathbf{f}-2\mu L^{-1}\mathbf{f}=0,$$

where we used the symmetry of both $L^{-1}M$ and L^{-1} in the preceding equality. This implies that $L^{-1}M\mathbf{f} = \mu L^{-1}\mathbf{f}$. Multiplying both sides by L, we obtain $M\mathbf{f} = \mu\mathbf{f}$ for some unique μ. We then have $|\mathbf{f}^\top L^{-1}M\mathbf{f}| = |\mathbf{f}^\top L^{-1}\mu\mathbf{f}| = |\mu|$, where the second equation follows from $|\mathbf{f}^\top L^{-1}\mathbf{f}| = 1$. $\square$

Step 4: In this step, we show $\left|\frac{R(W,\alpha,\mathbf{f})-R(\tilde{W},\alpha,\mathbf{f})}{R(W,\alpha,\mathbf{f})}\right| \leq \frac{\alpha\lambda_{\max}(W)\epsilon}{1-\alpha\lambda_{\max}(W)(1+\epsilon)}$.

Using Steps 1 to 3 yields

$$\begin{aligned}\left|\frac{R(W,\alpha,\mathbf{f})-R(\tilde{W},\alpha,\mathbf{f})}{R(W,\alpha,\mathbf{f})}\right| &\overset{(a)}{=} \max_{\mathbf{f}\,:\,\mathbf{f}^\top L^{-1}\mathbf{f}=1}\left|\mathbf{f}^\top L^{-1}M\mathbf{f}\right| \\ &\overset{(b)}{=} \max_{\mathbf{f}\,:\,\mathbf{f}^\top L^{-1}\mathbf{f}=1}\left|\mathbf{f}^\top L^{-1}\mu\mathbf{f}\right| = \max\{|\mu|\ :\ \mu=0 \text{ or } \mu \text{ is an eigenvalue of } M\} \\ &\overset{(c)}{\leq} \|M\|,\end{aligned} \tag{OA.4}$$

where (a) follows from Steps 1 and 2, (b) follows from Step 3, and (c) holds because $\|M\|_2 \geq \frac{\|M\mathbf{f}\|_2}{\|\mathbf{f}\|_2} = |\mu|$. We next show that $\|M\|_2 \leq \frac{\alpha\lambda_{\max}(W)\epsilon}{1-\alpha\lambda_{\max}(W)(1+\epsilon)}$. We can write

$$\begin{aligned}\|M\|_2 = \left\|I-(I-\alpha W)\left(I-\alpha\tilde{W}\right)^{-1}\right\|_2 &= \left\|\alpha(W-\tilde{W})(I-\alpha\tilde{W})^{-1}\right\|_2 \\ &\leq \left\|\alpha(W-\tilde{W})\right\|_2\left\|(I-\alpha\tilde{W})^{-1}\right\|_2 \\ &\overset{(a)}{\leq} \alpha\epsilon\|W\|_2\left\|(I-\alpha\tilde{W})^{-1}\right\|_2 \\ &\overset{(b)}{\leq} \alpha\epsilon\lambda_{\max}(W)\left\|(I-\alpha\tilde{W})^{-1}\right\|_2 \\ &\overset{(c)}{\leq} \frac{\alpha\epsilon\lambda_{\max}(W)}{\lambda_{\min}(W)(I-\alpha\tilde{W})} \\ &= \frac{\alpha\epsilon\lambda_{\max}(W)}{1-\alpha\lambda_{\max}(\tilde{W})} \\ &\overset{(d)}{\leq} \frac{\alpha\lambda_{\max}(W)\epsilon}{1-\alpha\lambda_{\max}(W)\,(1+\epsilon)},\end{aligned} \tag{OA.5}$$

where (a) follows from $\tilde{W} \in \mathcal{W}(W,\epsilon)$; (b) follows from that for symmetric non-negative matrices we have $\|W\|_2 = \rho(W) = \lambda_{\max}(W)$; (c) holds as $(I-\alpha\tilde{W})^{-1}$ is symmetric and non-negative (see Theorem OA.2 in the appendix), and therefore, we can write $\left\|(I-\alpha\tilde{W})^{-1}\right\|_2 = \lambda_{\max}\left((I-\alpha\tilde{W})^{-1}\right) = 1/\lambda_{\min}\left(I-\alpha\tilde{W}\right)$; and (d) follows from that $\tilde{W}$ is symmetric and non-negative, and therefore, we have $\lambda_{\max}(\tilde{W}) = \left\|\tilde{W}\right\|_2 \leq \left\|\tilde{W}-W\right\|_2 + \|W\|_2 \leq (1+\epsilon)\|W\|_2 = (1+\epsilon)\lambda_{\max}(W)$.

