Online Supplement to "Intertemporal Price Discrimination via Randomized Promotions": Proofs

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A. Proofs of Results in the Main Body

Proof of Lemma 1. Note that $V(p_t) = \max\{v - p_t, -c + \mathsf{E}[V(P)], 0\}$, which is equivalent to $V(p_t) = \max\{v - p_t, [-c + \mathsf{E}[V(P)]]^+\}$. It is easy to see that $v - p_t$ is decreasing in p_t, and $[-c + \mathsf{E}[V(P)]]^+$ is independent of p_t. On the one hand, $v - 0 = v > [-c + \mathsf{E}[V(P)]]^+$, because $V(p_t) = \max\{v - p_t, -c + \mathsf{E}[V(P)], 0\} \le v - p_t < v$. On the other hand, $v - v = 0 \le [-c + \mathsf{E}[V(P)]]^+$. Thus, there exists a threshold $\underline{p} \in (0, v]$ such that $v - p_t \ge [-c + \mathsf{E}[V(P)]]^+$ if and only if $p_t \le \underline{p}$. Consequently, $V(p_t)$ is in the form of Equation (1). $\square$

Proof of Proposition 1. Denote $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \le c\}$. Since no customer would make a purchase at an infinitely high price, we just consider the support of P as being finite. It is easy to verify that $\mathsf{E}[(v' - P)^+]$ is weakly increasing in v' and eventually strictly increasing in v' for v' higher than the upper end of P's finite support, $\mathsf{E}[(0 - P)^+] = 0 < c$ and $\lim_{v' \nearrow \infty} \mathsf{E}[(v' - P)^+] = \infty > c$. Hence, the existence of $\underline{v}^*$ is guaranteed.

Consider first when $v > \underline{v}^*$. We prove by contradiction that a customer with valuation v would be willing to wait and eventually purchase a unit of product in a future period. Suppose on the contrary, the customer would not wait by either purchasing the product now or leaving immediately. Under the assumption of zero utility from the outside option, the customer would purchase the product if and only if $v \ge p_t$. Thus, under the stipulation that the customer would not wait by either purchasing the product now or leaving immediately, we must have $\mathsf{E}[V(P)] = \mathsf{E}[(v - P)^+]$. In such a case, her expected utility from waiting, i.e., $-c + \mathsf{E}[V(P)]$, must be less than or equal to 0, which means that $\mathsf{E}[(v - P)^+] \le c$ due to $\mathsf{E}[V(P)] = \mathsf{E}[(v - P)^+]$. On the other hand, recall that by definition, $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \le c\}$. Because $v > \underline{v}^*$, we have $\mathsf{E}[(v - P)^+] > c$, which leads to a contradiction. Hence, a customer with valuation $v > \underline{v}^*$ would be willing to wait and eventually purchase a unit of product.

Next, we prove that the customer's purchase threshold $\underline{p}$ is equal to $\underline{v}^*$, meaning that a customer with valuation $v \geq \underline{v}^*$ would make a purchase if and only if the price in the focal period is less than or equal to $\underline{v}^*$. First, as shown above, we must have $-c + \mathsf{E}[V(P)] > 0$ when $v > \underline{v}^*$. Given a purchase theshold $\underline{p}$ (its existence is guranteed by Lemma 1), we have $\mathsf{E}[V(P)] = F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] + \bar{F}(\underline{p})[-c + \mathsf{E}[V(P)]]$, which implies that

$$\mathsf{E}[V(P)] = \frac{F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c}{F(\underline{p})} + c. \tag{OS.1}$$

Thus, we have $v - \underline{p} - (-c + \mathsf{E}[V(P)]) = (-F(\underline{p})\mathsf{E}[\underline{p} - P | P \leq \underline{p}] + c)/F(\underline{p}) \geq 0$, where the inequality is guaranteed by Lemma 1. This implies $\mathsf{E}[(\underline{p} - P)^+] = F(\underline{p})\mathsf{E}[\underline{p} - P | P \leq \underline{p}] \leq c$. As $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$ by definition, we conclude that $\underline{v}^* \geq \underline{p}$. Lastly, we prove by contradiction that $\underline{p} = \underline{v}^*$. Suppose $\underline{v}^* > \underline{p}$. Because $v - p_t \geq -c + \mathsf{E}[V(P)]$ if and only if $p_t \leq \underline{p}$, we have $v - \underline{v}^* < -c + \mathsf{E}[V(P)] = (F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c)/F(\underline{p})$, which is equivalent to $F(\underline{p})\mathsf{E}[\underline{v}^* - P | P \leq \underline{p}] > c$. Because $\underline{v}^* > \underline{p}$, we have $F(\underline{p})\mathsf{E}[\underline{v}^* - P | P \leq \underline{p}] \leq F(\underline{v}^*)\mathsf{E}[\underline{v}^* - P | P \leq \underline{v}^*] = \mathsf{E}[(\underline{v}^* - P)^+] \leq c$, which leads to a contradiction. Hence, the behavior of a customer with valuation $v > \underline{v}^*$ should be as characterized in Proposition 1.

