Online Appendix to the Paper "No Claim? Your Gain: Design of Residual Value Extended Warranties under Risk Aversion and Strategic Claim Behavior"

Lemma 1. *Given any* $x \geqslant y \geqslant 0$, $(e^{\gamma x} - e^{\gamma y})/\gamma$ *is increasing in* γ. *Moreover,* $(e^{\gamma x} - e^{\gamma y})/\gamma \geqslant x - y$ *if* $\gamma > 0$*;* $(e^{\gamma x} - e^{\gamma y})/\gamma \leqslant x - y$ *if* $\gamma < 0$.

Proof of Lemma 1. Notice that $(e^{\gamma x} - e^{\gamma y})/\gamma = (e^{\gamma(x-y)} - 1)/\gamma \cdot e^{\gamma y}$. Because both $(e^{\gamma(x-y)} - 1)/\gamma$ and $e^{\gamma y}$ are non-negative for any γ and $x \geqslant y$, and $e^{\gamma y}$ is increasing in γ for any given $y \geqslant 0$, then it is sufficient to show that $(e^{\gamma(x-y)} - 1)/\gamma$ is increasing in γ.
Consider its first-order derivative with respect to γ

$$\frac{\partial\big((e^{\gamma(x-y)} - 1)/\gamma\big)}{\partial\gamma} = \frac{\gamma(x-y)e^{\gamma(x-y)} - (e^{\gamma(x-y)} - 1)}{\gamma^2}.$$

Let $G(\gamma) := e^{\gamma(x-y)} - 1$. Then, $\partial G(\gamma)/\partial\gamma \geqslant 0$ and $\partial^2 G(\gamma)/\partial\gamma^2 \geqslant 0$ for any given $x \geqslant y$, so $G(\gamma)$ is increasing convex in γ. Note that $G(0) = 0$. Apparently, for any $\gamma \geqslant 0$,

$$G(\gamma) = G(0) + \int_0^\gamma G'(\tau)d\tau \leqslant \gamma G'(\gamma).$$

We note that the above inequality also holds for any $\gamma \leqslant 0$ because $G(\gamma) = G(0) - \int_\gamma^0 G'(\tau)d\tau \leqslant -(0 - \gamma)G'(\gamma) = \gamma G'(\gamma)$. Thus,

$$\frac{\partial\big((e^{\gamma(x-y)} - 1)/\gamma\big)}{\partial\gamma} = \frac{\gamma G'(\gamma) - G(\gamma)}{\gamma^2} \geqslant 0,$$

and given any $x \geqslant y \geqslant 0$, $(e^{\gamma x} - e^{\gamma y})/\gamma$ is increasing in γ. Consider the limit as γ goes to zero, $\lim_{\gamma\to 0}(e^{\gamma x} - e^{\gamma y})/\gamma = \lim_{\gamma\to 0}(xe^{\gamma x} - ye^{\gamma y}) = x - y$. Therefore, $(e^{\gamma x} - e^{\gamma y})/\gamma \geqslant x - y$ if $\gamma > 0$; $(e^{\gamma x} - e^{\gamma y})/\gamma \leqslant x - y$ if $\gamma < 0$.
Furthermore, there exist tighter bounds for $(e^{\gamma x} - e^{\gamma y})/\gamma$, e.g., for any $x \geqslant y \geqslant 0$,

$$\min(e^{\gamma x}, e^{\gamma y}) \cdot (x - y) \leqslant (e^{\gamma x} - e^{\gamma y})/\gamma = \int_y^x e^{\gamma\tau} d\tau \leqslant \max(e^{\gamma x}, e^{\gamma y}) \cdot (x - y).$$

Proof of Theorem 1. Apparently, the optimal claim policy has a threshold structure: it is optimal for a customer with risk attitude γ to place a claim at time t for a failure with repair cost C_t if and only if $C_t \geqslant g(t;\gamma,r)$. Moreover, it is straightforward that $g(t;\gamma,r)$ is decreasing in t and increasing in r, noting that t represents the time-to-go.
We next show the monotonic comparative statics of $g(t;\gamma,r)$ with respect to γ. Suppose $g(t;\gamma,r) \leqslant g(t;\gamma',r)$ at time t for any $\gamma > \gamma'$. We will next show $g(t+\delta;\gamma,r) \leqslant g(t+\delta;\gamma',r)$ for a sufficiently small $\delta > 0$. According to the differential equation (1),

$$\begin{aligned} g(t+\delta;\gamma,r) &= g(t;\gamma,r) - \frac{\lambda_t\delta}{\gamma}\big(E\big[e^{\gamma\cdot\min(C_t, g(t;\gamma,r))}\big] - 1\big) + o(\delta), \\ g(t+\delta;\gamma',r) &= g(t;\gamma',r) - \frac{\lambda_t\delta}{\gamma'}\big(E\big[e^{\gamma'\cdot\min(C_t, g(t;\gamma',r))}\big] - 1\big) + o(\delta). \end{aligned}$$

Then,

$$\begin{aligned} &\big(E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big] - 1\big)/\gamma - \big(E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}\big] - 1\big)/\gamma' \\ &\geqslant \big(E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))}\big] - 1\big)/\gamma' - \big(E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}\big] - 1\big)/\gamma' = \frac{1}{\gamma'}E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))} - e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}\big] \\ &\geqslant E\big[\Theta\cdot(\min(C_t,g(t;\gamma,r)) - \min(C_t,g(t;\gamma',r)))\big] \geqslant E\big[\Theta\cdot(g(t;\gamma,r) - g(t;\gamma',r))\big], \end{aligned}$$

where $\Theta = e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}$ if $\gamma' \geqslant 0$; $\Theta = e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))}$ if $\gamma \leqslant 0$. The first inequality holds because $E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big] - 1\big)/\gamma$ is increasing in γ by Lemma 1; the second inequality holds by a similar argument in the proof of Lemma 1; the third inequality holds because $\min(C_t, x) - \min(C_t, y) \leqslant (x - y)$ for any $x \geqslant y$.

Therefore, $g(t+\delta;\gamma,r) - g(t+\delta;\gamma',r) \leqslant \big(g(t;\gamma,r) - g(t;\gamma',r)\big) \cdot \Big(1 - \lambda_t \delta \Theta\Big) \leqslant 0$. Then, $g(t;\gamma,r)$ is decreasing with respect to the risk attitude γ.
The case with γ approaching $-\infty$:

$$g'(t;-\infty,r) = \lim_{\gamma\to-\infty} -\frac{\lambda_t}{\gamma}\Big(E\big[e^{\gamma\cdot\min(C_t, g(t;\gamma,r))}\big] - 1\Big) = 0.$$

Then, $g(t;-\infty,r) = g(0;-\infty,r) + \int_0^t g'(s;-\infty,r)ds = r$.
The case with γ approaching $+\infty$: for any positive time t

$$g'(t;\infty,r) = \lim_{\gamma\to+\infty} -\frac{\lambda_t}{\gamma}\Big(E\big[e^{\gamma\cdot\min(C_t, g(t;\gamma,r))}\big] - 1\Big) = -\lambda_t E\big[\min(C_t, g(t;\infty,r)) \cdot e^{\infty\cdot\min(C_t, g(t;\infty,r))}\big].$$

If $g(t;\infty,r) > 0$, then $g'(t;\infty,r) = -\infty$ and $g(t;\infty,r) = g(0;\infty,r) + \int_0^t g'(s;\infty,r)ds < 0$ for any positive t, which is impossible. Thus, the only solution to the above differential equation is $g(t;\infty,r) = 0$ for any positive t.