Invoking (OA.5) in (OA.4), we have the desired result. □

Proof of Proposition 3. First note that the largest eigenvalue of a symmetric matrix is a convex function of the matrix, i.e., $\lambda_{\max}(M)$ is a convex function of M. This is because, by definition, we have $\lambda_{\max}(M) = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^\top M\mathbf{x}$. Therefore, we have

$$\begin{aligned}\lambda_{\max}(\beta M_1 + (1-\beta)M_2) &= \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^\top(\beta M_1 + (1-\beta)M_2)\mathbf{x}\\ &= \max_{\|\mathbf{x}\|_2=1} (\beta\mathbf{x}^\top M_1\mathbf{x} + (1-\beta)\mathbf{x}^\top M_2\mathbf{x})\\ &\le \beta \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^\top M_1\mathbf{x} + (1-\beta)\max_{\|\mathbf{x}\|_2=1} \mathbf{x}^\top M_2\mathbf{x}\\ &= \beta\lambda_{\max}(M_1) + (1-\beta)\lambda_{\max}(M_2).\end{aligned}$$

Hence, $\lambda_{\max}(\beta\overline{W} + (1-\beta)\overline{RW})$ is convex in β.

We define $F(W, RW, \beta) = \beta\overline{W} + (1-\beta)\overline{RW}$. For any β, β' where $\beta' < \beta$, we have

$$\begin{aligned}F(W, RW, \beta') &= \beta'\overline{W} + (1-\beta')\overline{RW}\\ &= \frac{\beta'}{\beta}\left(\beta\overline{W} + (1-\beta)\overline{RW}\right) + (1-\frac{\beta'}{\beta})\overline{RW}\\ &= \frac{\beta'}{\beta}F(W, RW, \beta) + (1-\frac{\beta'}{\beta})\overline{RW}.\end{aligned}$$

Therefore, using the convexity of largest eigenvalue, we have

$$\lambda_{\max}(F(W, RW, \beta')) \le \frac{\beta'}{\beta}\lambda_{\max}(F(W, RW, \beta)) + (1-\frac{\beta'}{\beta})\lambda_{\max}(\overline{RW}) \le \lambda_{\max}(F(W, RW, \beta)),$$

where the last inequality is due to the fact that for any regular network RW, we have $\lambda_{\max}(\overline{RW}) = 1/n$, and by Lemma 1, $\lambda_{\max}(\overline{W}) \ge 1/n$. Hence, $\lambda_{\max}(\beta\overline{W} + (1-\beta)\overline{RW})$ is increasing in β.

Moreover, by (2), the revenue volatility is increasing and convex in the largest eigenvalue of the underlying network. Thus, the revenue volatility of $\beta\overline{W} + (1-\beta)\overline{RW}$ is increasing and convex in β because we just proved that $\lambda_{\max}(\beta\overline{W} + (1-\beta)\overline{RW})$ is increasing and convex in β. This completes the proof.

The second part follows because increasing the values of some entries (while keeping the rest of the entries the same) of a matrix with all non-negative entries increases its largest eigenvalue. In particular, given a network W with SW as a subgraph of W, we have

$$[\beta W+(1-\beta)SW]_{ij}=W_{ij}-(1-\beta)(W_{ij}-SW_{ij}).$$

Assuming that $SW\subset W$, we have $0\le W_{ij}-SW_{ij}\le W_{ij}$, implying that $[\beta W+(1-\beta)SW]_{ij}$ is increasing in β, completing the proof of the second part. $\square$

Proof of Proposition 4. We have

$$\begin{aligned}
&\mathbf{1}^\top(\sum_{k=0}^{2}\alpha^kF(W,RW,\beta)^k\sum_{i=1}^{\infty}C(i,k)\alpha^{2i}\sigma^{2i})\mathbf{1}\\
&=\mathbf{1}^\top I\mathbf{1}\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\mathbf{1}^\top\alpha F(W,RW,\beta)\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}\mathbf{1}\\
&\quad+\mathbf{1}^\top\alpha^2F(W,RW,\beta)^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\mathbf{1}\\
&=n\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\mathbf{1}^\top\alpha(\beta\overline{W}+(1-\beta)\overline{RW})\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}\mathbf{1}\\
&\quad+\mathbf{1}^\top\alpha^2(\beta\overline{W}+(1-\beta)\overline{RW})^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\mathbf{1}\\
&=n\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\alpha\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}+\alpha^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\left(\mathbf{1}^\top(\beta\overline{W}+(1-\beta)\overline{RW})^2\mathbf{1}\right)\\
&=n\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\alpha\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}+\left(\alpha^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\right)\left(\sum_{l=1}^{n}Deg_l^2(\beta\overline{W}+(1-\beta)\overline{RW})\right)\\
&=n\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\alpha\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}+\left(\alpha^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\right)\left(\sum_{l=1}^{n}\left(\beta(Deg_l(\overline{W})+\frac{1-\beta}{n}\right)^2\right)\\
&=n\sum_{i=1}^{\infty}\frac{1}{i+1}\binom{2i}{i}\alpha^{2i}\sigma^{2i}+\alpha\sum_{i=1}^{\infty}C(i,1)\alpha^{2i}\sigma^{2i}+\left(\alpha^2\sum_{i=1}^{\infty}C(i,2)\alpha^{2i}\sigma^{2i}\right)\left(\frac{1-\beta^2}{n}+\beta^2\sum_{l=1}^{n}Deg_l^2(\overline{W})\right).
\end{aligned}$$