Next consider the case when $v \leq \underline{v}^*$. We first show by contradiction that the customer would either purchase the product now or leave immediately. Suppose the customer would be willing to wait, which means that her expected utility from waiting must be greater than zero, i.e., $-c + \mathsf{E}[V(P)] > 0$. Based on Lemma 1, there exists a threshold $\underline{p}$ such that the customer would be willing to wait if and only if the price in the current period is greater than $\underline{p}$. Thus Equation (OS.1) still holds, which implies that $-c + \mathsf{E}[V(P)] = (F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] - c)/F(\underline{p}) > 0$. This is equivalent to $F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] > c$. Based on Lemma 1, we also know that $v - \underline{p} \geq -c + \mathsf{E}[V(P)] > 0$, i.e., $v > \underline{p}$. Coupling with the increasing monotonicity of $F(v')\mathsf{E}[v - P | P \leq v']$ in v' for any $v' \leq v$, we have $\mathsf{E}[(v - P)^+] = F(v)\mathsf{E}[v - P | P \leq v] \geq F(\underline{p})\mathsf{E}[v - P | P \leq \underline{p}] > c$, which implies that $v > \underline{v}^*$ because $\mathsf{E}[(v' - P)^+]$ is increasing in v' and $\underline{v}^* = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$. This contracts to the aforementioned condition that $v \leq \underline{v}^*$. Hence, a customer with valuation $v \leq \underline{v}^*$ would either purchase the product now or leave immediately. Under the assumption of zero utility from the outside option, her purchase threshold is simply her valuation v, meaning that the customer would purchase the product if and only if $v \geq p_t$. We thus obtain the announced results. $\square$

Proof of Lemma 2. (i) If $v > \underline{v}^*$, customers will wait until the price is no more than $\underline{p} = \underline{v}^*$, as shown in Proposition 1. The probability that customers purchase in the nth period after arrival is given by $\left(\bar{F}(\underline{p})\right)^{n-1} F(\underline{p})$, and if they indeed make a purchase in a period, the firm's expected profit is $\mathsf{E}[P|P \le \underline{p}]$. Consequently, the monopolist's expected profit ex ante is

$$\sum_{n=1}^{\infty} \left(\bar{F}(\underline{p})\right)^{n-1} F(\underline{p})\mathsf{E}[P|P \le \underline{p}] = \lim_{n\to\infty} F(\underline{p})\mathsf{E}[P|P \le \underline{p}]\frac{1-\left(\bar{F}(\underline{p})\right)^n}{1-\bar{F}(\underline{p})} = \mathsf{E}[P|P \le \underline{p}] = \mathsf{E}[P|P \le \underline{v}^*],$$

where the second equality is due to $\bar{F}(\underline{p}) < 1$.

(ii) If $v \le \underline{v}^*$, customers will either buy with a price no more than v or quit immediately. Consequently, the expected profit is $F(v)\mathsf{E}[P|P \le v]$. $\square$

Proof of Lemma 3. Note that $\underline{p}^i = \max\{v'|\mathsf{E}[(v'-P)^+] \le c_i\}$, $i = L, H$. Because $\mathsf{E}[(v'-P)^+]$ is increasing in v' and $c_L \ge c_H$, we have $\underline{p}^L \ge \underline{p}^H$. Next, we consider three mutually exclusive but collectively exhaustive cases: (a1) $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$; (b1) $v_H > \underline{p}^H$, $v_L \le \underline{p}^L$; (c1) $v_H > v_L > \underline{p}^L \ge \underline{p}^H$.

(a1) $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$. Due to Proposition 1, neither customers with a high-valuation nor customers with a low valuation would wait. Consequently, the firm's expected profit, as shown in Lemma 2, is given by

$$\begin{aligned}
&\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
&= \alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] + (1-\alpha)F(v_L)\mathsf{E}[P|P \le v_L] \\
&= F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] \\
&\le v_L F(v_L) + (1-\alpha)v_H[F(v_H) - F(v_L)] \le \max\{v_L, (1-\alpha)v_H\}F(v_H) \le \max\{v_L, (1-\alpha)v_H\},
\end{aligned}$$

where the first inequality is due to $\mathsf{E}[P|P \le v_L] \le \mathsf{E}[v_L|P \le v_L] \le v_L$ and $\mathsf{E}[P|v_L \le P \le v_H] \le \mathsf{E}[v_H|v_L \le P \le v_H] = v_H$. Thus, in the case when $\underline{p}^L \ge \underline{p}^H \ge v_H > v_L$, the expected profit from the optimal randomized pricing is no greater than that from the optimal static pricing policy.