Proof of Proposition 1. (a). Note that $g(t;\gamma,r)$ is decreasing in t and $g(0;\gamma,r) = r$. When the time-to-go t is very small, $g(t;\gamma,r)$ is sufficiently close to r, so it is optimal for the customer not to claim any failure since $g(t;\gamma,r) \approx r > c$. Then, the differential equation (1) becomes $g'(t;\gamma,r) = -\frac{\lambda_t}{\gamma}\big(e^{\gamma c} - 1\big)$. Solving the above differential equation and combining with the boundary condition $g(0;\gamma,r) = r$ yields $g(t;\gamma,r) = r - \frac{1}{\gamma}(e^{\gamma c} - 1)\Lambda(t)$. Denote the unique solution to the equation $r - 1/\gamma \cdot (e^{\gamma c} - 1)\Lambda(t) = c$ with respect to t by t^*. Then, for any $t \geqslant t^*$, it is optimal to claim all the failures so the differential equation (1) becomes $g'(t;\gamma,r) = -\frac{\lambda_t}{\gamma}\big(e^{\gamma g(t;\gamma,r)} - 1\big)$, with boundary condition $g(t^*;\gamma,r) = c$. Similarly, the unique solution to the above differential equation is $g(t;\gamma,r) = -\frac{1}{\gamma}\ln\big(1 - e^{-\Lambda(t)+\Lambda(t^*)} \cdot (1 - e^{-\gamma c})\big)$.
(b). Under the exponential distribution, the differential equation (1) can be rewritten as follows

$$g'(t;\gamma,r) = -\frac{\lambda_t}{\gamma-\mu}\left(e^{(\gamma-\mu)g(t;\gamma,r)} - 1\right).$$

It is straightforward to verify that function (3) satisfies the differential equation (1) and its boundary condition. In particular,

$$\lim_{\gamma\to\mu} g(t;\gamma,r) = \lim_{\gamma\to\mu} \frac{e^{-\Lambda(t)}e^{-(\gamma-\mu)r}r}{1 - e^{-\Lambda(t)} \cdot \big(1 - e^{-(\gamma-\mu)r}\big)} = e^{-\Lambda(t)}r.$$

Proof of Proposition 2. We first consider the case $\gamma > 0$. Consider the first-order derivative with respect to γ,

$$\begin{aligned}\frac{\partial w_{\mathtt{tw}}(t;\gamma)}{\partial\gamma} &= \frac{1}{\gamma^2 E\big[e^{\gamma R(t)}\big]}\Big(\gamma E[R(t)e^{\gamma R(t)}] - E[e^{\gamma R(t)}]\log\big(E[e^{\gamma R(t)}]\big)\Big) \\ &\geqslant \frac{1}{\gamma E\big[e^{\gamma R(t)}\big]}\Big(E[R(t)e^{\gamma R(t)}] - E[R(t)]E[e^{\gamma R(t)}]\Big).\end{aligned}$$

The inequality holds because of the Jensen's inequality: $\log\big(E[e^{\gamma R(t)}]\big) \leqslant E[\log(e^{\gamma R(t)})] = \gamma E[R(t)]$. For a similar reason, $E[R(t)e^{\gamma R(t)}] \geqslant E[R(t)]e^{\gamma E[R(t)]}$. We will next show a stronger result, i.e., $E[R(t)e^{\gamma R(t)}] \geqslant E[R(t)]E[e^{\gamma R(t)}]$. First, suppose that $R(t)$ takes values from the discrete set $\{R(t)^1, \ldots, R(t)^n\}$ with respective probabilities $\alpha_1, \ldots, \alpha_n$, where $\alpha_1 + \cdots + \alpha_n = 1$. Then,

$$\begin{aligned}&E[R(t)e^{\gamma R(t)}] - E[R(t)]E[e^{\gamma R(t)}] = \sum_{i=1}^n \alpha_i R(t)^i e^{\gamma R(t)^i} - \sum_{i=1}^n \alpha_i R(t)^i \sum_{j=1}^n \alpha_j e^{\gamma R(t)^j} \\ &= \sum_{i=1}^n \alpha_i R(t)^i \Big(e^{\gamma R(t)^i} - \sum_{j=1}^n \alpha_j e^{\gamma R(t)^j}\Big) = \sum_{i=1}^n \alpha_i R(t)^i \sum_{j=1}^n \alpha_j \left(e^{\gamma R(t)^i} - e^{\gamma R(t)^j}\right) \\ &= \sum_{(i,j)} \alpha_i\alpha_j \big(R(t)^i - R(t)^j\big)\left(e^{\gamma R(t)^i} - e^{\gamma R(t)^j}\right) \geqslant 0,\end{aligned}$$

where (i,j) and (j,i) are considered the same pair. The inequality holds because $\left(R(t)^i - R(t)^j\right) \cdot \left(e^{\gamma R(t)^i} - e^{\gamma R(t)^j}\right) \geqslant 0$ for each pair (i,j). Similarly, we can show that the inequality also holds when $R(t)$ has a continuous support set. Therefore, for $\gamma > 0$

$$\frac{\partial w_{\mathtt{tw}}(t;\gamma)}{\partial\gamma} \geqslant \frac{1}{\gamma E\big[e^{\gamma R(t)}\big]}\Big(\sum_{(i,j)} \alpha_i\alpha_j \big(R(t)^i - R(t)^j\big)\left(e^{\gamma R(t)^i} - e^{\gamma R(t)^j}\right)\Big) > 0.$$

Similarly, we can show that the above inequality also holds for $\gamma < 0$. Thus, $w_{\mathtt{tw}}(t;\gamma)$ is increasing in γ.
We next consider the case with γ approaching 0 as follows,

$$\lim_{\gamma\to 0} w_{\mathtt{tw}}(t;\gamma) = \frac{\lim_{\gamma\to 0}\partial \ln\left(E[e^{\gamma R(t)}]\right)\gamma}{\lim_{\gamma\to 0}\partial\gamma/\partial\gamma} = \lim_{\gamma\to 0}\frac{E[R(t)e^{\gamma R(t)}]}{E[e^{\gamma R(t)}]} = E[R(t)].$$

It completes the proof. □

Proof of Proposition 3. Let N_t be a random variable following the Poisson distribution with mean $\Lambda(t) = \int_0^t \lambda_s ds$. Notice that $\sum_{i=1}^{N_t} C_i$ is a compound Poisson random variable. It is known that the total repair cost $R(t)$ has the same distribution as $\sum_{i=1}^{N_t} C_i$ assuming that the repair costs are i.i.d., independent of the failure process (see, e.g., Ross 1995). By equation (5),

$$\begin{aligned} w_{\mathtt{tw}}(t;\gamma) \;&=\; \frac{1}{\gamma}\log\left(E[e^{\gamma R(t)}]\right) = \frac{1}{\gamma}\log\left(E[e^{\gamma\sum_{i=1}^{N_t} C_i}]\right) = \frac{1}{\gamma}\log\left(E\big[E[e^{\gamma\sum_{i=1}^{N_t} C_i}|N_t]\big]\right) \\ &=\; \frac{1}{\gamma}\log\left(E\big[\Pi_{i=1}^{N_t}E[e^{\gamma C_i}|N_t]\big]\right) = \frac{1}{\gamma}\log\left(E\big[M_C(\gamma)^{N_t}\big]\right) = \frac{1}{\gamma}\log\left(e^{\Lambda(t)\cdot(M_C(\gamma)-1)}\right) = \frac{\Lambda(t)\cdot(M_C(\gamma)-1)}{\gamma}, \end{aligned}$$

where $M_C(\gamma)$ is the moment generating function of the repair cost C, i.e., $M_C(\gamma) = E[e^{\gamma C}]$. The second last equality holds because $E[x^{N_t}] = M_{N_t}(\log(x)) = e^{\Lambda(t)\cdot(x-1)}$.
Apparently, for the constant repair cost

$$w_{\mathtt{tw}}(t;\gamma) \;=\; \Lambda(t)\cdot(M_C(\gamma)-1)/\gamma = \Lambda(t)(e^{\gamma c}-1)/\gamma.$$

