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Crosscutting Areas

Technical Note—Revenue Volatility Under Uncertain Network Effects

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Abstract. We study the revenue volatility of a monopolist selling a divisible good to consumers in the presence of local network externalities among consumers. The utility of consumers depends on their consumption level as well as those of their neighbors in a network through network externalities. In the eye of the seller, there exist uncertainties in the network externalities, which may be the result of unanticipated shocks or a lack of exact knowledge of the externalities. However, the seller has to commit to prices ex ante. We quantify the magnitude of revenue volatility under the optimal pricing in the presence of those random externalities. We consider both a given uncertainty set (from a robust optimization perspective) and a known uncertainty distribution (from a stochastic optimization perspective) and carry out the analyses separately. For a given uncertainty set, we show that the worst case of revenue fluctuation is determined by the largest eigenvalue of the matrix that represents the underlying network. Our results indicate that in networks with a smaller largest eigenvalue, the monopolist has a less volatile revenue. For the known uncertainty, we model the random noise in the form of a Wigner matrix and investigate large networks such as social networks. For such networks, we establish that the expected revenue is the sum of the revenue associated with the underlying expected network externalities and a term that depends on the noise variance and the weighted sum of all walks of different lengths in the expected network. We demonstrate that in a less connected network, the revenue is less volatile to uncertainties, and perhaps counterintuitively, the expected revenue increases with the level of uncertainty in the network. We show that a seller in the two settings favors the opposite type of network. In particular, if the underlying network is such that all the edge weights equal 1, the seller in the robust optimization setting prefers more asymmetry and the seller in the stochastic optimization setting prefers less asymmetry in the underlying network; by contrast, if the underlying network is such that the sum of all the edge weights is fixed, the seller in the robust optimization setting prefers less symmetry and the seller in the stochastic optimization setting prefers more asymmetry.

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1. Introduction

One assumption in the pricing literature on network games of strategic complementarities is that the underlying network structure is fixed and known; see, for example, Candogan et al. (2012), Bloch and Quérou (2013), and Cohen and Harsha (2020). However, in many practical settings, the seller, who sells to a market with local network externalities, may not know the exact structure of the network nor the exact level of influence between connected people. This network uncertainty can be manifested as forecast errors in defining the network. Moreover, even if a seller knows the exact number of an influencer's connections on a social network such as Facebook, Instagram, or LinkedIn, that seller may only have a rough idea of the exact level of this influence on different connections. In the aforementioned settings, there is a lack of understanding of the effects of network uncertainty on the seller's revenue. Our paper aims at filling this gap.

We consider a model of monopoly pricing in the presence of local network externalities and study the effects of random network perturbation on the monopolist's revenue. We focus on symmetric network structures—that is, the externality that one consumer has on another is the same within a pair. The price, consumption level, and weighted sum of neighbors' consumption levels in a network determines the consumer's utility. The underlying symmetric network structure is captured by a symmetric weight matrix, with entries being the weights of influence among consumers. The game consists of two stages: In the first stage, the monopolist selects the price offered to each consumer ex ante without knowing the true underlying network externalities or before the realization of these externalities. In the second stage, consumers choose their consumption levels simultaneously, knowing the true (local) network externalities, in anticipation of their neighbors' consumption levels.[1] Therefore, the revenue of the seller depends on the underlying (random) weighted network. We study the effects of variability in the underlying network structure on the seller's revenue.

We take two complementary perspectives to quantify the impact of the random or unobservable network structure on the seller's revenue in equilibrium. One is from the perspective of robust optimization with an uncertainty set of possible outcomes, in which we focus on the *worst-case* performance in revenue. The other is from the perspective of stochastic optimization with known uncertainty, in which we focus on the *expected* performance in revenue. In the robust optimization setting, we assume that the real weights of the symmetric network are within a given uncertainty set centered around the estimated ones. Specifically, we assume that the L_2 norm between the realized and the estimated networks is bounded by ϵ. We then characterize the worst-case revenue deviation from the revenue obtained based on the estimated network in terms of ϵ and the underlying network structure. In the stochastic optimization setting, we assume that the mean of the real network is given by the estimated network and that the uncertainty around the mean is captured by a Wigner matrix. We characterize the expected revenue deviation from the optimal revenue obtained based on the estimated network for very large networks. Our results could help sellers to estimate their revenue volatility when they conduct risk assessment. With this view, for a seller, our results can be interpreted as how volatile they estimates of the revenue could be when facing either unpredictable shocks or a lack of precise knowledge of the network structure.

For the robust optimization setting, we tightly characterize revenue volatility as a result of unanticipated changes in the underlying network. We show that the worst-case volatility for a bounded uncertainty set regarding the underlying externalities depends on the spectral properties of the underlying weighted network. In particular, the worst-case volatility relates to the largest eigenvalue of the underlying network. Building on this result and adopting the results from spectral graph theory (namely, the results in Wilfe 1967, Nosal 1970, Lovász and Pelikan 1973, and Hofmeister 1988), we study how the structural properties of the underlying network impact revenue volatility. In particular, we show that adding links and/or increasing the edge weights increases the revenue volatility. We show that among all connected networks with equal weights on all connected edges, a chain network has the smallest revenue volatility, whereas a complete network has the highest.

For the stochastic optimization setting, we characterize the expected revenue volatility for large networks, such as social networks, in the presence of random network effects. We show that revenue volatility depends on the noise variance and the weighted sum of all walks of different lengths in the expected network. Moreover, we show that the expected realized revenue grows with the uncertainty in the network. That is, the larger the revenue volatility is, the higher the seller's expected revenue is. In the analysis for this model, we establish asymptotic properties of the moments of a stochastic matrix that is the addition of a deterministic matrix and a Wigner matrix, which could be of independent interest.