(b1) $v_H > \underline{p}^H$, $v_L \le \underline{p}^L$. In this case, customers with a high-valuation will wait until the price is no higher than $\underline{p}^H$, but customers with a low valuation will quit the market immediately if the price is higher than their valuation v_L. According to Lemma 2, the monopolist's expected profit is given by

$$\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H].$$

Consider first when $\underline{p}^H \le v_L$. Then, we have

$$\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha v_L F(v_L) + (1-\alpha)\underline{p}^H \le \alpha v_L F(v_L) + (1-\alpha)v_L \le v_L,$$

where the first inequality is due to $\mathsf{E}[P|P \le v_L] \le \mathsf{E}[v_L|P \le v_L] \le v_L$ and $\mathsf{E}[P|P \le \underline{p}^H] \le \mathsf{E}[\underline{p}^H|P \le v_H] = \underline{p}^H$.

Next consider the circumstance when $v_H > \underline{p}^H > v_L$. The condition implies that $\mathsf{E}[P|P \le \underline{p}^H] \le \mathsf{E}[P|P \le v_H]$. Then, we have

$$\begin{aligned}
&\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \\
\le&\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le v_H] \le \alpha \frac{F(v_L)}{F(v_H)}\mathsf{E}[P|P \le v_L] + (1-\alpha)\mathsf{E}[P|P \le v_H] \\
=&\frac{\alpha F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H]}{F(v_H)} \le \frac{\max\{v_L, (1-\alpha)v_H\}F(v_H)}{F(v_H)} = \max\{v_L, (1-\alpha)v_H\},
\end{aligned}$$

where the second inequality is due to $0 < F(v_H) \le 1$, and the last inequality is based on the proof in (a1).

(c1) $v_H > v_L > \underline{p}^L \ge \underline{p}^H$. In this case, based on Proposition 1, both customers with a high-valuation and customers with a low valuation would be willing to wait. As a result, the expected profit, as shown in Lemma 2, is given by

$$\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \underline{p}^L + (1-\alpha)\underline{p}^H < v_L,$$

where the first inequality is due to $\mathsf{E}[P|P \le \underline{p}^L] \le \underline{p}^L$ and $\mathsf{E}[P|P \le \underline{p}^H] \le \underline{p}^H$, the second inequality is due to $v_L > \underline{p}^L \ge \underline{p}^H$.

We thus obtained the announced results. $\square$

Proof of Lemma 4. Based on Lemma 3, we can restrict our discussion to the situation when $c_L < c_H$. Note that $\underline{p}^i = \max\{v'|\mathsf{E}[(v'-P)^+] \le c_i\}$, $i = L, H$. Because $c_L < c_H$ implies that $\underline{p}^L < \underline{p}^H$, we need to consider four mutually exclusive but collective exhaustive cases: (a2) $v_L \le \underline{p}^L, v_H \le \underline{p}^H$; (b2) $v_L \le \underline{p}^L, v_H > \underline{p}^H$; (c2) $v_L > \underline{p}^L, v_H \le \underline{p}^H$; (d2) $v_L > \underline{p}^L, v_H > \underline{p}^H$.

Under the case (a2), based on Proposition 1, we know that neither high-valuation customers nor low-valuation customers would wait. Then following a similar approach as in the proof of case (a1) in Lemma 3, we can show that an optimal static pricing policy dominates randomized pricing policies. Under the case (b2), due to Proposition 1, low-valuation customers would never wait but high-valuation customers would wait in the system with the hope of a favorable price in future periods. The case is similar to the case (b1) in the proof of Lemma 3, and we can show that an optimal static pricing policy dominates randomized pricing policies following a similar approach.

Next we show in an auxiliary lemma below that case (c2) cannot happen under the condition that $\frac{v_L}{c_L} \leq \frac{v_H}{c_H}$.

LEMMA OS.1. $\frac{v_L}{c_L} > \frac{v_H}{c_H}$ *is a necessary condition for (c2).*

Proof of Lemma OS.1. By Proposition 1, (c2) implies that $F(v_L)\mathsf{E}[v_L - P|P \leq v_L] = \mathsf{E}[(v_L - P)^+] > c_L$ and $F(v_H)\mathsf{E}[v_H - P|P \leq v_H] = \mathsf{E}[(v_H - P)^+] \leq c_H$. Then, we have

$$c_H \geq F(v_H)\mathsf{E}[v_H - P|P \leq v_H] > F(v_L)\mathsf{E}[v_H - P|P \leq v_L] = v_H F(v_L) - F(v_L)\mathsf{E}[P|P \leq v_L],$$

and

$$c_L < F(v_L)\mathsf{E}[v_L - P|P \leq v_L] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \leq v_L].$$