Note that the balancing process does not affect the impact of zero- and first-degree connections on revenue volatility. Moreover, we have

$$\frac{\partial}{\partial\beta}\left(\frac{1-\beta^2}{n}+\beta^2\sum_{l=1}^{n}Deg_l^2(\overline{W})\right)=2\beta(\frac{-1}{n}+\sum_{l=1}^{n}Deg_l^2(\overline{W})),$$

and using Cauchy-Schwarz inequality, and since $\sum_{l=1}^{n} Deg_l(\overline{W}) = 1$, we have

$$1 = \sum_{l=1}^{n} Deg_l(\overline{W}) \leq \sum_{l=1}^{n} Deg_l^2(\overline{W}) \sum_{l=1}^{n} 1,$$

implying that $\sum_{i=1}^{n} Deg_i^2(\overline{W}) \geq \frac{1}{n}$. Therefore, $\frac{-1}{n} + \sum_{i=1}^{n} Deg_i^2(\overline{W})$ is a positive constant, and the absolute value of revenue volatility is increasing in β, completing the proof. $\square$

B.2. Proof of Theorem 2.

In the rest of this section, we prove Theorem 2. We first show a lemma, establishing that under Assumption 3, the matrix $I - \alpha(W_n + G_n)$ is non-singular and that the Taylor series of $\left(I - \alpha(W_n + G_n)\right)^{-1}$ is convergent. In particular, we show that $\rho(\alpha(W_n + G_n)) < 1$ almost surely, which implies that the Taylor series expansion converges (see Theorem OA.2 in the appendix). We then write $(I - \alpha(W_n + G_n))^{-1}$ in its Taylor series expansion and characterize the expected revenue volatility as a function of the standard deviation of the noise σ, parameter α, and the network structure W_n.

Lemma OA.2. *Given a sequence of symmetric matrices W_n and a sequence of symmetric random noise matrices G_n, if G_n satisfies Assumption 3 and $\alpha < \frac{1}{\rho(W_n)+2\sigma}$, then as $n \to \infty$, $\rho(W_n + G_n) < 1$ almost surely, and the matrix $I - \alpha(W_n + G_n)$ is invertible.*

Next, we estimate $R(W_n, G_n, \alpha, \mathbf{b})$ using the Taylor series approximation. We let $H(W_n, G_n) = (I - \alpha(W_n + G_n))^{-1}$ and use Lemma OA.2 to write

$$H(W_n, G_n) = \sum_{i=0}^{\infty} \alpha^i (W_n + G_n)^i. \tag{OA.6}$$

For a Wigner matrix G_n, as a part of the proof of the Wigner semicircle law, it has been shown that matrix moments are convergent in expectation (see, e.g., Anderson et al. 2010, Chapter 2) and

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\operatorname{Tr} G_n^k\right] = \begin{cases} \sigma^k C(k/2) & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd.} \end{cases} \tag{OA.7}$$

If W_n and G_n were commuting, we could use the binomial expansions to simplify $H(W_n, G_n)$ and then calculate $\mathbb{E}\left[G_n^k\right]$ separately to measure the volatility as a function of the network structure and the noise distribution. However, W_n and G_n are not necessarily commuting, which prevents us from using this argument. This requires developing new techniques. We next provide a generalization of the characterization of the Wigner matrix moment convergence and then use it to establish Lemma OA.2. In this regard, we first introduce a combinatorial interpretation of $H(W_n, G_n)$ by defining a corresponding graph that we refer to as the *white-gray graph*.

Definition OA.1 (White-gray Graph). Given two matrices W_n and G_n, the corresponding white-gray weighted graph $WG(W_n, G_n)$ is a weighted graph with n nodes in which between any pair of nodes, there is a gray edge and a white edge. The weight of the white edge between nodes i and j is αW_{ij}, and the weight of the gray edge between nodes i and j is αG_{ij}. More precisely, the weighted multi-graph $WG(W_n, G_n) = (V, \mathcal{E})$ where $V = \{1, \ldots, n\}$ and

$$\mathcal{E} = \mathcal{E}^w \cup \mathcal{E}^g = \{(i,j)_w \cup (i,j)_g, i, j \in V\},$$

$$w((i,j)_w) = \alpha W_{ij},\ \ w((i,j)_g) = \alpha G_{ij},$$

where the edge $e = (i,j)_w$ denotes a white edge between i and j (similarly, $e(i,j)_g$ denotes a gray edge), and $w(e)$ denotes the weight of an edge e.

Definition OA.2. In the white-gray graph $WG(W_n, G_n)$, a walk of length k between v_0 and v_k is an alternating sequence of nodes and edges, $T = (v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k)$, where the *endpoints* of e_i are v_i and v_{i+1} (i.e., e_i is the edge between v_i and v_{i+1}). We next introduce the notions we use for these walks:

- *Starting node* $s(\cdot)$ *and ending node* $d(\cdot)$: We refer to v_0 as the starting node of T and denote it by $s(T)$. We refer to v_k as the ending node of T and denote it by $d(T)$.

- $V(\cdot)$ *and* $E(\cdot)$: We define $V(T)$ and $E(T)$ as the set of distinct nodes and distinct edges used in T, respectively.

- *The length of a walk* $|\cdot|$: The length of a walk denotes the number of edges used in the walk, i.e., $|T| = k$.

- *White walk, gray walk*: A walk T is a white walk if all edges traversed in T are white. A walk T is a gray walk if all edges traversed are gray.