The two risk assessment perspectives embrace network/revenue volatility in the opposite direction. The robust optimization perspective treats the volatility as revenue reduction in the worst case, whereas the stochastic optimization perspective views the volatility as revenue enhancement in expectation. For both settings, we relate revenue volatility to the structural properties of the underlying symmetric network. We show that a seller in both settings favors the opposite type of network. In particular, if the underlying network is such that all the edge weights equal 1, increasing asymmetries in the underlying network reduces revenue volatility. Hence, in such a network, the seller in the robust optimization setting prefers more asymmetry in the underlying network, whereas the seller in the stochastic optimization setting prefers less asymmetry in the underlying network. However, if the sum of all the edge weights is fixed, decreasing asymmetries in the underlying network reduces revenue volatility. Hence, in such a network, the opposite preferences hold.

1.1. Literature Review

There exist extensive studies on the strategic and welfare implications of network externalities; see Acemoglu et al. (2016) for a survey.

Most related to our paper is the literature on revenue management and pricing in the presence of *local* network externalities; see, for example, Candogan et al. (2012) and Bloch and Quérou (2013). In particular, Candogan et al. (2012) study the optimal pricing strategy of a monopolistic seller selling a divisible good (service) to consumers with local network externalities, which serves as the base of our model. Their model takes the form of a network game among agents that interact locally and is related to a series of papers on such games—for example, Ballester et al. (2006), Bramoullé and Kranton (2007), Corbo et al. (2007), Hartline et al. (2008), Galeotti and Goyal (2009), and Bramoullé et al. (2014). As a competitive version of Candogan et al. (2012), Chen et al. (2018) study the competitive pricing decisions by firms that sell substitutable goods to consumers under local network externalities. The authors show that consumers with a more central role receive lower prices in equilibrium. Moreover, the authors characterize how the firm's profits change as a function of the structure of the underlying network. We extend this stream of literature—in particular, Candogan et al. (2012)—by assuming that the firm faces uncertainty in the underlying network that influences consumer purchases, which in turn impacts the firm's profit.

In a seminal paper, Galeotti et al. (2010) consider a game over the network where the individuals' payoff depends on both their own actions and the actions of their neighbors. The authors assume that the network structure is partially known to the agents and characterize how this structure, the nature of externalities, and the level of information affect the equilibrium outcome and individual payoffs. By contrast, we assume that the network structure is known by consumers; however, the seller has only a noisy estimate of this structure.

Fainmesser and Galeotti (2016) consider the scenario in which the monopolist has partial information about consumers' network externalities. In particular, the authors consider asymmetric networks and compute the monopolist's value of information as the difference in its profits when it price-discriminates between consumers based on the knowledge of the structure versus when it offers a uniform price to all consumers. The authors adopt a mean-field type of approach and assume a network generated from the indegree/outdegree distributions. Zhang and Chen (2020) study the optimal nonlinear pricing in social networks in the presence of local externalities. The authors assume that individuals know the local network structure but that the seller only knows the global network. For the Erdős-Rényi graph, the authors show that the optimal pricing policy is a uniform price. The authors further show that nonlinear pricing allows the seller to respond more effectively to changes in network topology and economic factors compared with linear pricing. In contrast to these papers in the line of Galeotti et al. (2010), our paper has a different information structure. Fainmesser and Galeotti (2016) and Zhang and Chen (2020) assume that other than their own degree of connections, consumers face uncertainty about the rest of the network encapsulated by a degree distribution. However, in our paper, consumers can observe the exact network structure, though the sellers face uncertainty when estimating their profits. Furthermore, in our paper, the uncertainty that the seller faces is captured by either a bounded uncertainty set or random uncertainty in the form of a Wigner matrix. Besides, both our paper and Fainmesser and Galeotti (2016) study linear pricing, whereas Zhang and Chen (2020) focus on nonlinear pricing, which allows the firm to screen the network information.

Cohen and Harsha (2020) study a variant of Candogan et al. (2012) with an indivisible good and derive efficient approaches to optimally solving the problem. With the same form of local network effects, Zhou and Chen (2015; 2016) extend Ballester et al. (2006) to settings where the players are partitioned into leaders and followers engaging in a two-stage competition game. Zhou and Chen (2018) further extend their model to study the optimal decisions of the move sequence and pricing by a seller. By contrast, we study the impact of network uncertainty on the seller who faces a market of consumers with local network externality. We provide tight characterizations of the impact of uncertainties on revenue for both known and unknown uncertainties.

More recently, Huang et al. (2021) and Zhang and Chen (2020) have also studied pricing for a large random network. However, their focus is on the benefit of using discriminative or nonlinear pricing as opposed to uniform pricing, whereas our focus is on the revenue volatility under the optimal discriminative prices.

Another stream of literature studies the marketing and pricing decisions under general network benefit functions of a *global* nature (i.e., the network benefit of a customer depends on the behavior of all (previous) customers). Du et al. (2016; 2018) and Wang and Wang (2017) incorporate global network effects into the discrete choice model and then study the optimal pricing decisions. Moreover, Hu et al. (2015) investigate the operations and marketing policies when a firm sells to a market of sequentially arriving customers under global network effects. Hu et al. (2020) study the joint decisions of pricing and sales information disclosure when a firm sells a good to forward-looking customers who time their purchases under global network effects.