The above two inequalities suggest that

$$\frac{c_H}{c_L} > \frac{v_H - \mathsf{E}[P|P \leq v_L]}{v_L - \mathsf{E}[P|P \leq v_L]} > \frac{v_H}{v_L},$$

where the second inequality is due to $(v_H - \mathsf{E}[P|P \leq v_L])\, v_L - (v_L - \mathsf{E}[P|P \leq v_L])\, v_H = (v_H - v_L)\mathsf{E}[P|P \leq v_L] > 0$. We thus obtain the announced result. $\square$

Lastly, we show that the optimal static pricing policy outperforms randomized pricing policies under (d2), i.e., $v_L > \underline{p}^L, v_H > \underline{p}^H$, and $\frac{c_H}{c_L} \leq \frac{v_H}{v_L}$. By Propositions 1, we know that customers would either purchase immediately or wait. Due to Lemma 2, the corresponding expected profit is given by

$$\alpha\mathsf{E}[P|P \leq \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \leq \underline{p}^H].$$

On the one hand, $\underline{p}^L = \max\{v'|\mathsf{E}[(v'-P)^+] \leq c_L\}$, and then for any $v > \underline{p}^L$, we have $\mathsf{E}[(v-P)^+] > c_L$. Consequently, $F(v_L)\mathsf{E}[v_L - P|P \leq v_L] = \mathsf{E}[(v_L - P)^+] > c_L$ due to $v_L > \underline{p}^L$. On the other hand, $\underline{p}^H = \max\{v'|\mathsf{E}[(v' - P)^+] \leq c_H\}$, which implies $F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \leq \underline{p}^H] = \mathsf{E}[(\underline{p}^H - P)^+] \leq c_H$. We also observe that $\frac{c_H}{c_L} > \frac{\underline{p}^H}{v_L}$, which can be shown with the same approach as that in the proof of Lemma OS.1.

Consider first when $\underline{p}^H \le v_L$. We have $\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \underline{p}^L + (1-\alpha)\underline{p}^H < v_L \le \max\{v_L, (1-\alpha)v_H\}$. That is, the firm is worse off under a randomized pricing policy.

Next consider the scenario when $\underline{p}^H > v_L$. We start by showing that $F(v_L) < \frac{c_H - c_L}{\underline{p}^H - v_L}$. This is true because

$$\begin{aligned} c_H - c_L &> F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \le \underline{p}^H] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] \\ &> F(v_L)\mathsf{E}[\underline{p}^H - P|P \le v_L] - F(v_L)\mathsf{E}[v_L - P|P \le v_L] = (\underline{p}^H - v_L)F(v_L), \end{aligned}$$

where the first inequality is due to $F(v_L)\mathsf{E}[v_L - P|P \le v_L] > c_L$ and $F(\underline{p}^H)\mathsf{E}[\underline{p}^H - P|P \le \underline{p}^H] \le c_H$, and the second inequality is due to $\underline{p}^H > v_L$. As a result, we have

$$\begin{aligned} &\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)\underline{p}^H \\ <&\alpha \left(v_L - \frac{c_L}{F(v_L)}\right) + (1-\alpha)\underline{p}^H < \alpha \left(v_L - \frac{c_L}{c_H - c_L}(\underline{p}^H - v_L)\right) + (1-\alpha)\underline{p}^H \\ =&\alpha \frac{c_H}{c_H - c_L} v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H. \end{aligned}$$

The first inequality is due to $\underline{p}^L < v_L$, the second inequality is due to $v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L] = F(v_L)\mathsf{E}[v_L - P|P \le v_L] > c_L$, and the third inequality is due to $F(v_L) < \frac{c_H - c_L}{\underline{p}^H - v_L}$.

If $(1-\alpha) - \alpha \frac{c_L}{c_H - c_L} \le 0$, then we have

$$\alpha \frac{c_H}{c_H - c_L} v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H < \alpha \frac{c_H}{c_H - c_L} v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right) v_L = v_L \le \max\{v_L, (1-\alpha)v_H\},$$

where the first inequality is due to $v_L < \underline{p}^H$. Otherwise, i.e., if $(1-\alpha) - \alpha \frac{c_L}{c_H - c_L} > 0$, we have

$$\begin{aligned} &\alpha \frac{c_H}{c_H - c_L} v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\underline{p}^H < \alpha \frac{c_H}{c_H - c_L} v_L + \left(1 - \alpha - \frac{\alpha c_L}{c_H - c_L}\right)\frac{c_H}{c_L} v_L \\ =&(1-\alpha)\frac{c_H}{c_L} v_L \le (1-\alpha)v_H \le \max\{v_L, (1-\alpha)v_H\}. \end{aligned}$$

The first inequality is due to $\frac{c_H}{c_L} > \frac{\underline{p}^H}{v_L}$, and the second inequality is due to $\frac{c_H}{c_L} \le \frac{v_H}{v_L}$. $\square$

Proof of Proposition 2. Recall the four mutually exclusive but collective exhaustive cases considered in the proof of Lemma 4: (a2) $v_L \le \underline{p}^L, v_H \le \underline{p}^H$; (b2) $v_L \le \underline{p}^L, v_H > \underline{p}^H$; (c2) $v_L > \underline{p}^L, v_H \le \underline{p}^H$; (d2) $v_L > \underline{p}^L, v_H > \underline{p}^H$. We already show that a randomized pricing policy cannot outperform an optimal static pricing policy under (a2) and (b2). Under the case (c2), low-valuation customers would make a purchase immediately or wait in the system, while high-valuation customers would either purchase the product upon arrival or leave immediately. Thus, all we need to show is that case (d2) is dominated by either (c2) or an optimal static pricing policy.