Moreover, $w_{\mathtt{tw}}(t;\gamma)\to\Lambda(t)c$ as $\gamma\to 0$ because $\lim_{\gamma\to 0}(e^{\gamma c}-1)/\gamma = c$.
For the exponential distributed repair cost,

$$w_{\mathtt{tw}}(t;\gamma) = \frac{\Lambda(t)\cdot(M_C(\gamma)-1)}{\gamma} = \frac{\Lambda(t)}{\mu-\gamma}.$$

The last equality holds because $M_C(\gamma) = \mu/(\mu-\gamma)$ for $\gamma < \mu$. Apparently, $w_{\mathtt{tw}}(t;\gamma)\to\Lambda(t)/\mu$ as $\gamma\to 0$. □

Proof of Proposition 4. The willingness-to-pay $w_{\mathtt{rvw}}(t;\gamma,r)$ is the quality such that the utility of buying and not buying an RVW is indifferent, taking into account the possible out-of-pocket cost and refund for the RVW, i.e.,

$$E[-e^{-\gamma(v-R(t))}] = -e^{-\gamma(v-w_{\mathtt{rvw}}(t;\gamma,r)+g(t;\gamma,r))}.$$

Combining with equation (5) yields $w_{\mathtt{rvw}}(t;\gamma,r) = w_{\mathtt{tw}}(t;\gamma) + g(t;\gamma,r)$.
(a). First, we consider the risk-neutral case, i.e., $\gamma = 0$. Note that $w_{\mathtt{tw}}(t;0) = E[R(t)]$ by Proposition 2, so we only need to show $h(t;0,r) = g(t;0,r) + E[R(t)]$. Recall that

$$\frac{\partial}{\partial t}\big(g(t;0,r)+E[R(t)]\big) = \lambda_t E\big[C_t - \min(C_t, g(t;0,r))\big] = \lambda_t E\big[\big(C_t - g(t;0,r)\big)^+\big].$$

Plugging $h(t;0,r) = g(t;0,r) + E[R(t)]$ into equation (4) results in

$$h'(t;0,r) = \lambda_t \Pr(C_t \geqslant g(t;0,r))\Big\{E[C_t|C_t\geqslant g(t;0,r)] - g(t;0,r)\Big\} = \lambda_t E\big[\big(C_t - g(t;0,r)\big)^+\big].$$

Combining it with the boundary condition $h(0;0,r) = g(0;0,r) + E[R(0)] = r$, we have obtained $h(t;0,r) = g(t;0,r) + E[R(t)]$ at any $t\geqslant 0$ for any given refund r.
Next, we consider the case of $\gamma > 0$. Suppose $h(t;\gamma,r)\geqslant g(t;\gamma,r)+E[R(t)]$ at some $t\geqslant 0$ for given $\gamma>0$ and r. We will next show that for any sufficiently small $\delta>0$, $h(t+\delta;\gamma,r)\geqslant g(t+\delta;\gamma,r)+E[R(t+\delta)]$. By the differential equations (1) and (4),

$$\begin{aligned} &h(t+\delta;\gamma,r) - g(t+\delta;\gamma,r) - E[R(t+\delta)] = h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)] + \delta\lambda_t\Bigg(\int_{g(t;\gamma,r)}^{\infty} c_t dF_t(c_t) \\ &\quad -(1-F_t(g(t;\gamma,r)))\cdot(h(t;\gamma,r)-E[R(t)]) + (-1+E[e^{\gamma\min(C_t,g(t;\gamma,r))}])/\gamma - E[C_t]\Bigg) + o(\delta) \\ &= h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)] + \delta\lambda_t\Bigg(-\int_0^{g(t;\gamma,r)} c_t dF_t(c_t) - (1-F_t(g(t;\gamma,r)))\cdot(h(t;\gamma,r)-E[R(t)]) \end{aligned}$$

$$+ (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma\Big) + o(\delta)$$
$$= \big(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]\big)\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r)))\big) + \delta\lambda_t\Big(- E[\min(C_t, g(t;\gamma,r))]$$
$$+ (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma\Big) + o(\delta)$$
$$\geqslant \big(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]\big)\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r)))\big) \geqslant 0.$$

The first inequality holds because $(-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma \geqslant E[\min(C_t, g(t;\gamma,r))]$ for any $\gamma > 0$ by Lemma 1; the second inequality holds because $h(t;\gamma,r) \geqslant g(t;\gamma,r) + E[R(t)]$. Therefore, $h(t;\gamma,r) \geqslant g(t;\gamma,r) + E[R(t)]$ at any $t \geqslant 0$ for given $\gamma > 0$ and r. Similarly, we can show $h(t;\gamma,r) \leqslant g(t;\gamma,r) + E[R(t)]$ at any $t \geqslant 0$ for given $\gamma < 0$ and r.
For the comparison between $h(t;\gamma,r)$ and $w_{\texttt{rvw}}(t;\gamma,r)$, suppose $h(t;\gamma,r) \geqslant g(t;\gamma,r) + w_{\texttt{tw}}(t;\gamma)$ at some $t \geqslant 0$ for given $\gamma < 0$ and r. We will next show that for a sufficiently small $\delta > 0$, $h(t+\delta;\gamma,r) \geqslant g(t+\delta;\gamma,r) + w_{\texttt{tw}}(t+\delta;\gamma)$.
The willingness-to-pay for the TW can be expressed as follows

$$w_{\texttt{tw}}(t+\delta;\gamma) = \frac{1}{\gamma}\log\big(E[e^{\gamma R(t+\delta)}]\big) = \frac{1}{\gamma}\log\big(E[e^{\gamma R(t)} \cdot e^{\gamma R(t,t+\delta)}]\big) = w_{\texttt{tw}}(t;\gamma) + \frac{\lambda_t \delta}{\gamma}\big(E[e^{\gamma C_t}] - 1\big) + o(\delta), \quad (21)$$

where $R(t, t+\delta)$ represents the total repair cost from time-to-go $t+\delta$ to t. The equality (21) holds because

$$\begin{aligned}\log\big(E[e^{\gamma R(t,t+\delta)}]\big) &= \log\big(\lambda_t\delta E[e^{\gamma C_t}] + (1-\lambda_t\delta) + o(\delta)\big) = \log\big(1 + \lambda_t\delta(E[e^{\gamma C_t}] - 1) + o(\delta)\big)\\ &= \lambda_t\delta(E[e^{\gamma C_t}] - 1) + o(\delta).\end{aligned}$$