1.2. Notation

For any matrix $M \in \mathbb{R}^{m\times n}$, we use $[M]_{ij}$ or M_{ij} to denote the entry at the ith row and jth column. For a matrix M, $M \geq 0$ means $M_{ij} \geq 0$ for all i, j. We show vectors with boldface letters and numerals (e.g., the vector of all 1s is denoted by $\mathbf{1}$). The dimension of vectors and matrices is specified only when it is not clear from the context. We denote by $\mathbf{e}_i$ the vector with ith entry equal to 1 and the rest of the entries equal to 0. For any integer $n \in \mathbb{N}$, we let $[n] = \{1, \ldots, n\}$. We denote the transposes of vector $\mathbf{x}$ and matrix M by $\mathbf{x}^\top$ and $M^\top$, respectively. We denote a weighted undirected network by $(\mathcal{V}, W)$, where $\mathcal{V} = \{1, \ldots, n\}$ represents the set of nodes and W_{ij} represents the weight of the edge between nodes i and j. We consider undirected, symmetric networks with nonnegative edge weights (i.e., $W = W^\top$ and $W \geq 0$). We use $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$, respectively, to denote the largest and smallest absolute values among the eigenvalues of the matrix M and use $\mathbf{v}_{\max}(M)$ and $\mathbf{v}_{\min}(M)$ to denote their corresponding eigenvectors.[2]

1.3. Outline

In Section 2, we first describe the model, and then, for a known symmetric network, we review the equilibrium outcome, the optimal pricing, and the optimal revenue as a function of the underlying network. In Section 3, we study the revenue volatility for the robust optimization model with a given uncertainty set of network perturbations. In Section 4, we characterize revenue volatility for the stochastic optimization model with a stochastic perturbation. We provide concluding remarks in Section 5. All proofs are presented in the online appendix.

2. Pricing with Known Network Externalities

In this section, we review the network pricing game with no uncertainty and when the underlying network is known. We then present the existence of equilibrium and the characterization of the optimal discriminative pricing, as shown in Candogan et al. (2012).

2.1. Pricing for Network Externalities Game

The market consists of a set of agents, $\mathcal{V} = \{1, \ldots, n\}$, residing on a connected network $(\mathcal{V}, W)$.[3] We let $W_{ij} \geq 0$ denote the strength of the peer effect between agents i and j; we assume that the network is *symmetric* (i.e., $W_{ij} = W_{ji}$ for all pairs of i and j). This assumption represents a setting in which the externality between two agents depends on the extent of their two-way interactions and is hence symmetric.

We consider a two-stage game between a monopolist and n agents. In the first stage, the monopolist introduces a divisible good in the market and chooses a vector $\mathbf{p} \in \mathbb{R}^n$ of prices, where p_i is the price offered to the ith agent. In the second stage, the underlying network is realized, and agents choose the amount of goods to purchase at their announced price. For each $i \in \mathcal{V}$, we let $x_i \in \mathbb{R}_+$ denote the amount of goods agent i decides to purchase and refer to it as the *action* of agent i. We denote the vector of all actions by $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. The utility of each agent depends on that agent's price, choice, the choice of actions by other agents, and the underlying network W. The utility of agent $i \in \mathcal{V}$ is given by

$$u_i(\mathbf{x}; W, \mathbf{b}, \mathbf{p}) = -\frac{1}{2}x_i^2 + b_i x_i + \alpha \sum_{j \in \mathcal{V} \setminus \{i\}} W_{ij} x_i x_j - p_i x_i,$$

where the first two terms denote the value obtained by agent i's own consumption, which is concave in the purchased quantity; the third term represents the positive network effect obtained from the consumption by agent i's neighbors; and the last term is the purchase cost. We assume $\alpha \in \mathbb{R}_+$. We also let $b_i \in \mathbb{R}$ be an agent-specific parameter that represents the optimal individual consumption quantity of agent i in the absence of network externalities when $p_i = 0$. We refer to b_i as the primary value of agent i. The network externalities, encoded by the weight matrix W, capture the pattern and strength of interactions among agents.

In the first stage, the monopolist sets price vector $\mathbf{p}$ to maximize the profit:

$$\max_{\mathbf{p}} \sum_i x_i p_i - c x_i,$$

where p_i denotes the price offered to agent i and c denotes the marginal production cost. For any $W, \alpha, \mathbf{b}$, and $\mathbf{p}$, we let $(W, \alpha, \mathbf{b}, \mathbf{p})$ denote the consumption game among n agents in the second stage. In equilibrium, all agents simultaneously decide on their consumption quantity x_i to maximize their utility given the offered prices in anticipation of the consumption levels of all other agents, $\mathbf{x}_{-i}$; that is, $x_i \in \arg\max_{y_i \in \mathbb{R}^+} u_i(y_i, \mathbf{x}_{-i}; W, \mathbf{b}, \mathbf{p})$, where $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ denotes the actions of all agents except agent i.

2.2. Equilibrium Characterization of the Network Game

Similar to Candogan et al. (2012), we make the following assumptions throughout the paper.

Assumption 1 (Regularity). *Given the weight matrix W, we assume α is small enough such that $\frac{1}{\alpha} > \max_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} W_{ij}$.*

Assumption 2 (Positive Consumption with No Network Effect). *We assume that $\mathbf{b} > c\mathbf{1}$.*

Assumption 1 ensures that, in equilibrium, the consumption levels are bounded, and there exists a subgame perfect equilibrium. Assumption 2 ensures that if the item takes its lowest price (i.e., $p_i = c$ for all i), agents find it beneficial to consume even without any network effect.

We use the following notation.

Definition 1 (Consumption Game Equilibrium). For any consumption game $(W, \alpha, \mathbf{b}, \mathbf{p})$ with a unique consumption equilibrium, we denote by $\mathbf{x}(W, \alpha, \mathbf{b}, \mathbf{p})$ its equilibrium, with a slight abuse of notation. The subgame perfect equilibrium of the two-stage pricing-consumption game is denoted by $(\mathbf{p}, \mathbf{x})$, and the optimal revenue of the monopolist in this equilibrium is denoted by $R(W, \alpha, \mathbf{b})$.

We next review the consumption equilibrium for a given price vector $\mathbf{p}'$ in the second stage and the optimal pricing decision of the seller in the first stage based on Candogan et al. (2012).