Based on Proposition 1 and Lemma 2, we can derive the optimal expected profit under case (d2) by solving the following optimization problem.

$$\begin{aligned} \max_{F(p)} \quad & \alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H]. \\ \text{s.t.} \quad & \underline{p}^L = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_L\}, \\ & \underline{p}^H = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_H\}, \\ & \mathsf{E}[(v_L - P)^+] > c_L, \quad \mathsf{E}[(v_H - P)^+] > c_H. \end{aligned} \tag{OS.2}$$

If $\underline{p}^H \le v_L$, we have $\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \le \alpha \underline{p}^L + (1-\alpha)\underline{p}^H < v_L$, which suggests that the optimal randomized pricing policy under case (d2) reduces to a static pricing policy. Thus we only need to restrict our attention to the case when $\underline{p}^H > v_L$. Suppose $F(p)$ is an optimal solution to problem (OS.2). Because no customer would buy at a price greater than v_H, the upper bound of an optimal distribution will be no more than v_H, i.e., $F(v_H) = 1$.

Based on Proposition 1 and Lemma 2, the optimal expected profit under case (c2) can be derived by solving problem (2). We construct a price distribution $G(p)$ as follows: $G(p)$ is the same as $F(p)$ for any $p \le \underline{p}^H$, and it takes one single value of v_H for any $p > \underline{p}^H$, with a probability mass equal to $1 - F(\underline{p}^H)$. We first show that $G(p)$ is a feasible solution of problem (2). Because $G(p)$ is equal to $F(p)$ for any $p \le v_L < \underline{p}^H$, we have $\mathsf{E}_G[(v_L - P)^+] > c_L$ and $\underline{p}^L = \max\{v' | \mathsf{E}[(v'-P)^+] \le c_L\} = \max\{v' | \mathsf{E}_G[(v'-P)^+] \le c_L\}$. On the other hand, $\mathsf{E}_G[v_H - P | \underline{p}^H < P \le v_H] = 0$ as $G(p)$ takes only one value of v_H for any $p > \underline{p}^H$. Consequently, $G(v_H)\mathsf{E}_G[v_H - P|P \le v_H] = G(\underline{p}^H)\mathsf{E}_G[v_H - P|P \le \underline{p}^H] = F(\underline{p}^H)\mathsf{E}[v_H - P|P \le \underline{p}^H] \le c_H$, where the last equality is due to the definition of $\underline{p}^H$. Thus, $G(p)$ is a feasible solution of problem (2).

Lastly, we prove that case (d2) is dominated by case (c2) by showing that the expected profit from $F(p)$ is lower than that from $G(p)$:

$$
\begin{aligned}
&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)\mathsf{E}[P|P \le \underline{p}^H] \\
=&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(\underline{p}^H)\mathsf{E}[P|P \le \underline{p}^H] + (1-\alpha)(1-F(\underline{p}^H))\mathsf{E}[P|P \le \underline{p}^H] \\
<&\alpha \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(\underline{p}^H)\mathsf{E}[P|P \le \underline{p}^H] + (1-\alpha)(1-F(\underline{p}^H))v_H \\
=&\alpha \mathsf{E}_G[P|P \le \underline{p}^L] + (1-\alpha)G(\underline{p}^H)\mathsf{E}_G[P|P \le \underline{p}^H] + (1-\alpha)(1-G(\underline{p}^H))\mathsf{E}_G[P|\underline{p}^H < P \le v_H] \\
=&\alpha \mathsf{E}_G[P|P \le \underline{p}^L] + (1-\alpha)G(v_H)\mathsf{E}_G[P|P \le v_H],
\end{aligned}
$$

where the inequality is due to $\underline{p}^H < v_H$, and the last equality is due to $G(v_H) = F(v_H) = 1$. We thus obtain the announced result. $\square$