The last equality holds because of the Taylor expansion. Then, consider

$$h(t+\delta;\gamma,r) - g(t+\delta;\gamma,r) - w_{\texttt{tw}}(t+\delta;\gamma) = h(t;\gamma,r) - g(t;\gamma,r) - w_{\texttt{tw}}(t;\gamma) + \delta\lambda_t\Big(\int_{g(t;\gamma,r)}^{\infty} c_t dF_t(c_t)$$
$$- (1 - F_t(g(t;\gamma,r)))\cdot(h(t;\gamma,r) - E[R(t)]) + (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma - (E[e^{\gamma C_t}] - 1)/\gamma\Big) + o(\delta)$$
$$= h(t;\gamma,r) - g(t;\gamma,r) - w_{\texttt{tw}}(t;\gamma) + \delta\lambda_t\Big(E[C_t] - E[\min(C_t, g(t;\gamma,r))]$$
$$- (1 - F_t(g(t;\gamma,r)))\cdot(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]) + (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma - (E[e^{\gamma C_t}] - 1)/\gamma\Big) + o(\delta)$$
$$= \big(h(t;\gamma,r) - g(t;\gamma,r) - w_{\texttt{tw}}(t;\gamma)\big)\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r)))\big) + \delta\lambda_t\Big(- (1 - F_t(g(t;\gamma,r)))(w_{\texttt{tw}}(t;\gamma) - E[R(t)])$$
$$- E[\min(C_t, g(t;\gamma,r))] + E[e^{\gamma \min(C_t, g(t;\gamma,r))}]/\gamma - \big(- E[C_t] + E[e^{\gamma C_t}]/\gamma\big)\Big) + o(\delta)$$
$$\geqslant \big(h(t;\gamma,r) - g(t;\gamma,r) - w_{\texttt{tw}}(t;\gamma)\big)\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r)))\big) \geqslant 0.$$

The first inequality holds because $w_{\texttt{tw}}(t;\gamma) \leqslant E[R(t)]$ for any $t \geqslant 0$ and $\gamma < 0$, and by Lemma 1, for any given $\gamma < 0$,

$$\big(E[e^{\gamma C_t}] - E[e^{\gamma \min(C_t, g(t;\gamma,r))}]\big)/\gamma \leqslant E[C_t] - E[\min(C_t, g(t;\gamma,r))];$$

the second inequality holds because $h(t;\gamma,r) \geqslant g(t;\gamma,r) + w_{\texttt{tw}}(t;\gamma)$. Therefore, $h(t;\gamma,r) \geqslant w_{\texttt{rvw}}(t;\gamma,r)$ at any $t \geqslant 0$ for given $\gamma < 0$ and r. Similarly, we can prove $h(t;\gamma,r) \leqslant w_{\texttt{rvw}}(t;\gamma,r)$ at any $t \geqslant 0$ for given $\gamma > 0$ and r.
(b). First, consider the case of $\gamma < 0$. Suppose $h(t;\gamma,r) - g(t;\gamma,r) \leqslant h(t;\gamma,r') - g(t;\gamma,r')$ for some $t \geqslant 0$ and $r > r'$. Then,

$$\big(h(t+\delta;\gamma,r) - g(t+\delta;\gamma,r)\big) - \big(h(t+\delta;\gamma,r') - g(t+\delta;\gamma,r')\big) = \big(h(t;\gamma,r) - g(t;\gamma,r)\big) - \big(h(t;\gamma,r') - g(t;\gamma,r')\big)$$
$$+ \delta\lambda_t\Big(\int_{g(t;\gamma,r)}^{\infty} c_t dF_t(c_t) - (1 - F_t(g(t;\gamma,r)))\cdot(h(t;\gamma,r) - E[R(t)]) + (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma$$
$$- \int_{g(t;\gamma,r')}^{\infty} c_t dF_t(c_t) + (1 - F_t(g(t;\gamma,r')))\cdot(h(t;\gamma,r') - E[R(t)]) - (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r'))}])/\gamma\Big)$$
$$= \big(h(t;\gamma,r) - g(t;\gamma,r)\big)\cdot\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r)))\big) - \big(h(t;\gamma,r') - g(t;\gamma,r')\big)\cdot\big(1 - \delta\lambda_t(1 - F_t(g(t;\gamma,r')))\big)$$
$$+ \delta\lambda_t\Big(E[R(t)] - E[\min(C_t, g(t;\gamma,r))] + (-1 + E[e^{\gamma \min(C_t, g(t;\gamma,r))}])/\gamma$$

$$
\begin{aligned}
&- \big(E[R(t)] - E[\min(C_t, g(t;\gamma,r'))]\big) - \big(F_t(g(t;\gamma,r)) - F_t(g(t;\gamma,r'))\big)\cdot E[R(t)] - (-1 + E[e^{\gamma\min(C_t,g(t;\gamma,r'))}])/\gamma\Big)\\
=& \Big(\big(h(t;\gamma,r) - g(t;\gamma,r)\big) - \big(h(t;\gamma,r') - g(t;\gamma,r')\big)\Big)\cdot\big(1-\delta\lambda_t(1-F_t(g(t;\gamma,r')))\big)\\
&+\delta\lambda_t\bigg(-E[\min(C_t,g(t;\gamma,r))] + E[e^{\gamma\min(C_t,g(t;\gamma,r))}]/\gamma - \Big(-E[\min(C_t,g(t;\gamma,r'))] + E[e^{\gamma\min(C_t,g(t;\gamma,r'))}]/\gamma\Big)\\
&+\big(F_t(g(t;\gamma,r)) - F_t(g(t;\gamma,r'))\big)\cdot\big(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]\big)\bigg)\\
\leqslant& \Big(\big(h(t;\gamma,r) - g(t;\gamma,r)\big) - \big(h(t;\gamma,r') - g(t;\gamma,r')\big)\Big)\cdot\big(1-\delta\lambda_t(1-F_t(g(t;\gamma,r')))\big)\leqslant 0.
\end{aligned}
$$

The first inequality holds because $h(t;\gamma,r) \leqslant g(t;\gamma,r) + E[R(t)]$ for $\gamma < 0$, and by Lemma 1,

$$
\big(E[e^{\gamma\min(C_t,g(t;\gamma,r))}] - E[e^{\gamma\min(C_t,g(t;\gamma,r'))}]\big)/\gamma \leqslant E[\min(C_t,g(t;\gamma,r))] - E[\min(C_t,g(t;\gamma,r'))];
$$

the second inequality holds because $h(t;\gamma,r) - g(t;\gamma,r) \leqslant h(t;\gamma,r') - g(t;\gamma,r')$. Therefore, $h(t;\gamma,r) - g(t;\gamma,r)$ is decreasing in r for any given $\gamma < 0$. Similarly, we can prove that $h(t;\gamma,r) - g(t;\gamma,r)$ is increasing in r for any given $\gamma > 0$.

Obviously, $g(t;\gamma,r)\to\infty$ as $r\to\infty$. We have already known that $h(t;\gamma,r) - g(t;\gamma,r)$ is monotonic in r for any given γ and is bounded from below and above, i.e., $w_{\texttt{tw}}(t;\gamma) > h(t;\gamma,r) - g(t;\gamma,r) > E[R(t)]$ for any $\gamma > 0$ and $w_{\texttt{tw}}(t;\gamma) < h(t;\gamma,r) - g(t;\gamma,r) < E[R(t)]$ for any $\gamma < 0$, so it converges to a limit as r approaches infinity. To show its limit, we consider its derivative with respect to t. From differential equations (1) and (4),

$$
\begin{aligned}
\frac{\partial\big(h(t;\gamma,r) - g(t;\gamma,r)\big)}{\partial t} &= \lambda_t \Pr(C_t > g(t;\gamma,r))\Big\{E[C_t|C_t > g(t;\gamma,r)] - \big(h(t;\gamma,r) - E[R(t)]\big)\Big\}\\
&\quad + \frac{\lambda_t}{\gamma}\Big(-1 + E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big]\Big)\\
&= \lambda_t\frac{E[e^{\gamma C_t}]-1}{\gamma} - \lambda_t\int_{g(t;\gamma,r)}^{\infty}\big(\frac{e^{\gamma C_t}}{\gamma} - C_t\big) - \big(\frac{e^{\gamma g(t;\gamma,r)}}{\gamma} - g(t;\gamma,r)\big)dF_t(C_t)\\
&\quad + \lambda_t\Pr(C_t > g(t;\gamma,r))\cdot\big(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]\big).
\end{aligned}
$$