Proposition 1 (The Pricing-Consumption Game (Candogan et al. 2012)). *Suppose Assumptions 1 and 2 hold. We then have the following:*

(a) *The consumption game $(W, \alpha, \mathbf{b}, \mathbf{p}')$ is an exact potential game, and the myopic best response among agents converges to the unique consumption equilibrium. Moreover, if $\mathbf{b} > \mathbf{p}'$, the consumptions in the equilibrium are strictly positive and satisfy $\mathbf{x}(W, \alpha, \mathbf{b}, \mathbf{p}') = (I - \alpha W)^{-1}(\mathbf{b} - \mathbf{p}')$.*

(b) *The equilibrium outcome of the pricing-consumption two-stage game $(W, \alpha, \mathbf{b})$ is*

$$(\mathbf{p}, \mathbf{x}) = \left(\frac{\mathbf{b} + c\mathbf{1}}{2}, (I - \alpha W)^{-1}\left(\frac{\mathbf{b} - c\mathbf{1}}{2}\right)\right),$$

and the optimal revenue of the monopolist is

$$R(W, \alpha, \mathbf{b}) = \left(\frac{\mathbf{b} - c\mathbf{1}}{2}\right)^{\top} (I - \alpha W)^{-1}\left(\frac{\mathbf{b} - c\mathbf{1}}{2}\right).$$

Proposition 1 implies that even though for symmetric networks the optimal price vector is independent of the underlying network structure, the firm's revenue is a function of the network structure captured by the matrix of $(I - \alpha W)^{-1}$, which represents how well connected the agents are in the network.

3. Pricing with Unknown Uncertain Network Externalities

In this section, we take a robust optimization approach and characterize how much the revenue changes as a function of the maximum change in the expected network externalities.

3.1. Revenue Volatility: Uncertainty Set

To state the setting and the results, we first introduce some notation and definitions. We show that the maximum volatility as a result of unknown uncertainties depends on the largest eigenvalue of the network. We measure the size of the uncertainty by the L_2 norm of the difference between the estimated and realized networks and refer to this size as the perturbation.[4] In this section, for notational convenience, we refer to $\lambda_{\max}(W)$ and $\lambda_{\min}(W)$ as $\lambda_{\max}$ and $\lambda_{\min}$, respectively. For other matrices, such as $\tilde{W}$, we use $\lambda_{\max}(\tilde{W})$ explicitly.

Definition 2 (ϵ-Perturbed Matrices). For a given perturbation level ϵ, we define the uncertainty set of ϵ-perturbed matrices (or networks) as

$$\mathcal{W}(W, \epsilon) \triangleq \{\tilde{W} \,:\, \|W - \tilde{W}\|_2 \le \epsilon \|W\|_2, \tilde{W} \ge 0, \tilde{W}^{\top} = \tilde{W}\}.$$

For instance, if the seller's forecast of the externalities is W and any possible true externalities matrix $\tilde{W}$ satisfies $W \in \mathcal{W}(\tilde{W}, \epsilon)$, then the forecast accuracy is at least ϵ (i.e., the relative distance between W and $\tilde{W}$ is at most ϵ). We characterize the revenue volatility among all ϵ-perturbed networks and highlight how it depends on the underlying network characteristics. Before stating the results, we define revenue volatility.

Definition 3 (Marginal Change and Revenue Volatility). For a given network W and perturbed network $\tilde{W}$, we denote by $\mathrm{RV}(W, \tilde{W})$ the maximum marginal change in the revenue because of the change in network externalities, where the maximum is taken over all possible values of $\mathbf{b}$:

$$\mathrm{RV}(W, \tilde{W}) \triangleq \sup_{\mathbf{b} > c\mathbf{1}} \frac{|R(W, \alpha, \mathbf{b}) - R(\tilde{W}, \alpha, \mathbf{b})|}{R(W, \alpha, \mathbf{b})}.$$

We denote by $\mathrm{RV}(W, \epsilon)$ the maximum relative effect on the seller's revenue from a perturbation drawn from the set $\mathcal{W}(W, \epsilon)$. For a perturbation level ϵ, we define the revenue volatility for a given uncertainty set as

$$\mathrm{RV}(W, \epsilon) \triangleq \max_{\tilde{W} \in \mathcal{W}(W, \epsilon)} \mathrm{RV}(W, \tilde{W}).$$

Because $\mathrm{RV}(W, \tilde{W})$ is the maximum relative change in the revenue for any given perturbation, where the maximization is taken over all possible primary values $\mathbf{b}$, $\mathrm{RV}(W, \epsilon)$ represents the worst-case change in the revenue over all ϵ-level perturbations, and hence, it captures how volatile the revenue is when the underlying network is drawn from the uncertainty set.

As implied by Proposition 1, the seller's ex ante pricing decision stays unchanged in the presence of perturbation at $\mathbf{p} = (\mathbf{b} + c\mathbf{1})/2$. However, consumers' consumption levels will change as they observe the perturbed network effects. These changes will subsequently impact the revenue of the seller.

3.2. Tight Upper Bound on Revenue Volatility

Our key result for a given uncertainty set, presented next, tightly characterizes revenue volatility over the set of ϵ-perturbed networks as a function of the largest eigenvalue of the estimated network.

Theorem 1. *Suppose Assumptions 1 and 2 hold. For any given ϵ such that*

$$\epsilon < \frac{1}{\alpha \lambda_{\max}(W)} - 1, \tag{1}$$

we have that for all $\tilde{W} \in \mathcal{W}(W, \epsilon)$, an interior consumption

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equilibrium exists. Moreover, we have

$$\mathrm{RV}(W,\epsilon)=\frac{\alpha\lambda_{\max}(W)\epsilon}{1-\alpha\lambda_{\max}(W)(1+\epsilon)}. \tag{2}$$

The tight bound on the impact of a perturbation within an uncertainty set on the revenue characterized in Theorem 1 has the following implications.