Proof of Lemma 5. Suppose $F(p)$ is a feasible solution for the optimization problem (3). We can construct another distribution, say $G(p)$, as follows. $G(p)$ is equal to $F(p)$ for any $p \le v_L$. However, under $G(p)$, the distribution has only one value, namely v_H, above v_L, and its corresponding probability mass is equal to $1 - F(v_L)$. Because $G(p)$ is equal to $F(p)$ for any $p \le v_L$, it is easy to verify that $G(v_L)\mathsf{E}_G[v_L - P|P \le v_L] > c_L$ holds. On the other hand, $\mathsf{E}_G[v_H - P|v_L < P \le v_H] = 0$ as $G(p)$ has only one value of v_H for any $p > v_L$. Consequently, $G(v_H)\mathsf{E}_G[v_H - P|P \le v_H] = G(v_L)\mathsf{E}_G[v_H - P|P \le v_L] + [G(v_H) - G(v_L)]\mathsf{E}_G[v_H - P|v_L < P \le v_H] = G(v_L)\mathsf{E}_G[v_H - P|P \le v_L] = F(v_L)\mathsf{E}[v_H - P|P \le v_L] \le F(v_H)\mathsf{E}[v_H - P|P \le v_H] \le c_H$. Thus, $G(p)$ is also a feasible solution. To establish the announced result, we next show that the expected profit under $G(p)$ always dominates that under $F(p)$. That is,

$$
\begin{aligned}
&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_L)\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_H) - F(v_L)]\mathsf{E}[P|v_L < P \le v_H] \\
\le&\alpha \mathsf{E}_G[P|P \le v_L] + (1-\alpha)G(v_L)\mathsf{E}_G[P|P \le v_L] + (1-\alpha)v_H(1-F(v_L)) \\
=&\alpha \mathsf{E}_G[P|P \le v_L] + (1-\alpha)G(v_H)\mathsf{E}_G[P|P \le v_H],
\end{aligned}
$$

where the inequality is due to $F(v_H) \le 1$ and $\mathsf{E}[P|v_L < P \le v_H] \le \mathsf{E}[v_H|v_L < P \le v_H] = v_H$, and the second equality is due to $v_H[1 - F(v_L)] = v_H[1 - G(v_L)] = [G(v_H) - G(v_L)]\mathsf{E}_G[P|v_L < P \le v_H]$. The inequality is strict if $F(p)$ takes any value other than v_H within the interval $(v_L, v_H]$. We thus obtain the announced result. $\square$

Proof of Lemma 6. Denote $\beta = F(v_L)$ and $U(\beta) = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta} - (1 -$

$\alpha)\beta(v_H - v_L)$. We first show that $U(\beta)$ is an upper bound for the expected profit from the optimization problem (3). Suppose $F(p)$ is an optimal solution for (3). Then, the expected profit under $F(p)$ satisfies

$$\begin{aligned}
&\alpha\mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] \\
=&\alpha\mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_L)\mathsf{E}[P|P \le v_L] + (1-F(v_L))v_H] \\
<&\alpha v_L - \frac{\alpha c_L}{F(v_L)} + (1-\alpha)[v_L F(v_L) - c_L + (1-F(v_L))v_H] \\
=&\alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \frac{\alpha c_L}{\beta} - (1-\alpha)\beta(v_H - v_L),
\end{aligned}$$

where $\beta = F(v_L)$. The first equality is due to Lemma 5, and the inequality is due to $c_L < F(v_L)\mathsf{E}[v_L - P|P \le v_L] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L]$.

(i) $\frac{\partial U(\beta)}{\partial \beta} = \frac{\alpha c_L}{\beta^2} - (1-\alpha)(v_H - v_L)$. If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge 1$, $\frac{\partial U(\beta)}{\partial \beta}$ is guaranteed to be greater than or equal to 0, because $\beta \le 1$. That is, $U(\beta)$ is increasing in β when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge 1$. Consequently, $U(\beta) \le \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \alpha c_L - (1-\alpha)(v_H - v_L) = v_L - c_L \le \max\{v_L, (1-\alpha)v_H\}$.

(ii) Because $\frac{\partial U(\beta)}{\partial \beta} = \frac{\alpha c_L}{\beta^2} - (1-\alpha)(v_H - v_L)$ is decreasing in β, $U(\beta)$ is concave in β. $\frac{\partial U(\beta)}{\partial \beta} = 0$ is realized at $\beta = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$. Recall that any feasible solution to Problem (3) satisfies $\beta < \frac{c_H - c_L}{v_H - v_L}$. Consequently, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, the maximum of $U(\beta)$ is realized at $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$, and thus $U(\beta) \le U_1 = U(\beta^*)$. Otherwise, $U(\beta) \le U(\frac{c_H - c_L}{v_H - v_L} - \delta) = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L} - \eta(\delta)$, where $\lim_{\delta \searrow 0} \eta(\delta) = 0$. Thus, we obtain the announced results. $\square$

Proof of Proposition 3. Recall that the optimal profit given by Problem (2) is bounded from above by U_1 if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$, or by U_2 if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} \ge \frac{c_H - c_L}{v_H - v_L}$. Next we show that the two-point distribution shown in the theorem is a feasible solution to Problem (2), and the corresponding profit converges to the upper bounds when δ converges to 0.