- *A section of a walk*: Given a walk T and two nodes v_i and v_j on it ($i < j$), we denote the *section* of a walk between v_i and v_j by $T_{v_i v_j}$.

- *Weight of a walk,* $w(\cdot)$: We define the weight of a walk as the product of the weights of the edges on the walk, i.e., $w(v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k) = \prod_{i=0}^{k-1} w(e_i)$.

- Finally, given a white-gray network $WG(W_n, G_n)$, we let $\mathcal{P}_{ij}$ denote the set of all walks, with starting node i and ending node j. Note that $\mathcal{P}_{ij}$ is an infinite collection of walks with all possible lengths of 1 up to ∞.

In Figure OA.1(a), we display a white-gray graph with four nodes, and in Figure OA.1(b), we highlight a walk $T = (v_0 = 1, e_0 = (1,2)_w, v_1 = 2, e_1 = (2,3)_g, v_2 = 3, e_2 = (3,4)_g, v_3 = 4, e_3 = (4,3)_g, v_4 = 3, e_4 = (3,2)_w, v_5 = 2)$ from node 1 ($s(T) = 1$) to node 2 ($d(T) = 2$). The length of T, $|T| = 5$, $V(T) = \{1,2,3,4\}$ and $E(T) = \{(1,2)_w, (2,3)_g, (3,4)_g, (3,2)_w\}$. The section of the walk $T_{v_2=3, v_5=2} = (3, (3,4)_g, 4, (4,3)_g, 3, (3,2)_w, 2)$, and the section of the walk $T_{v_4=3, v_5=2} = \{3, (3,2)_w, 2\}$. In this example, nodes 2 and 3 are visited twice in T.

We next present a simple connection between each entry of matrix $H(W_n, G_n)$ and the walks in the white-gray graph $WG(W_n, G_n)$.

Lemma OA.3. *Given a white-gray network $WG(W_n, G_n)$, the (i,j)-th entry of $H(W_n, G_n)$ is*

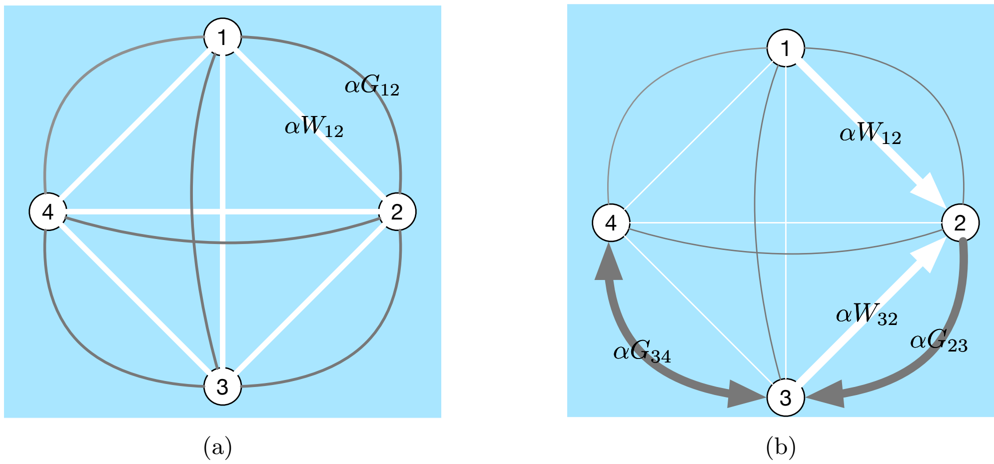
$$H_{ij} = \sum_{p \in \mathcal{P}_{ij}} w(p),$$

where $\mathcal{P}_{ij}$ is the collection of walks between i and j in $WG(W_n, G_n)$.

Proof of Lemma OA.3. Using Eq. (OA.6), we have

$$H_{ij} = \sum_{k=0}^{\infty} \left[(\alpha(W_n + G_n))^k \right]_{ij} = \sum_{k=0}^{\infty} \sum_{p \in \mathcal{P}_{ij}, |p| = k} w(p) = \sum_{p \in \mathcal{P}_{ij}} w(p),$$

Figure OA.1 (a) A white-gray graph $WG(W_4, G_4)$ with 4 nodes. (b) A walk $T = (v_0 = 1, e_0 = (1,2)_w, v_1 = 2, e_1 = (2,3)_g, v_2 = 3, e_2 = (3,4)_g, v_3 = 4, e_3 = (4,3)_g, v_4 = 3, e_4 = (3,2)_w, v_5 = 2)$ of length 5 between 1 and 2.