Apparently, $\Pr(C_t > g(t;\gamma,r))\cdot\big(h(t;\gamma,r) - g(t;\gamma,r) - E[R(t)]\big)\to 0$ as $g(t;\gamma,r)\to\infty$ because $h(t;\gamma,r) - g(t;\gamma,r)$ is bounded. Assume that the moment generating function is finite, i.e., $E[e^{\gamma C_t}]$ is finite. Then,

$$
\lim_{g(t;\gamma,r)\to\infty}\int_{g(t;\gamma,r)}^{\infty}\big(\frac{e^{\gamma C_t}}{\gamma} - C_t\big) - \big(\frac{e^{\gamma g(t;\gamma,r)}}{\gamma} - g(t;\gamma,r)\big)dF_t(C_t) = 0.
$$

The equality holds because the limit of each integral in the above equation is equal to zero. Thus, it holds that

$$
\frac{\partial\big(h(t;\gamma,\infty) - g(t;\gamma,\infty)\big)}{\partial t} = \lambda_t\frac{E[e^{\gamma C_t}]-1}{\gamma}.
$$

Then, for any given γ, we have

$$
h(t;\gamma,\infty) - g(t;\gamma,\infty) = \int_0^t \lambda_s\frac{E[e^{\gamma C_s}]-1}{\gamma}dF_s(C_s) = w_{\texttt{tw}}(t;\gamma).
$$

The second equality holds because

$$
\begin{aligned}
\frac{\partial w_{\texttt{tw}}(t;\gamma)}{\partial t} &= \lim_{\delta\to 0}\frac{w_{\texttt{tw}}(t+\delta;\gamma) - w_{\texttt{tw}}(t;\gamma)}{\delta\gamma} = \lim_{\delta\to 0}\frac{\log\big(E[e^{\gamma R(t,t+\delta)}]\big)}{\delta\gamma} = \lim_{\delta\to 0}\frac{\log\big(\lambda_t\delta E[e^{\gamma C_t}] + (1-\lambda_t\delta) + o(\delta)\big)}{\delta\gamma}\\
&= \lim_{\delta\to 0}\frac{\lambda_t\delta(E[e^{\gamma C_t}]-1) + o(\delta)}{\delta\gamma} = \frac{\lambda_t(E[e^{\gamma C_t}]-1)}{\gamma}.
\end{aligned}
$$

It completes the proof. □

Proof of Theorem 2. The maximum price of the TW is equal to its willingness-to-pay. Then, the profit of offering a TW is equal to

$$w_{\texttt{tw}}(T;\gamma) - E[R(T)].$$

For risk-averse customers with $\gamma > 0$, we have

$$w_{\texttt{rvw}}(T;\gamma,r) - h(T;\gamma,r) = w_{\texttt{tw}}(T;\gamma) + g(T;\gamma,r) - h(T;\gamma,r) < w_{\texttt{tw}}(T;\gamma) - E[R(t)].$$

The inequality holds for any $r > 0$ by Proposition 4. For the RVW provider with a positive refund r earns less profit than the TW, so the RVW degenerates to a TW in a homogeneous market with risk-averse customers, i.e., $r^* = 0$.
Similarly, for risk-seeking customers, i.e., $\gamma < 0$,

$$w_{\texttt{rvw}}(T;\gamma,r) - h(T;\gamma,r) = w_{\texttt{tw}}(T;\gamma) + g(T;\gamma,r) - h(T;\gamma,r) > w_{\texttt{tw}}(T;\gamma) - E[R(t)].$$

Notice that the TW loses money for $\gamma < 0$, so the RVW can balance the revenue and the support cost by offering a sufficiently large refund because Proposition 4 shows that $h(T;\gamma,r) - g(T;\gamma,r) \to w_{\texttt{tw}}(T;\gamma)$ as $r \to \infty$. □

Proof of Theorem 3. To investigate the profitability of the RVW over the TW, we consider the following comparison

$$\max_r \left\{w_{\texttt{tw}}(T;\gamma_b) + g(T;\gamma_b,r) - h(T;\gamma_a,r)\right\} > w_{\texttt{tw}}(T;\gamma_b) - E[R(T)].$$

Or equivalently $\min_r \left\{h(T;\gamma_a,r) - g(T;\gamma_b,r)\right\} < E[R(T)]$. By Theorem 4, $h(T;\gamma_a,r)$ is first increasing in γ_a for $\gamma_a \leqslant 0$ and then is decreasing in it for $\gamma_a \geqslant 0$. Then, $\max_r \left\{h(T;\gamma_a,r) - g(T;\gamma_b,r)\right\}$ is also first increasing and then decreasing in γ_a. Denote the two solutions to the following equation with respect to γ_a,

$$\min_r \left\{h(T;\gamma_a,r) - g(T;\gamma_b,r)\right\} = E[R(T)]$$

by $\underline{\gamma}_a$ and $\overline{\gamma}_a$, and $\underline{\gamma}_a \leqslant \overline{\gamma}_a$. Therefore, the RVW is strictly more profitable than the TW if and only if $\gamma_a < \underline{\gamma}_a$ or $\gamma_a > \overline{\gamma}_a$. □

Proof of Proposition 5. Let $\alpha^H \left(w_{\texttt{tw}}(T;\gamma^H) - E[R(t)]\right) = w_{\texttt{tw}}(T;\gamma^L) - E[R(T)]$. We have

$$w_{\texttt{tw}}(T;\gamma^H) = \frac{w_{\texttt{tw}}(T;\gamma^L) - \alpha^L E[R(T)]}{\alpha^H}.$$

By Proposition 2, $w_{\texttt{tw}}(T;\gamma^H)$ is increasing in γ^H, so there exists a unique solution to equation (5) with respect to γ^H, denoted by $\widehat{\gamma}^H$. We remark that if $\gamma^H \geqslant \widehat{\gamma}$, it is more profitable for the provider to only serve type-H customers by charging price $w_{\texttt{tw}}(T;\gamma^H)$. □

Proof of Theorem 4. We will prove this theorem by induction.
(a). Suppose that for a given refund r, $w_{\texttt{rvw}}(t;\gamma,r) \geqslant w_{\texttt{rvw}}(t;\gamma',r)$ at time-to-go t for any $\gamma > \gamma'$. We will next show that the inequality also holds at time-to-go $t+\delta$ for a sufficiently small $\delta > 0$, i.e., $w_{\texttt{rvw}}(t+\delta;\gamma,r) \geqslant w_{\texttt{rvw}}(t+\delta;\gamma',r)$.
According to the differential equation (1), we have

$$g(t+\delta;\gamma,r) = g(t;\gamma,r) - \frac{\lambda_t \delta}{\gamma}\Big(E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big] - 1\Big) + o(\delta).$$

Combing with equation (21), we have

$$g(t+\delta;\gamma,r) + w_{\texttt{tw}}(t+\delta;\gamma) = g(t;\gamma,r) + w_{\texttt{tw}}(t;\gamma) + \lambda_t\delta \frac{E[e^{\gamma C_t}] - E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma} + o(\delta).$$