For the feasibility, we notice that $\mathsf{E}[(v_L - P)^+] = \beta^*(v_L - \underline{p}^*) = c_L + \beta^*\eta(\delta) > c_L$, and $\mathsf{E}[(v_H - P)^+] = \beta^*(v_H - \underline{p}^*) = \beta^*(v_H - v_L) + c_L + \beta^*\eta(\delta) \le c_H$, which is due to $0 < \beta^* < \frac{c_H - c_L}{v_H - v_L}$ and $\eta(\delta) \searrow 0$. Thus the two-point distribution is a feasible solution to Problem (2).

Next we prove the optimality of the two-point distribution. Consider first when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L}$. The expected profit is given by

$$
\begin{aligned}
&\alpha\mathsf{E}[P|P\le \underline{p}^*]+(1-\alpha)F(v_H)\mathsf{E}[P|P\le v_H]\\
=&\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-\frac{\alpha c_L}{\beta^*}-(1-\alpha)\beta^*(v_H-v_L)-[\alpha+(1-\alpha)\beta^*]\eta(\delta)\\
=&U_1-(\alpha+(1-\alpha)\beta^*)\eta(\delta).
\end{aligned}
$$

Next when $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}\ge\frac{c_H-c_L}{v_H-v_L}$, the expected profit is given by

$$
\begin{aligned}
&\alpha\mathsf{E}[P|P\le \underline{p}^*]+(1-\alpha)F(v_H)\mathsf{E}[P|P\le v_H]\\
=&\alpha v_L+(1-\alpha)v_H-(1-\alpha)c_L-\frac{\alpha c_L}{\beta^*}-(1-\alpha)\beta^*(v_H-v_L)-[\alpha+(1-\alpha)\beta^*]\eta(\delta)\\
=&U_2-\left[\alpha v_L\frac{v_H-v_L}{c_H-c_L}\frac{1}{\frac{c_H-c_L}{v_H-v_L}-\delta}-(1-\alpha)(v_H-v_L)\right]\delta-[\alpha+(1-\alpha)\beta^*]\eta(\delta).
\end{aligned}
$$

We thus obtain the announced result. $\square$

Proof of Proposition 4. To facilitate the discussion, we first prove an auxiliary lemma as described below.

LEMMA OS.2. *(i) The necessary and sufficient conditions for $U_1>v_L$ and $U_1>(1-\alpha)v_H$ are given by $\sqrt{\frac{c_L}{v_H-v_L}}<\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ and $\sqrt{\frac{c_L}{v_H-v_L}}<\sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}}-1\right)$, respectively;*

(ii) The necessary and sufficient conditions for $U_2>v_L$ and $U_2>(1-\alpha)v_H$ are given by $\frac{c_H}{v_H-v_L}+\frac{\alpha c_L}{(1-\alpha)(c_H-c_L)}<1$ and $\alpha\frac{v_Lc_H-c_Lv_H}{c_H-c_L}>(1-\alpha)c_H$, respectively.

Proof of Lemma OS.2. (i) $U_1-v_L=(1-\alpha)(v_H-v_L)-(1-\alpha)c_L-2\sqrt{\alpha(1-\alpha)c_L(v_H-v_L)}=(1-\alpha)(v_H-v_L)\left[1-\frac{c_L}{v_H-v_L}-2\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}\right]$. Thus, $U_1>v_L$ if and only if $\sqrt{\frac{c_L}{v_H-v_L}}<\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. On the other hand, $U_1-(1-\alpha)v_H=\alpha v_L-(1-\alpha)c_L-2\sqrt{\alpha(1-\alpha)c_L(v_H-v_L)}=(1-\alpha)(v_H-v_L)\left[\frac{\alpha v_L}{(1-\alpha)(v_H-v_L)}-\frac{c_L}{v_H-v_L}-2\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}\right]$. Thus, $U_1>(1-\alpha)v_H$ if and only if $\sqrt{\frac{c_L}{v_H-v_L}}<\sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}}-1\right)$.

(ii) $U_2-v_L=(1-\alpha)(v_H-v_L)-(1-\alpha)c_H-\alpha c_L\frac{v_H-v_L}{c_H-c_L}=(1-\alpha)(v_H-v_L)\left[1-\frac{c_H}{v_H-v_L}-\frac{\alpha c_L}{(1-\alpha)(c_H-c_L)}\right]$. Thus, $U_2>v_L$ if and only if $\frac{c_H}{v_H-v_L}+\frac{\alpha c_L}{(1-\alpha)(c_H-c_L)}<1$. On the other hand, $U_2-(1-\alpha)v_H=\alpha v_L-(1-\alpha)c_H-\alpha c_L\frac{v_H-v_L}{c_H-c_L}=\alpha\frac{v_Lc_H-c_Lv_H}{c_H-c_L}-(1-\alpha)c_H$. Thus, $U_2>(1-\alpha)v_H$ if and only if $\alpha\frac{v_Lc_H-c_Lv_H}{c_H-c_L}>(1-\alpha)c_H$. $\square$