Note. (a) Between every pair (i,j) of nodes, there is a white edge with weight αW_{ij} and a gray edge with weight αG_{ij}. (b) The starting node of T is $s(T) = 1$, and the ending node of T is $d(T) = 2$. $V(T) = \{1,2,3,4\}$. The weight of T, $w(T) = \alpha^5 W_{12} G_{23} G_{34}^2 W_{32}$. The section of the walk $T_{v_2=3, v_5=2} = (3, (3,4)_g, 4, (4,3)_g, 3, (3,2)_w, 2)$.

completing the proof. $\square$

Lemma OA.3 states that the entry (i,j) of the matrix H represents the sum of all weights of the walks between i and j in the white-gray graph $WG(W_n, G_n)$. If there were no additional noise, i.e., G_n is the zero matrix, then H_{ij} would equal the sum of all weights of walks on white edges from i to j. This is because the weight of any walk with at least one gray edge would be 0. For random noise matrix G_n, using Assumption 3, $\mathbb{E}[G_{ij}] = 0$ for all $i,j \in V$. Therefore, for any walk that traverses a gray edge once, the expected weight is 0. However, the expected weight of a walk traversing a gray edge more than once is no longer 0 and should be accounted for when computing the sum of all weights of the walks between any two nodes.

In order to find a succinct representation of H, we next define a partition of walks into grouped-walks, as follows.

Definition OA.3 (Grouped-walks). We define the grouped-walks from a starting node s to an ending node d as a collection of walks in $WG(W_n, G_n)$ formed by alternating gray and white sections. In addition to the starting node and ending node (s,d), a group-walk is specified by a sequence of k white walks $(T_1,\ldots,T_k)$ and the length of the gray walks in between the white walks. The length of the gray walks is compactly denoted by a vector $\delta \in (\mathbb{N}\cup\{0\})^{k+1}$, where δ_{i-1} is the length of the gray section before T_i and $\delta_i > 0$ for $1 \le i \le k$. More precisely, we have

$$\text{grouped-walks}\left((s,d),(T_1,\ldots,T_k),\delta\right) =$$
$$\left\{p = (Q_0, T_1, Q_1, T_2, \ldots, T_k, Q_k) \mid p \in \mathcal{P}_{sd}, \forall i, |Q_i| = \delta_i, \forall e \in E(Q_i) \ : \ e \in \mathcal{E}^g \right\},$$

where $\mathcal{P}_{sd}$ is the set of walks in $WG(W_n, G_n)$ from s to d and $\mathcal{E}^g$ is the set of all gray edges in $WG(W_n, G_n)$. T_i's represent the white sections grouped-walks, and Q_i's represent the gray sections of grouped-walks (the edges used in Q_i are all gray). If $\delta_0 = 0$, then $|Q_0| = 0$, which means the walks starting with T_1. Furthermore, the expected weight of the grouped-walks, denoted by $\mathcal{GP}$, is defined as

$$w(\mathcal{GP}) = \mathbb{E}\left[\sum_{p\in\mathcal{GP}} w(p)\right].$$

The following lemma shows a key property of the weight of a grouped-walks.

Lemma OA.4. *Let G_n be a Wigner matrix with $\mathbb{E}[G_{12}^2] = \frac{\sigma^2}{n}$. For any $\epsilon > 0$, there exists N such that for all $n \ge N$, $w\left(\text{grouped-walks}((s,d),(T_1,\ldots,T_k),\delta)\right) < \epsilon$ if at least one of $\delta_1,\ldots,\delta_k$ is an odd number or the concatenation of the white sections, i.e., $(T_1,T_2,\ldots,T_k)$ does not form a walk from s to d.*

Proof of Lemma OA.4. We let $\mathcal{GP}$ denote the set of grouped-walks$((s,d),(T_1,\ldots,T_k),\delta)$. Let $\delta = \sum_{i=0}^{k}\delta_i$ denote the total number of gray edges in the grouped-walks $\mathcal{GP}$. Using Definition OA.1 and Assumption 3, the weight of the gray edge (i,j) is $\alpha G_{ij} = \frac{\alpha}{\sqrt{n}} Y_{ij}$. For a gray section Q, let $Y(Q) = \prod_{e\in E(Q)} Y_e$. We can write

$$\mathbb{E}[w(\mathcal{GP})] = \left(\prod_{j=1}^{k} w(T_j)\right)\left(\frac{\alpha}{\sqrt{n}}\right)^{\delta} \mathop{\mathbb{E}}_{\substack{|Q_i|=\delta_i,\\ \forall e\in E(Q_i)\ :\ e\in\mathcal{E}^g,\\ s(Q_i)=d(T_i),\\ d(Q_i)=s(T_{i+1})}} \left[\prod_{i=1}^{k} Y(Q_i)\right]. \tag{OA.8}$$

Using Assumption 3, we have $\mathbb{E}[Y_{ij}]=0$. Therefore, for any sequence of gray sections $(Q_0,\ldots,Q_k)$ in which at least one edge appears exactly once, we obtain $\mathbb{E}[\prod_{i=0}^{k} Y(Q_i)]=0$. Therefore, we have the following properties.

Property 1: Any gray edge in a walk $p\in\mathcal{GP}$ with non-zero expected weight is traversed at least twice.