Therefore,

$$\begin{aligned}
&\Big(g(t+\delta;\gamma,r) + w_{\texttt{tw}}(t+\delta;\gamma)\Big) - \Big(g(t+\delta;\gamma',r) + w_{\texttt{tw}}(t+\delta;\gamma')\Big) = \Big(g(t;\gamma,r) + w_{\texttt{tw}}(t;\gamma)\Big) \\
&\quad - \Big(g(t;\gamma',r) + w_{\texttt{tw}}(t;\gamma')\Big) + \lambda_t\delta\Big(\frac{E[e^{\gamma C_t}] - E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma} - \frac{E[e^{\gamma' C_t}] - E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}\big]}{\gamma'}\Big) + o(\delta) \\
&\geqslant \Big(g(t;\gamma,r) + w_{\texttt{tw}}(t;\gamma)\Big) - \Big(g(t;\gamma',r) + w_{\texttt{tw}}(t;\gamma')\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_t\delta\Big(\frac{E[e^{\gamma C_t}]-E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma}-\frac{E[e^{\gamma' C_t}]-E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma'}\Big)+o(\delta)\\
&\geqslant\Big(g(t;\gamma,r)+w_{\texttt{tw}}(t;\gamma)\Big)-\Big(g(t;\gamma',r)+w_{\texttt{tw}}(t;\gamma')\Big)\geqslant 0.
\end{aligned}
$$

The first inequality holds because $g(t;\gamma,r)\leqslant g(t;\gamma',r)$ and $E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma',r))}\big]/\gamma'\geqslant E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))}\big]/\gamma'$; the second inequality holds because by Lemma 1,

$$
\frac{E[e^{\gamma C_t}]-E\big[e^{\gamma\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma}\geqslant\frac{E[e^{\gamma' C_t}]-E\big[e^{\gamma'\cdot\min(C_t,g(t;\gamma,r))}\big]}{\gamma'};
$$

the last inequality holds because $g(t;\gamma,r)+w_{\texttt{tw}}(t;\gamma)\geqslant g(t;\gamma',r)+w_{\texttt{tw}}(t;\gamma')$. Thus, for any given time-to-go t and refund r, $w_{\texttt{rvw}}(t;\gamma,r)$ is decreasing with respect to γ no matter whether γ is positive or negative. Notice that $w_{\texttt{rvw}}(t;\gamma,r)=w_{\texttt{tw}}(t;\gamma)+g(t;\gamma,r)$ and $g(t;\gamma,r)$ is decreasing in γ, so $w_{\texttt{rvw}}(t;\gamma,r)$ is increasing at a lower rate than $w_{\texttt{tw}}(t;\gamma)$.

(b). We first consider the risk-averse customers, i.e., $\gamma>0$. Suppose that for a given refund r, $h(t;\gamma,r)\leqslant h(t;\gamma',r)$ for any $\gamma>\gamma'>0$. Then, we compare $h(t+\delta;\gamma,r)$ and $h(t+\delta;\gamma',r)$. From the differential equation (4),

$$
\begin{aligned}
h(t+\delta;\gamma,r)\;&=\;h(t;\gamma,r)+\delta\lambda_t\Pr(C_t>g(t;\gamma,r))\Big\{E[C_t|C_t>g(t;\gamma,r)]-\big(h(t;\gamma,r)-E[R(t)]\big)\Big\}+o(\delta)\\
&=\;h(t;\gamma,r)+\delta\lambda_t\left(\int_{g(t;\gamma,r)}^{\infty}c_t dF_t(c_t)-(1-F_t(g(t;\gamma,r))\big(h(t;\gamma,r)-E[R(t)]\big)\right)+o(\delta).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&h(t+\delta;\gamma,r)-h(t+\delta;\gamma',r)=h(t;\gamma,r)-h(t;\gamma',r)+\delta\lambda_t\Bigg(\int_{g(t;\gamma,r)}^{g(t;\gamma',r)}c_t dF_t(c_t)+\big(F_t(g(t;\gamma',r))-F_t(g(t;\gamma,r))\big)\\
&\quad\cdot E[R(t)]-\big(h(t;\gamma,r)-h(t;\gamma',r)\big)+\big(F_t(g(t;\gamma,r))h(t;\gamma,r)-F_t(g(t;\gamma',r))h(t;\gamma',r)\big)\Bigg)+o(\delta)\\
&\leqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)+\delta\lambda_t\Big(\big(F_t(g(t;\gamma',r))-F_t(g(t;\gamma,r))\big)\cdot\big(g(t;\gamma',r)+E[R(t)]\big)\\
&\quad+\big(F_t(g(t;\gamma,r))h(t;\gamma,r)-F_t(g(t;\gamma',r))h(t;\gamma',r)\big)\Big)\\
&\leqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)+\delta\lambda_t\Big(\big(F_t(g(t;\gamma',r))-F_t(g(t;\gamma,r))\big)\cdot\big(g(t;\gamma',r)+E[R(t)]-h(t;\gamma',r)\big)\Big)\\
&\leqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)\leqslant 0.
\end{aligned}
$$

The first inequality holds because $g(t;\gamma,r)\leqslant g(t;\gamma',r)$; the second inequality holds because $h(t;\gamma,r)\leqslant h(t;\gamma',r)$; the third inequality holds because $g(t;\gamma',r)+E[R(t)]-h(t;\gamma',r)\leqslant 0$ for $\gamma'>0$ by Proposition 4. Therefore, for any given time-to-go t and refund r, $h(t;\gamma,r)$ is decreasing in γ for risk-averse customers.

Next, consider the risk-seeking customers, i.e., $\gamma<0$. Suppose that for a given refund r, $h(t;\gamma,r)\geqslant h(t;\gamma',r)$ at some $t\geqslant 0$ for any $\gamma'<\gamma<0$. Similarly, we have

$$
\begin{aligned}
&h(t+\delta;\gamma,r)-h(t+\delta;\gamma',r)=h(t;\gamma,r)-h(t;\gamma',r)+\delta\lambda_t\Bigg(\int_{g(t;\gamma,r)}^{g(t;\gamma',r)}c_t dF_t(c_t)+\big(F_t(g(t;\gamma',r))\\
&\quad-F_t(g(t;\gamma,r))\big)E[R(t)]\\
&\geqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)+\delta\lambda_t\Big(\big(F_t(g(t;\gamma',r))-F_t(g(t;\gamma,r))\big)\cdot\big(g(t;\gamma,r)+E[R(t)]\big)\\
&\quad+\big(F_t(g(t;\gamma,r))h(t;\gamma,r)-F_t(g(t;\gamma',r))h(t;\gamma',r)\big)\Big)\\
&\geqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)+\delta\lambda_t\Big(\big(F_t(g(t;\gamma',r))-F_t(g(t;\gamma,r))\big)\cdot\big(g(t;\gamma,r)+E[R(t)]-h(t;\gamma,r)\big)\Big)\\
&\geqslant\big(h(t;\gamma,r)-h(t;\gamma',r)\big)(1-\delta\lambda_t)\geqslant 0.
\end{aligned}
$$

The first inequality holds because $g(t;\gamma,r)\leqslant g(t;\gamma',r)$; the second inequality holds because $h(t;\gamma,r)\geqslant h(t;\gamma',r)$; the third inequality holds because $g(t;\gamma,r)+E[R(t)]-h(t;\gamma,r)\geqslant 0$ for $\gamma<0$ by Proposition 4. Thus, for given time-to-go t and refund r, $h(t;\gamma,r)$ is increasing in γ for risk-seeking customers. □

Proof of Theorem 5. For $\gamma^L < \gamma^H \leqslant \widehat{\gamma}$, the TW captures both type-L and type-H customers and its profit is equal to $w_{\mathtt{tw}}(T;\gamma^L) - E[R(T)]$. An RVW with refund r and price $w_{\mathtt{rvw}}(T;\gamma^L,r)$ captures both market segments. Consider the following inequality

$$\max_r \Big\{ w_{\mathtt{tw}}(T;\gamma^L) + g(T;\gamma^L,r) - \alpha^L h(T;\gamma^L,r) - \alpha^H h(T;\gamma^H,r) \Big\} > w_{\mathtt{tw}}(T;\gamma^L) - E[R(T)],$$

or equivalently,

$$\min_r \Big\{ -g(T;\gamma^L,r) + \alpha^L h(T;\gamma^L,r) + \alpha^H h(T;\gamma^H,r) \Big\} < E[R(T)]. \tag{22}$$