First, revenue volatility is increasing in α. Recall that α is the coefficient of the network externalities and represents the significance of the network externality compared with the primary value of an agent for consumption. Therefore, when α is larger, the consumption utility of agents depends more on the network externalities; hence, the revenue volatility is increasing in α. Consequently, when two monopolists face the same level of uncertainty, the monopolist who is selling a product with a higher dependency on the social network (i.e., larger α) would experience, in the worst case, a more significant change in the revenue.

Second, by increasing ϵ, the uncertainty set of possible perturbations expands, and thus, the revenue volatility increases. In view of Theorem 1, the monopolist should invest more in gathering information on the true network externalities to reduce the size ϵ of the uncertainty set and, as a result, to mitigate the revenue volatility.

Third, revenue volatility is increasing in the largest eigenvalue $\lambda_{\max}(W)$. The largest eigenvalue of W (with n nodes) can be expressed in terms of the dynamic mean of node degrees of the network as $\lambda_{\max}(W)=\lim_{k\to\infty}(N_k(W)/n)^{1/k}$, where $N_k(W)$ represents the weighted sum of walks in W of length k, and the weight of each walk is proportional to the product of the weights of the edges on the walk (Cvetković 1971). Intuitively, $\lambda_{\max}(W)$ represents an aggregate value of the influence of agents on each other. Such a higher value represents a larger aggregate influence and leads to a larger impact of any given network perturbation of any size.

Therefore, the revenue volatility increases in $\lambda_{\max}(W)$. Note that higher connectivity often implies a higher expected revenue for the monopolist. Our result states that higher connectivity can also imply a higher revenue volatility.

3.3. Impact of the Network Structure

In this subsection, we explain how revenue volatility depends on the network structure by combining Theorem 1 and the spectral properties of the underlying network.

Proposition 2. *Suppose Assumption 1 holds. For any $\epsilon>0$, we have:*

(a) *Increasing the weight of any edge increases the revenue volatility. In particular, for any unweighted network (i.e., $W_{ij}\in\{0,1\}$, with a weight of 1 for all existing edges), adding links to the network increases the revenue volatility.*

(b) *Among all connected unweighted tree networks, the chain network has the smallest revenue volatility, and the star network has the largest revenue volatility.*

Proposition 2(a) follows from the fact that the largest eigenvalue increases in the (nonnegative) weights of the edge of the network. Proposition 2(b) follows theorems 2 and 3 from Lovász and Pelikan (1973), stating that among connected trees on n vertices, the chain network has the lowest largest eigenvalue, and the star network has the highest largest eigenvalue. Proposition 2 implies that as the network becomes denser, the externalities among consumers increase, resulting in more revenue volatility for any perturbation in the network. Hence, among unweighted networks, the complete network, with all agents connected, has the largest revenue volatility. Moreover, the chain network has the lowest revenue volatility because the connections are spread out, implying that a given perturbation to the network structure has the smallest impact on the overall consumption and hence the revenue.

In order to better relate the revenue volatility to the network properties, we adopt some basic results from spectral graph theory—more specifically, the results of Nosal (1970), Hofmeister (1988), and Wilfe (1967). We first introduce two definitions and then summarize and restate the results that we will use.

Definition 4. For any network whose weights are represented by the matrix W, we let

$$d_{\max}(W)=\max_{1\le i\le n}\sum_{j=1}^{n}W_{ij},\qquad d_{\mathrm{avg}}(W)=\frac{1}{n}\sum_{i,j=1}^{n}W_{ij}.$$

Lemma 1 (Wilfe 1967, Nosal 1970, Hofmeister 1988). *For any adjacency matrix W of an undirected network (either weighted or unweighted), we have $d_{\mathrm{avg}}(W)\le\lambda_{\max}(W)\le d_{\max}(W)$.*

Invoking this lemma in Theorem 1, we obtain the following (with the proof omitted for brevity).

Corollary 1. *Given a network W, suppose Assumption 1 holds and $\epsilon<\frac{1}{\alpha d_{\max}(W)}-1$. We have*

$$\frac{\epsilon\alpha}{\frac{1}{d_{\mathrm{avg}}(W)}-\alpha(1+\epsilon)}\le \mathrm{RV}(W,\epsilon)\le\frac{\epsilon\alpha}{\frac{1}{d_{\max}(W)}-\alpha(1+\epsilon)}.$$

Moreover, for all d-regular networks (i.e., networks with all nodes of degree d), we have

$$\mathrm{RV}(W,\epsilon)=\frac{\epsilon\alpha}{\frac{1}{d}-\alpha(1+\epsilon)}.$$

Corollary 1 implies that for a given number of edges in the network, denoted by m, the least volatile network is the $2m/n$-regular network (recall that n is the number of nodes/agents in the network). This is because for any network W, $\lambda_{\max}(W)\ge d_{\mathrm{avg}}(W)=2m/n$, and for the $2m/n$-regular network RG, Lemma 1 implies $\lambda_{\max}(RG) = d_{\max}(RG) = d_{\text{avg}}(RG) = 2m/n$. This result shows that from a robust perspective for revenue maximization with endogenous prices, given the same average degree, the most preferred underlying network is the regular (balanced) network.

We next illustrate the results of Theorem 1 and Corollary 1 with numerical examples.

Example 1. We suppose $\epsilon < 2/3$ and $\alpha = 1/5$ and compare the revenue volatility of a star network with that of a complete network, both with four nodes. In the star network, the adjacency matrix S has $S_{i1} = 1$ for all $i \in \{2,3,4\}$ (and $s_{ij} = 0$ otherwise). We have $\lambda_{\max}(S) = \sqrt{3}$ so that $\text{RV}(S,\epsilon) = \sqrt{3}\epsilon/(5 - \sqrt{3} - \sqrt{3}\epsilon)$. In the complete network, the adjacency matrix C has $C_{ij} = 1$ for all $i,j \in \{1,\ldots,4\}, i \neq j$ (and $C_{ii} = 0$ for all i). We have $\text{RV}(C,\epsilon) = 3\epsilon/(2 - 3\epsilon)$.