Property 2: The expected weight of grouped-walks with δ gray edges, for which the number of distinct nodes (excluding the starting and ending nodes of each gray section) used by gray edges is no more than half of the number of gray edges, goes to zero as n grows. Formally, we will show for a given grouped-walks with k white walks,

$$\lim_{n\to\infty} \mathop{\mathbb{E}}_{\substack{|Q_i|=\delta_i,\\ \forall e\in E(Q_i)\,:\,e\in\mathcal{E}^g,\\ s(Q_i)=d(T_i),\\ d(Q_i)=s(T_{i+1})}} \left[w(\mathcal{GP})\;:\;|\cup_{i=0}^{k}\left(V(Q_i)-\{s(Q_i),d(Q_i)\}\right)|<\frac{\delta}{2}\right]=0. \tag{OA.9}$$

The contribution of white edges to the weight of $\mathcal{GP}$ is a constant. We next estimate the number of ways that one can form the gray sections of $\mathcal{GP}$. First note that for a given set S of nodes, the number of ways of creating a sequence of k sections of a total length δ is at most of order $|S|^{\delta}$. Additionally, note that the starting and ending nodes of each section Q_i are fixed and that the generated sequence represents the sections without their first and last edge. Therefore, in the sequence generated by S, the number of ways of selecting s nodes is at most n^s. Assumption 3 implies that all finite moments of Y_{ij} are bounded. Letting $\beta=\max_{1\le l\le\delta}(\mathbb{E}[Y_{ij}^l])^{\frac{1}{l}}$ yields

$$\begin{aligned}
\mathop{\mathbb{E}}_{\substack{|Q_i|=\delta_i,\\ \forall e\in E(Q_i),e\in\mathcal{E}^g,\\ s(Q_i)=d(T_i),\\ d(Q_i)=s(T_{i+1})}} \left[w(\mathcal{GP}):|\cup_{i=0}^{k}\left(V(Q_i)-\{s(Q_i),d(Q_i)\}\right)|<\frac{\delta}{2}\right] &\le \left(\prod_{i=1}^{k} w(T_i)\right)\left(\frac{\alpha}{\sqrt{n}}\right)^{\delta}\sum_{s=0}^{\delta/2-1} n^s s^{\delta}\beta^{\delta}\\
&\le \left(\prod_{i=1}^{k} w(T_i)\right)(\alpha\beta\delta)^{\delta}\sum_{s=0}^{\delta/2-1} n^{s-\frac{\delta}{2}}\\
&\le \left(\prod_{i=1}^{k} w(T_i)\right)(\alpha\beta\delta)^{\delta}\left(\frac{\delta}{2}n^{-\frac{\delta}{2}}n^{\frac{\delta}{2}-1}\right)
\end{aligned}$$

$$= \frac{1}{n}\left(\frac{\delta}{2}\left(\prod_{i=1}^{k} w(T_i)\right)(\alpha\beta\delta)^{\delta}\right).$$

Taking the limit of both sides as $n \to \infty$ shows Property 2. Property 2 implies that when n is sufficiently large, we should consider only walks $p \in \mathcal{GP}$ with at least $\delta/2$ distinct nodes on gray edges (excluding the starting and ending nodes of the sections). Combining Properties 1 and 2, we show the following property.

Property 3: For any walk $p = (Q_0, T_1, \ldots, T_k, Q_k) \in \mathcal{GP}$ that satisfies Properties 1 and 2, we have the following:

- The sequence of sections $(T_1, \ldots, T_k)$ is a walk from i to j.
- Each section Q_i is a closed walk in the form of a tree in which each edge is traversed exactly twice.

We next prove this property. Given $p = (Q_0, T_1, \ldots, T_k, Q_k)$, let t denote the number of connected components formed by the union of gray sections $(Q_1, \ldots, Q_k)$. Note that each gray section is a connected component; therefore, the number of connected components formed by the union of gray components is no more than $k+1$, i.e., $t \leq k+1$. We refer to each connected component corresponding to a subset of gray sections as a partition and denote all partitions by the vector $\pi = (\pi_1, \ldots, \pi_t)$, where t denotes the number of connected components. We next find an upper bound on the number of free distinct nodes on gray sections (i.e., distinct nodes excluding the starting and ending nodes of each gray section). Property 1 implies that the number of distinct edges on gray walks is at most $\frac{\delta}{2}$. Moreover, the number of free distinct nodes used in a graph with t connected components and $\frac{\delta}{2}$ edges is no more than $\frac{\delta}{2} + t$, and the equality holds only if every component is a tree. Furthermore, given partition π, in each connected component, at least one node is not free (corresponding to the starting and ending nodes of the gray section in that connected component). Therefore, for a given partition π with δ edges, the number of distinct nodes is bounded by

$$|\cup_{i=0}^{k}\left(V(Q_i)-\{s(Q_i),d(Q_i)\}\right)|\leq \frac{\delta}{2}+t-t=\frac{\delta}{2}. \tag{OA.10}$$

Note that in each partition, if the number of pre-specified nodes is more than one, then the number of distinct free nodes we can choose from becomes strictly less than $\frac{\delta}{2}$, which, by Property 2, contributes asymptotically 0 to the expected weight of the grouped-walks. To have exactly one pre-specified node in each connected component, the starting and ending nodes of all sections in each connected component should be identical, i.e., given the partition π with exactly t connected components, we have

$$\forall 1\leq i\leq t,\quad |\cup_{j\in\pi_i}\{s(Q_j),d(Q_j)\}|=1,$$

i.e., all walks in the same connected component are closed walks, and all of them share one node as their starting and ending nodes. Therefore, for a walk $p\in\mathcal{GP}$ that satisfies Properties 1 and 2, we have $d(T_i)=s(Q_i)=d(Q_i)=s(T_{i+1})$, i.e., $(T_1,\ldots,T_k)$ is a walk from i to j.