Since $h(T;\gamma^H,r)$ is decreasing in γ^H for any given r by Theorem 4, then $\min_r \big\{ -g(T;\gamma^L,r) + \alpha^L h(T;\gamma^L,r) + \alpha^H h(T;\gamma^H,r) \big\}$ is also decreasing in γ^H. Therefore, there exists a threshold $\underline{\gamma}^H$ such that inequality (22) holds for any $\gamma^H > \underline{\gamma}^H$. Apparently, $\underline{\gamma}^H < \widehat{\gamma}^H$.
We have already shown that the RVW is strictly more profitable than the TW for $\gamma^H \in \big(\underline{\gamma}^H, \widehat{\gamma}^H\big)$. We remark that the RVW may be more profitable for γ^H varying in an even larger range. □

Proof of Proposition 6. (a). We will use sample path argument to show the monotonic property. For a Poisson process with a stationary failure rate λ^H, the optimal claim policy has a threshold $g(t;\lambda^H,r)$ for each failure at time t. Assume that the customer with a stationary failure rate λ^L adopts the same policy with threshold $g(t;\lambda^H,r)$. Apparently, the expected refund net of out-of-pocket repair cost taking into account the risk attitude is greater than that under the failure process with a rate λ^H because each failure occurs with a lower probability. Therefore, $g(t;\lambda^H,r)$, that is the net value corresponding to rate λ^L under the optimal claim policy is even higher.
(b). For a given refund r, suppose $w_{\mathtt{rvw}}(t;\lambda^H,r) \geqslant w_{\mathtt{rvw}}(t;\lambda^L,r)$ at some $t \geqslant 0$. Next, we will show that the inequality also holds at time $t+\delta$ for a sufficiently small $\delta > 0$, i.e., $w_{\mathtt{rvw}}(t+\delta;\lambda^H,r) \geqslant w_{\mathtt{rvw}}(t+\delta;\lambda^L,r)$. Similar to the proof of Theorem 4, we have the following equations

$$\begin{aligned} w_{\mathtt{rvw}}(t+\delta;\lambda^H,r) &= w_{\mathtt{rvw}}(t;\lambda^H,r) + \lambda^H \delta \frac{E\big[e^{\gamma C_t}\big] - E\Big[e^{\gamma\cdot\min(C_t, g(t;\lambda^H,r))}\Big]}{\gamma} + o(\delta), \\ w_{\mathtt{rvw}}(t+\delta;\lambda^L,r) &= w_{\mathtt{rvw}}(t;\lambda^L,r) + \lambda^L \delta \frac{E\big[e^{\gamma C_t}\big] - E\Big[e^{\gamma\cdot\min(C_t, g(t;\lambda^L,r))}\Big]}{\gamma} + o(\delta). \end{aligned}$$

Comparing them, we have

$$w_{\mathtt{rvw}}(t+\delta;\lambda^H,r) \geqslant w_{\mathtt{rvw}}(t+\delta;\lambda^L,r).$$

The inequality holds because $w_{\mathtt{rvw}}(t;\lambda^H,r) \geqslant w_{\mathtt{rvw}}(t;\lambda^L,r)$, $\lambda^H \geqslant \lambda^L$ and $g(t;\lambda^H,r) \leqslant g(t;\lambda^L,r)$. Notice that the net value $g(t;\lambda,r)$ of the RVW is decreasing in failure rate λ, so the willingness-to-pay for the RVW is increasing at a lower rate than that for the TW. □

Proof of Theorem 6. (a). Notice that the willingness-to-pay can be expressed as follows

$$\widetilde{w}_{\mathtt{rvw}}(T;\lambda,r) = w_{\mathtt{tw}}(T;\lambda) + \widetilde{g}(T;\lambda,r) = \frac{\Lambda(T)\cdot(M_C(\gamma)-1)}{\gamma} - \frac{1}{\gamma}\ln\left((1-e^{-\Lambda(T)}) + e^{-\gamma r - \Lambda(T)}\right).$$

Consider its first-order derivative with respect to λ

$$\frac{\partial \widetilde{w}_{\mathtt{rvw}}(T;\lambda,r)}{\partial\lambda} = \frac{T}{\gamma}\left(M_C(\gamma) - 1 - \frac{e^{-\Lambda(T)} - e^{-\gamma r - \Lambda(T)}}{(1-e^{-\Lambda(T)}) + e^{-\gamma r - \Lambda(T)}}\right) > \frac{T}{\gamma}\left(M_C(\gamma) - \frac{1}{(1-e^{-\Lambda(T)})}\right) \geqslant 0.$$

The last inequality holds because of the condition in part (a) of Theorem 6.
(b). Dividing the both sides of equation (15) by $(e^{\widetilde{g}(t;\lambda,r)}-1)/\gamma$, taking integrals with respect to t and combining the boundary condition yields the closed-form solution for $\widetilde{g}(t;\lambda,r)$

$$\widetilde{g}(t;\lambda,r) = -\frac{1}{\gamma}\ln\left((1-e^{-\Lambda(t)}) + e^{-\gamma r - \Lambda(t)}\right).$$

For notational convenience, let $\Pi_{\mathtt{rvw}}(r) = \widetilde{w}_{\mathtt{rvw}}(T;\lambda^L,r) - \big(\alpha^L \widetilde{h}_{\mathtt{rvw}}(T;\lambda^L,r) + \alpha^H \widetilde{h}_{\mathtt{rvw}}(T;\lambda^H,r)\big)$. It can be further expressed by

$$\Pi_{\mathtt{rvw}}(r) = w_{\mathtt{tw}}(T;\lambda^L) - \big(\alpha^L\lambda^L + \alpha^H\lambda^H\big)Tc - \frac{1}{\gamma}\ln\Big((1-e^{-\lambda^L T}) + e^{-\gamma r - \lambda^L T}\Big) - \big(\alpha^L e^{-\lambda^L T} + \alpha^H e^{-\lambda^H T}\big)r.$$

Consider the derivatives of $\Pi_{\texttt{rvw}}(r)$ with respect to r.

$$\begin{aligned}\frac{\partial \Pi_{\texttt{rvw}}(r)}{\partial r} &= \frac{e^{-\gamma r-\lambda^L T}}{(1-e^{-\lambda^L T})+e^{-\gamma r-\lambda^L T}} - \left(\alpha^L e^{-\lambda^L T}+\alpha^H e^{-\lambda^H T}\right),\\ \frac{\partial^2 \Pi_{\texttt{rvw}}(r)}{\partial r^2} &= -\frac{\gamma(1-e^{-\lambda^L T})\cdot e^{-\gamma r-\lambda^L T}}{\left((1-e^{-\lambda^L T})+e^{-\gamma r-\lambda^L T}\right)^2}<0.\end{aligned}$$

So, the profit function $\Pi_{\texttt{rvw}}(r)$ is strictly concave in r. Solving the first-order condition $\partial\Pi_{\texttt{rvw}}(r)/\partial r=0$ yields the optimal refund

$$r^*=\frac{1}{\gamma}\ln\left(\frac{e^{-\lambda^L T}\cdot\left(1-(\alpha^L e^{-\lambda^L T}+\alpha^H e^{-\lambda^H T})\right)}{(1-e^{-\lambda^L T})\cdot(\alpha^L e^{-\lambda^L T}+\alpha^H e^{-\lambda^H T})}\right).$$