Because, for $\epsilon \in [0,2/3]$, we have $\text{RV}(C,\epsilon) - \text{RV}(S,\epsilon) \geq 0$ and increasing in ϵ, the revenue volatility of the complete network is larger than that of the star network, as established by Proposition 2(a). □

Example 2. Suppose we have 11 agents with $\alpha = 1/20$ and $\epsilon = 1/2$. If the underlying network externality matrix is a complete network with weights equal to 1, denoted by C, then $\lambda_{\max}(C) = 10$, and the revenue volatility is 1. If the underlying network externality matrix is a star network with weights equal to 1, denoted by S, then we have $\lambda_{\max}(S) = 3.16$, and the revenue volatility becomes 0.1. In Figure 1(a), we display the revenue volatility for a network externality matrix given by $\beta S + (1-\beta)C$ as a function of $\beta \in (0,1)$. We see that as β increases, the revenue volatility decreases. This is because as β increases, the sum of the edge weights decreases, and as Proposition 2(a) establishes, the revenue volatility decreases. In Figure 1(b), we display the revenue volatility for a network externality matrix given by $\beta\overline{S} + (1-\beta)\overline{C}$ as a function of $\beta \in (0,1)$. Here, $\overline{S}$ and $\overline{C}$ are normalized network externality matrices of star and complete networks, respectively, such that the sum of the weights in both networks is equal to 1 (i.e., $\overline{C} = C/\sum_{i,j}C_{ij}$ and $\overline{S} = S/\sum_{i,j}S_{ij}$); in contrast to the previous case, the revenue volatility increases in β. This is because as β increases while the sum of the edge weights of the network is fixed, Corollary 1 establishes that a regular network (the complete network in this example) is less volatile. Note that the revenue volatility is monotonically increasing and convex in the largest eigenvalue of the underlying network. Moreover, $\lambda_{\max}(\beta\overline{S} + (1-\beta)\overline{C})$ is increasing and convex in β (see Proposition 3(a)). Therefore, the revenue volatility is increasing and convex in β. □

Now we formalize the observation from the previous example about the impact of network asymmetry when the sum of all weights is fixed. Given a regular network RW and any other network W, we define $\overline{RW} = RW/\sum_{i,j}RW_{ij}$ and $\overline{W} = W/\sum_{i,j}W_{ij}$. We next show the following result.

Proposition 3. *Given any network W and any regular network RW on the same set of vertices, we have that*

(a) *$\lambda_{\max}(\beta\overline{W} + (1-\beta)\overline{RW})$ is increasing and convex in β, and hence, the revenue volatility of $\beta\overline{W} + (1-\beta)\overline{RW}$ is increasing and convex in β; and*

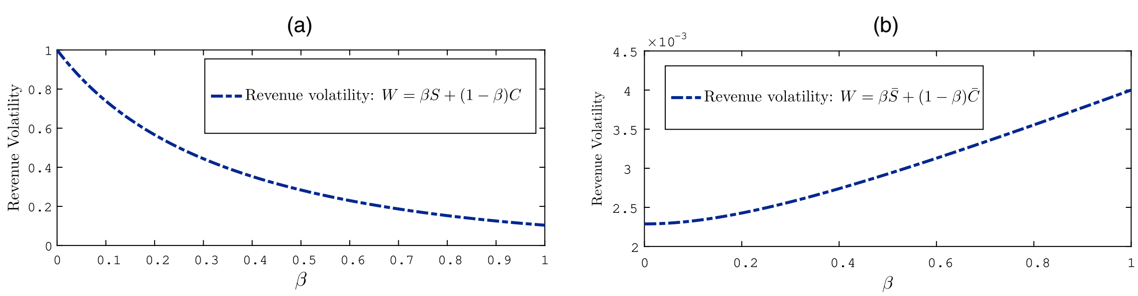
(b) *$\lambda_{\max}(\beta W + (1-\beta)SW)$ is increasing in β, where SW is a subgraph of W over the same set of vertices as W, and hence, the revenue volatility of $\beta W + (1-\beta)SW$ is increasing in β.*

Note that as β increases, the network $\beta\overline{W} + (1-\beta)\overline{RW}$ becomes less balanced. Then, Theorem 1 and Proposition 3 imply that when the sum of all weights is fixed, the revenue volatility increases as the network becomes less balanced.

4. Pricing with Known Uncertain Network Externalities

In this section, we characterize the impact of stochastic uncertainty on the revenue in large networks. We suppose that the monopolist estimates the value of externalities among agents. Then, the monopolist forecasts

Figure 1. (Color online) (a) The Revenue Volatility for the Network Externality Matrix $W = \beta S + (1-\beta)C$ as a Function of β, Where S Is the Network Externality Matrix of a Star Network with 11 Nodes (and Weights 1) and C Is the Network Externality Matrix of a Complete Network with 11 Nodes (and Weights 1); (b) the Revenue Volatility for the Network Externalities $W = \beta\overline{S} + (1-\beta)\overline{C}$ as a Function of β, Where $\overline{S} = S/\sum_{i,j=1}^{n} S_{ij}$ and $\overline{C} = C/\sum_{i,j=1}^{n} C_{ij}$



Note. In both panels, we have $\alpha = \frac{1}{20}$ and $\epsilon = \frac{1}{2}$.

the revenue based on the expected value of the externalities. As mentioned, the optimal price ex ante would be the same as in the setting with deterministic network externalities. However, the realized externalities could differ from the estimates, leading to a different revenue from that under the estimated externalities.

4.1. Revenue Volatility: Known Uncertainty

In a setting with n consumers, we let $\tilde{W}_n = W_n + G_n$ denote the matrix of actual network externalities, where G_n is a Wigner matrix of small random noises with mean 0, and W_n is the underlying matrix of expected network externalities (i.e., the estimate of the monopolist).