We next show that each section Q_i is a tree that is traversed exactly twice. To satisfy Inequality (OA.10) with equality, we must have the following: (a) every connected component should be a tree, and therefore, the distinct edges forming every walk also form a tree, and (b) all gray sections in the same connected component should have only one common node, which is the starting node of these sections. Therefore, each section is a tree, and there is no common node between the sections in the same connected component. Finally, each edge should be traversed exactly twice (to have exactly $\delta/2$ distinct edges). Therefore, each Q_i is a closed walk over a tree traversing each edge of the tree once in each direction, completing the proof of Property 3.

Combining Properties 1, 2, and 3, we complete the proof of Lemma OA.4. □

We now proceed with the proof of Theorem 2. Using Lemma OA.4, when calculating w(group-path), in the limit when n goes to infinity, it is sufficient to only consider grouped-walks between (i,j) with the following properties:

- All gray sections are closed walks in the form of a tree in which each edge is traversed exactly twice;
- The white sections form a proper walk from i to j.

We next count, for every white walk between i and j, the number of ways of choosing gray walks of a given length such that all the preceding properties are satisfied.

LEMMA OA.5. *Given a white walk of length ω between i and j, the number of walks that satisfy Property 3 with $2g$ gray edges is $\prod_{i=1}^{g}(n-i)C(g,\omega)$, where $C(g,\omega)$ is defined recursively as*

$$
\begin{aligned}
&C(i,k) = \sum_{t=0}^{k}\sum_{j=1}^{i} c(j-1)C(i-j,k-t), && i,k>0,\\
&C(i,0) = c(i) = \frac{1}{i+1}\binom{2i}{i},\\
&C(0,k) = 1,
\end{aligned}
$$

and $c(j-1)$ is the $(j-1)$-th Catalan number. In the limit as $n \to \infty$, the number of walks that satisfy Property 3 with $2g$ gray edges divided by $n^g C(g,\omega)$ goes to one.

Proof of Lemma OA.5. For any of the grouped-walks with $2g$ gray edges and ω white edges, Property 3 implies that the gray sections are closed walks and that there are no common nodes between the walks representing two distinct gray sections (except for the origin of the walks). Moreover, each gray section has a tree structure. Consider the grouped-walks between i and j. We partition these walks into two groups: (1) the group-walks starting with a gray section and (2) the group-walks starting with a white section.

(1) Grouped-walks starting with a gray section:

In this case, since all gray walks are closed, the gray section starting from i should also return to i. Let $2t$ denote the number of edges after which the gray section returns to i for the first time. Therefore, to form a gray walk that satisfies the above properties, first, an ordered set of t distinct nodes out of n nodes should be chosen. This ordering can be selected and permuted in

$\prod_{i=1}^{t}(n-i)$ ways. We also have $\lim_{n\to\infty} \frac{\prod_{i=1}^{t}(n-i)}{n^t} = 1$. Since the walk structure is a tree and each edge is traversed exactly twice, it is implied that the first edge that is traversed by the walk is traversed in the opposite direction at $2t$. (This is because the structure is a tree, and therefore, the number of paths between i and the first node after i in the given order is unique.) Therefore, if we remove the first and last edges, we still have a closed walk of a tree form, formed over the chosen sequence of length $2t-2$ (in this way, we ensure that the first time that the gray section returns to i is exactly after traversing $2t$ edges). It is well known that given an ordered sequence of nodes, the number of ways of making tree-like closed walks that visit these nodes for the first time in the sequence order is equal to $c(t-1)$, where $c(t) = \sum_{i=1}^{t} c(i-1)c(t-i)$ is the Catalan number, representing the number of proper parentheses sequences with t open parentheses.[6] After taking $2t$ edges and returning to i, we have $2(g-t)$ more edges to traverse in the grouped-walks starting from i, which we compute recursively. In particular, the rest of the grouped-walks can be formed in n^{g-t} times $C(g-t,\omega)$ ways. Therefore, given ω and g, the number of grouped-walks is n^g times $\sum_{t=1}^{g} c(t-1)C(g-t,\omega)$, where t denotes the number of edges before the first time revisiting i, $c(t-1)$ denotes the number of ways to make tree-like closed walks of length $t-1$ given an ordered set of t nodes, and $C(g-t,\omega)$ denotes the number of ways to make the grouped-walks with $g-t$ gray edges given ω, fixing the order of first-time visits to nodes.

(2) Grouped-walks starting with a white section:

Let j denotes the number of white edges of the white walk appearing in the first section of the grouped-walks (i.e., the edge $j+1$ should be gray). Therefore, the rest of such grouped-walks start with a gray section, which we explained how to count in the previous part. In particular, as explained there, we first select an ordered set of g distinct nodes out of n nodes. This ordering can be selected and permuted in $\prod_{i=1}^{g}(n-i)$ ways. We also have $\lim_{n\to\infty} \frac{\prod_{i=1}^{g}(n-i)}{n^g} = 1$. Therefore,

[6] One can show that there is an equivalence between Dyck words of length $2(t-1)$ and these walks. In particular, whenever an edge is visited for the first time in the walk, an x is added to the string, and whenever an edge is visited for the second time, a y will be added to the sequence.

the number of grouped-walks starting with a white section with exactly j edges is asymptotically equal to $n^g \sum_{t=1}^{g} C(g-t,\omega-j)c(t-1)$. Note that the summation ensures that edge $j+1$ is gray.