It can be verified that $r^*>0$ because for any $\lambda^H>\lambda^L$,

$$e^{-\lambda^L T}\cdot\left(1-(\alpha^L e^{-\lambda^L T}+\alpha^H e^{-\lambda^H T})\right)>(1-e^{-\lambda^L T})\cdot(\alpha^L e^{-\lambda^L T}+\alpha^H e^{-\lambda^H T}).$$

(c). Solving the equation $w_{\texttt{tw}}(T;\lambda^L)-\left(\alpha^L\lambda^L+\alpha^H\lambda^H\right)Tc=\alpha^H\left(w_{\texttt{tw}}(T;\lambda^H)-\lambda^H Tc\right)$ for λ^H yields

$$\widehat{\lambda}^H=\lambda^L\cdot\frac{(M_C(\gamma)-1)/\gamma-\alpha^L c}{\alpha^H(M_C(\gamma)-1)/\gamma}.$$

To show that the RVW is always strictly more profitable than the TW, we compare $\Pi(r^*)$ to the profit of the TW, which is equal to $w_{\texttt{tw}}(T;\lambda^L)-\left(\alpha^L\lambda^L+\alpha^H\lambda^H\right)Tc$, i.e.,

$$\Pi(r^*)>w_{\texttt{tw}}(T;\lambda^L)-\left(\alpha^L\lambda^L+\alpha^H\lambda^H\right)Tc.$$

Notice that the RVW degenerates to the TW when the refund is equal to zero, i.e., $\Pi(0)=w_{\texttt{tw}}(T;\lambda^L)-\left(\alpha^L\lambda^L+\alpha^H\lambda^H\right)Tc$. Because $\Pi(r)$ is strictly increasing in r and $r^*>0$, the above strict inequality holds. □

Proof of Proposition 7. (a). We can use the similar arguments as in the proof of Proposition 6 to show the monotonic property with respect to the repair cost.
(b). For a given refund r, suppose that $w_{\texttt{rvw}}(t;C^H,r)>w_{\texttt{rvw}}(t;C^L,r)$ at some $t\geqslant 0$. Next, we will show $w_{\texttt{rvw}}(t+\delta;C^H,r)\geqslant w_{\texttt{rvw}}(t+\delta;C^L,r)$. Similarly,

$$w_{\texttt{rvw}}(t+\delta;C,r)=w_{\texttt{rvw}}(t;C,r)+\lambda_t\delta\frac{E\left[e^{\gamma C_t}\right]-E\left[e^{\gamma\cdot\min(C_t,g(t;C,r))}\right]}{\gamma}+o(\delta).$$

We next compare

$$\begin{aligned}w_{\texttt{rvw}}(t+\delta;C^H,r) &= w_{\texttt{rvw}}(t;C^H,r)+\lambda_t\delta\frac{E\left[e^{\gamma C^H}\right]-E\left[e^{\gamma\cdot\min(C^H,g(t;C^H,r))}\right]}{\gamma}+o(\delta),\\ w_{\texttt{rvw}}(t+\delta;C^L,r) &= w_{\texttt{rvw}}(t;C^L,r)+\lambda_t\delta\frac{E\left[e^{\gamma C^L}\right]-E\left[e^{\gamma\cdot\min(C^L,g(t;C^L,r))}\right]}{\gamma}+o(\delta).\end{aligned}$$

Moreover, we have

$$E\left[e^{\gamma C^H}\right]-E\left[e^{\gamma\cdot\min(C^H,g(t;C^H,r))}\right]\geqslant E\left[e^{\gamma C^H}\right]-E\left[e^{\gamma\cdot\min(C^H,g(t;C^L,r))}\right]\geqslant E\left[e^{\gamma C^L}\right]-E\left[e^{\gamma\cdot\min(C^L,g(t;C^L,r))}\right].$$

The first inequality holds because $g(t;C^H,r)\leqslant g(t;C^L,r)$; the second inequality holds because function $\left(e^{\gamma x}-e^{\gamma\min(x,g)}\right)$ is increasing in x and C^H is stochastically larger than C^L. Therefore, for any given refund r

$$w_{\texttt{rvw}}(t+\delta;C^H,r)\geqslant w_{\texttt{rvw}}(t+\delta;C^L,r),$$

which completes the proof. □

Proof of Proposition 8. For any given refund r, problem (17) is a linear program. From the constraints, we have $p_{\texttt{rvw}}<p_{\texttt{tw}}+g(T;\gamma^L,r)\leqslant w_{\texttt{tw}}(T;\gamma^H)+g(T;\gamma^L,r)$. Recall that $p_{\texttt{rvw}}\leqslant w_{\texttt{rvw}}(T;\gamma^L,r)=w_{\texttt{tw}}(T;\gamma^L)+g(T;\gamma^L,r)$. Because $w_{\texttt{tw}}(T;\gamma^L)\leqslant w_{\texttt{tw}}(T;\gamma^H)$, then the maximum price of the RVW is $p_{\texttt{rvw}}=w_{\texttt{rvw}}(T;\gamma^L,r)$. Because $p_{\texttt{tw}}\leqslant p_{\texttt{rvw}}-g(T;\gamma^H,r)$, then the maximum price of the TW is $p_{\texttt{tw}}=w_{\texttt{rvw}}(T;\gamma^L,r)-g(T;\gamma^H,r)$. □

Proof of Theorem 7. Comparison between problems (18) and (13) yields that the warranty menu earns strictly more profit than the RVW alone for any given refund r, i.e.,

$$\max_r \Big\{ g(T;\gamma^L,r) - \alpha^L h(T;\gamma^L,r) - \alpha^H \big(g(T;\gamma^H,r) + E[R(T)] \big) \Big\} > \max_r \Big\{ g(T;\gamma^L,r) - \alpha^L h(T;\gamma^L,r) - \alpha^H h(T;\gamma^H,r) \Big\}.$$

The inequality holds because $h(T;\gamma^H,r) > g(T;\gamma^H,r) + E[R(T)]$ for any risk-averse customer and any positive refund r by Proposition 4.

To show the profit advantage of the menu over the TW alone, consider the following inequality

$$\max_r \ \Big\{ w_{\mathtt{tw}}(T;\gamma^L) + g(T;\gamma^L,r) - \alpha^L h(T;\gamma^L,r) - \alpha^H \big(g(T;\gamma^H,r) + E[R(T)] \big) \Big\} > w_{\mathtt{tw}}(T;\gamma^L) - E[R(T)],$$

or equivalently,

$$\min_r \ \Big\{ - g(T;\gamma^L) + \alpha^L h(T;\gamma^L,r) + \alpha^H g(T;\gamma^H,r) \Big\} < \alpha^L E[R(T)]. \tag{23}$$

Since $g(T;\gamma^H,r)$ is decreasing in γ^H for any given r by Theorem 1, then $\min_r \big\{ - g(T;\gamma^L,r) + \alpha^L h(T;\gamma^L,r) + \alpha^H g(T;\gamma^H,r) \big\}$ is also decreasing in γ^H. Therefore, there exists a threshold $\underline{\gamma}'^H$ such that inequality (23) holds for any $\gamma^H > \underline{\gamma}'^H$. Comparing inequalities (22) and (23) and recalling $h(T;\gamma^H,r) > g(T;\gamma^H,r) + E[R(T)]$ for $\gamma^H > 0$ by Proposition 4, we have $\underline{\gamma}'^H < \underline{\gamma}^H$. It completes the proof. ◻