Our goal is to characterize how much the stochastic uncertainty affects the asymptotic revenue in expectation. In particular, we show that even though the additional noise (i.e., G_n) has a mean of 0, the expected revenue of the seller is not equal to the estimated revenue according to the expected value of externalities (i.e., W_n). We then exactly characterize the expected revenue in the presence of such noises. We first define Wigner matrices; see Anderson et al. (2010, chap. 2).

Definition 5 (Wigner Matrix with Finite Moments). Matrix $G_n \in \mathbb{R}^{n\times n}$ is called a Wigner matrix if $G_n = n^{-1/2}Y_n$, where Y_n is a symmetric matrix such that for all $i < j$, $[Y_n]_{ij}$ are independent and identically distributed zero-mean random variables with moments equal to $\mathbb{E}([Y_n]_{12}^k)$, and for all i, $[Y_n]_{ii}$ are independent and identically distributed zero-mean random variables with moments equal to $\mathbb{E}([Y_n]_{11}^k)$. Moreover, all moments of Y_n are finite; that is, for all $k \geq 1$, we have $\max(\mathbb{E}[|[Y_n]_{ij}|^k], \mathbb{E}[|[Y_n]_{ii}|^k]) < \infty$. We denote the variance of off-diagonal entries by σ^2 (i.e., $\sigma^2 = \mathbb{E}[([Y_n]_{12})^2]$).

Note that we require the additional assumption that all finite moments of the entries of matrix Y_n are bounded. Wigner matrices typically only require the second moments to be bounded.

In what follows, we make an assumption regarding the noise matrix.

Assumption 3. *For a network with n nodes, the noise matrix G_n is a Wigner matrix with finite moments. Moreover, for all $1 \leq i \leq n$, we assume that $[G_n]_{ii} = 0$, the supports of the random variables are such that $W_n + G_n \geq 0$, and $\rho(W_n)$ is finite.*

Assumption 3 on the noise matrix captures a setting in which the inaccuracies in estimating the externalities among different pairs of consumers are identically distributed with zero mean and bounded moments. Note that there are $n-1$ random variables (i.e., $[G_n]_{ij}$, $j \in [n]$, $j \neq i$) that are involved in the utility of consumer i. Then, the normalization of Y_n by $\sqrt{n}$ guarantees that for a large n, the overall uncertainty in consumers' utilities does not grow with n. In particular, similar to the central limit theorem, the normalization by $\sqrt{n}$ enables the characterization of the limit of this matrix and its spectral properties as n grows (see Wigner 1958). Finally, the assumption $W_n + G_n \geq 0$ guarantees that the externalities are nonnegative. For instance, if $\min_{ij}[W_n]_{ij} > 0$ and $\min_{ij}[G_n]_{ij} \geq -\min_{ij}[W_n]_{ij}$, then $W_n + G_n \geq 0$.

Note that the seller's optimal ex ante pricing decision is always $\mathbf{p} = (\mathbf{b} + c\mathbf{1})/2$. This is because the matrix $W_n + G_n$ is symmetric (see, e.g., Candogan et al. 2012, corollary 1). Therefore, the optimal price remains the same even when there is uncertainty regarding the network structure. However, the consumers' consumption level will depend on the realized and observed network. This will subsequently impact the realized revenue of the seller. We define the expected revenue volatility as the expectation of the difference between the revenue obtained from W_n and the revenue obtained from the uncertain matrix of network externalities $W_n + G_n$.

Definition 6 (Expected Revenue Volatility Under Random Noise). For a given random matrix G_n, define

$$\Delta^R(W_n, G_n) = \mathbb{E}[R(W_n, \alpha, \mathbf{b}) - R(W_n + G_n, \alpha, \mathbf{b})]$$

as the expected difference between the revenue estimate when using the expected externalities and the actual revenue obtained under the realized network externalities.

4.2. Tight Characterization of Revenue Volatility

In the next theorem, we explicitly characterize, for each $\mathbf{b}$, the expected revenue volatility as a function of W_n, α, and the variance of the entries of G_n (i.e., $\sigma^2 = \mathbb{E}\big[([Y_n]_{12})^2\big]$). Recall that $\rho(W_n)$ is the spectral radius of the matrix W_n of the expected weights with n consumers.

Theorem 2. *Suppose the random noise matrix G_n satisfies Assumption 3. Given an underlying network of externalities with mean strengths W_n, if $\sigma \leq 1/(2\alpha) - \rho(W_n)/2$, we have*[5]

$$\Delta^R(W_n, G_n) = -\left(\frac{\mathbf{b} - c\mathbf{1}}{2}\right)^\top \left(\sum_{k=0}^{\infty} \alpha^k W_n^k \sum_{i=1}^{\infty} C(i,k)\alpha^{2i}\sigma^{2i}\right)\left(\frac{\mathbf{b} - c\mathbf{1}}{2}\right) + O\left(\frac{1}{n}\right), \tag{3}$$

with

$$\begin{aligned} C(i,k) &= \sum_{t=0}^{k}\sum_{j=1}^{i} c(j-1)C(i-j, k-t), \qquad i,k > 0,\\ C(i,0) &= c(i) = \frac{1}{i+1}\binom{2i}{i},\\ C(0,k) &= 1, \end{aligned} \tag{4}$$

where $c(j-1)$ is the $(j-1)$th Catalan number.

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Theorem 2 provides an exact asymptotic characterization of the expected revenue of the seller and explains how to compute its value. The proof of this theorem is presented in Online Appendix B.2, and its intuition is as follows. We first expand $R(W_n, \alpha, \mathbf{b}) - R(W_n + G_n, \alpha, \mathbf{b})$ (i.e., the difference between the revenue estimate when using the expected externalities and the actual revenue obtained given the realized network) in terms of the entries of matrices W_n and G_n. This expansion has a summation over multiple terms, each involving products of multiple entries of W_n and G_n. By using properties of Wigner matrices, we first identify a collection of terms whose expectation is 0. We then properly group the remaining terms (i.e., those with a nonzero mean) and use a combinatorial counting argument to prove their coefficient.