Combining groups (1) and (2), we obtain

$$C(g,\omega) = \sum_{t=1}^{g} c(t-1)C(g-t,\omega) + \sum_{j=1}^{\omega}\sum_{t=1}^{g} C(g-t,\omega-j)c(t-1) = \sum_{j=0}^{\omega}\sum_{t=1}^{g} C(g-t,\omega-j)c(t-1),$$

completing the proof. Note that $C(g,\omega)$ is a generalization of the Catalan number and is equivalent to the solution of the following combinatorial problem: "Given ω identical balls and g pairs of parentheses, count the number of distinct arrangement of balls and parentheses such that the sequence of parentheses is proper (i.e., at any time, the number of open parentheses is at least as large as that of the closed ones) and balls are only located where the numbers of open and closed parentheses are equal." $\square$

Combining Lemmas OA.3, OA.4, and OA.5, we have

$$\begin{aligned}
\Delta^R(W_n,G_n) &= \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \sum_{k=0}^{\infty}\alpha^k (W_n)^k \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) - \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \sum_{k=0}^{\infty}\alpha^k (W_n+G_n)^k \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) \\
&= \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \sum_{k=0}^{\infty}\alpha^k (W_n)^k C(0,k) \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) - \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \sum_{k=0}^{\infty}\alpha^k (W_n+G_n)^k \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) \\
&= \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \left(\sum_{k=0}^{\infty}\alpha^k (W_n)^k C(0,k) - \sum_{k=0}^{\infty}\alpha^k W_n^k \sum_{i=0}^{\infty} C(i,k)\alpha^{2i}\sigma^{2i}\right) \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) + O(\frac{1}{n}) \\
&= -\left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right)^\top \left(\sum_{k=0}^{\infty}\alpha^k W_n^k \sum_{i=1}^{\infty} C(i,k)\alpha^{2i}\sigma^{2i}\right) \left(\frac{\mathbf{b}-c\mathbf{1}}{2}\right) + O(\frac{1}{n}),
\end{aligned}$$

where the third equation follows from Lemma OA.3, Property 3, and Lemma OA.5, completing the proof of Theorem 2.

Proof of Lemma OA.2. We use Theorem OA.3 of Bai and Yin (1988), presented in the appendix, to show that $\rho(W_n+G_n)<1$. By definition, $G_n = \frac{Y_n}{\sqrt{n}}$, where Y_n is a symmetric matrix, and $E[Y_{11}]=0$, $E[Y_{12}^2]=\sigma^2<\infty$, and $E[Y_{12}^k]<\infty$. Using Theorem OA.3 of Bai and Yin (1988) implies that

$$\lambda_{\max}(G_n) = \lambda_{\max}(\frac{Y_n}{\sqrt{n}}) = 2\sigma, \quad a.s. \tag{OA.11}$$

Moreover, using Assumption 3, we have that $W_n + G_n$ is symmetric and non-negative, which implies that

$$\rho(W_n + G_n) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } W_n + G_n\} = \lambda_{\max}(W_n + G_n) \tag{OA.12}$$

(see Horn and Johnson 1990, Chapter 8). Moreover, since both W_n and G_n are symmetric, using Weyl's inequality (see Horn and Johnson 1990, Theorem 4.3.1), we have

$$\lambda_{\max}(W_n + G_n) \leq \lambda_{\max}(W_n) + \lambda_{\max}(G_n). \tag{OA.13}$$

Combining the assumption $\alpha < \frac{1}{\rho(W_n)+2\sigma}$ with Equations (OA.11), (OA.12), and (OA.13), we have

$$\rho(\alpha(W_n + G_n)) = \alpha\lambda_{\max}(W_n + G_n) \leq \alpha\lambda_{\max}(W_n) + \alpha\lambda_{\max}(G_n) \leq \alpha(\rho(W_n) + 2\sigma) \leq 1,$$

completing the proof. $\square$

References

Anderson, Greg W, Alice Guionnet, Ofer Zeitouni. 2010. *An introduction to random matrices*. Cambridge Studies in Advanced Mathematics, vol. 118. Cambridge University Press, New York, NY.

Bai, Zhi-Dong, Yong-Qua Yin. 1988. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a wigner matrix. *Annals of Probability* **16** (4) 1729–1741.

Bertsekas, Dimitri P. 1999. *Nonlinear Programming*. Athena Scientific, Belmont, MA.

Candogan, Ozan, Kostas Bimpikis, Asuman Ozdaglar. 2012. Optimal pricing in networks with externalities. *Operations Research* **60** (4) 883–905.

Horn, Roger A, Charles R Johnson. 1990. *Matrix Analysis*. Cambridge University Press, Cambridge, UK.

Monderer, Dov, Lloyd S. Shapley. 1996. Potential games. *Games and Economic Behavior* **14**(1) 124–143.