We next demonstrate and discuss the implications of this theorem. First, note that, as (3) immediately implies, the expected revenue in the presence of uncertainty for the seller is *higher* than the revenue obtained from the expected network. This observation can be generalized for a general random matrix W by noting the convexity of $R(W)$ in w_{ij} for all i, j and applying Jensen's inequality separately for each entry w_{ij}. Nevertheless, Theorem 2, in particular, (3), exactly characterizes that the asymptotic magnitude of the revenue volatility increases with $(\mathbf{b} - c\mathbf{1})^\top W_n^k (\mathbf{b} - c\mathbf{1})$, which leads to a higher expected revenue. Note that the entries of the matrix W_n^k represent the number of walks of length k between any pair of two nodes. Therefore, $(\mathbf{b} - c\mathbf{1})^\top W_n^k (\mathbf{b} - c\mathbf{1})$ is a weighted sum of the overall number of walks of length k. As a result, increasing the number of walks in a network increases the magnitude of the expected revenue volatility.

We next study the expected revenue volatility under the stochastic noise by fixing the sum of the network weights but varying the balance of the network. Recall that for any network W, we define $\overline{W} = W / \sum_{i,j} W_{ij}$, and let $F(W, RW, \beta) = \beta \overline{W} + (1 - \beta)\overline{RW}$, where RW is a regular network on the same vertices as W. In the next proposition, we show that when we consider up to the second-order-connectivity cascade effect, the absolute value of the right-hand side of (3) increases with β. This result implies that for networks with a fixed sum of weights, the absolute value of the revenue volatility, up to the second-order-connectivity cascade effect, is decreasing as the network becomes more balanced.

Proposition 4. *Suppose* $(\mathbf{b} - c\mathbf{1})/2 = \mathbf{1}$. *For a given network* W *and for any regular network* RW *on the same set of vertices,* $\mathbf{1}^\top (\sum_{k=0}^{2} \alpha^k F(W, RW, \beta)^k \sum_{i=1}^{\infty} C(i,k) \alpha^{2i} \sigma^{2i}) \mathbf{1}$ *is increasing in* β.

5. Conclusion

We consider the optimal pricing problem of a monopolist in selling a network good and focus on the effects of uncertainty in the underlying network structure. The firm makes inferences on the underlying network structure to decide on the pricing decisions and obtain an estimate of the corresponding revenue. However, firms may never know the true network externalities ex ante and therefore may have some uncertainty about their optimal actions and the resulting revenue. For the quadratic individual utility with local network externalities, the optimal pricing decisions stay the same regardless of the network uncertainty. We quantify the impact of this uncertainty on the optimal revenue and characterize its dependence on the underlying network structure. In particular, we take both robust and stochastic optimization perspectives, with the former characterized by an uncertainty set and the latter by a probabilistic distribution. In the former case, we suppose that the estimated matrix of network externalities is at most ϵ away from the true matrix of externalities and then characterize the worst-case relative revenue difference, referred to as revenue volatility. We establish that revenue volatility depends on the largest eigenvalue of the network externality matrix, which provides a natural ranking of networks in terms of the worst-case deviation in their revenues. In the latter setting, we consider uncertainties in the form of a modified Wigner matrix and characterize the expected revenue volatility. The characterization depends on a weighted sum of all walks of different lengths in the expected underlying network. From both perspectives, we relate revenue volatility to the structural properties of the estimated/expected network. In particular, if all the edge weights equal 1, we show that for both settings, when the underlying network is less balanced, revenue volatility is lower. However, if the sum of all the edge weights is fixed, we show that for both settings, when the underlying network is more balanced, revenue volatility is lower. Nevertheless, the two perspectives favor revenue volatility in the opposite direction. This is because they reflect the two sides of the same coin, with the former on the risk under network uncertainty and the latter on the associated risk premium, though they require different information structures.

A key assumption that simplifies our analysis is that the underlying network structure is symmetric; that is, the network externality one user imposes on another is mutual. In the setting with unknown uncertain network externalities, this symmetry assumption allows us to simplify the conditions to be expressed only in terms of the spectral radius or the maximum eigenvalue of the externality matrix, under which we can quantify the revenue volatility. The results could be much more complicated without the symmetry assumption, which we leave for future research. In the setting with known uncertain network externalities, this symmetry assumption allows us to resort to and build on the asymptotic properties of Wigner matrices (in particular, Bai and Yin 1988, see theorem OA.3).

Endnotes

[1] Even if consumers only have local network information, the equilibrium consumption can be achieved through a *tâtonnement* best-response process regardless of the starting point.

[2] Note that the eigenvector corresponding to the largest eigenvalue may not be unique. For the largest eigenvalue, we select the corresponding vector with nonnegative entries. Note that such a vector exists because of the Rayleigh quotient definition of the largest eigenvalue when the entries of the matrix are all nonnegative. For the smallest eigenvalue, we choose one of the corresponding vectors arbitrarily.

[3] The connectivity assumption is without loss of generality. If the network is not connected, we can treat each connected component as a separate connected network.

[4] Throughout the paper, we adopt the L_2 norms. The L_2 norm of a matrix $W \in \mathbb{R}^{n\times n}$ is $\|W\|_2 = \sup_{\mathbf{x}\neq 0}\frac{\|W\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$, where for any vector $\mathbf{x} = (x_1, \ldots, x_n)$, its L_2 norm is $\|\mathbf{x}\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$. Note that for symmetric matrices with nonnegative entries, $\|W\|_2 = \lambda_{\max}(W)$.

[5] A term $f(n)$ is $O(1/n)$ means that there exist n_0 and C such that for all $n \geq n_0$, $|f(n)| \leq C/n$.

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