Now we are ready to prove the main results. Consider first when $v_L\ge(1-\alpha)v_H$. Recall that, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}<\frac{c_H-c_L}{v_H-v_L}$ and $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}<1$, the profit with an optimal randomized pricing policy is U_1, based on Lemma 6. The profit with an optimal static pricing policy is v_L when

$v_L \geq (1-\alpha)v_H$. To this end, we only need to compare U_1 and v_L. Based on Lemma OS.2(i), $U_1 > v_L$ if and only if $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. Moreover, $\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}} < \sqrt{\frac{1-\alpha}{\alpha}}$, which is due to $\sqrt{\alpha}(1-\sqrt{\alpha}) < 1-\alpha = (1+\sqrt{\alpha})(1-\sqrt{\alpha})$. As such, $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < 1$ is guaranteed by the condition $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$.

If $1 > \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} \geq \frac{c_H-c_L}{v_H-v_L}$, the profit with an optimal randomized pricing policy is approximately U_2, by Lemma 6. Thus, we only need to compare U_2 and v_L. According to Lemma OS.2(ii), $U_2 > v_L$ if and only if $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} < 1$. Moreover, we can show that $\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)} < \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)}$ because $\frac{c_H-c_L}{v_H-v_L} < \frac{c_H}{v_H-v_L} < 1$, where the second inequality is due to $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} < 1$. Consequently, $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < 1$ is guaranteed by the condition $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} < 1$.

Consider next when $v_L \leq (1-\alpha)v_H$, i.e., $\alpha v_L \leq (1-\alpha)(v_H - v_L)$. The profit with an optimal static pricing policy is $(1-\alpha)v_H$. Based on Assumption (S), we have $c_L < v_L$, and thus $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \sqrt{\frac{\alpha v_L}{(1-\alpha)(v_H-v_L)}} \leq 1$. Then following a similar approach as described in the preceding paragraphs, we can obtain the conditions that an optimal randomized pricing policy dominates an optimal static pricing policy. That is, if $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$, the condition is given by $\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}} - 1\right)$; Otherwise, the condition is given by $\alpha\frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$.

To sum up, an optimal randomized pricing policy outperforms an optimal static pricing policy if $\frac{v_L}{c_L} > \frac{v_H}{c_H}$ and the following conditions hold.

(i) $v_L \geq (1-\alpha)v_H$.

- (a) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$, $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$;
- (b) Otherwise, $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} < 1$.

(ii) $v_L < (1-\alpha)v_H$.

- (a) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$, $\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}} - 1\right)$;
- (b) Otherwise, $\alpha\frac{v_L c_H - c_L v_H}{c_H - c_L} > (1-\alpha)c_H$.

When c_L is sufficiently small, either the set of conditions (i)-(a) or (ii)-(a) is satisfied, and we thus obtain the desired result. □

Proof of Corollary 1. Under the optimal randomized pricing policy, the expected profit from low valuation and high-valuation customers are given by $\mathsf{E}[P|P \leq \underline{p}^*]$ and $F(v_H)\mathsf{E}[P|P \leq v_H]$, respectively. The surplus of low-valuation customers is $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\leq \underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L$, according to Equation (OS.1), and the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$. Based on Lemma 5, $F(v_H)=1$, and thus $F(v_H)\mathsf{E}[P|P\leq v_H]+\mathsf{E}[(v_H-P)^+]=v_H$. Consequently, the total social welfare under an optimal randomized pricing policy is given by

$$
\begin{aligned}
&\alpha\left(\mathsf{E}[P|P\leq \underline{p}^*]+\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\leq \underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H\\
=&\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H.
\end{aligned}
$$

When $v_L<(1-\alpha)v_H$, the optimal static price is v_H and the corresponding social welfare is $(1-\alpha)v_H$, which is less than $\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H$. In this case, the surplus of all customers is equal to zero under the optimal static pricing policy, which is thus less than that under an optimal randomized pricing policy. On the other hand, when $v_L\geq(1-\alpha)v_H$, the optimal static price is v_L and the corresponding social welfare is $\alpha v_L+(1-\alpha)v_H$, which is greater than $\alpha\left(v_L-\frac{c_L}{F(\underline{p}^*)}+c_L\right)+(1-\alpha)v_H$. In this case, the surplus for low-valuation customers is also zero under the optimal static pricing policy, while customer surplus for high-valuation customers is given by v_H-v_L. Recall that, under the optimal randomized pricing policy, the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$, which is less than or equal to c_H according to the constraint in the optimization problem (2). Thus, the surplus of high-valuation customers under the optimal randomized pricing policy would be lower when $c_H<v_H-v_L$, and we obtain the announced results. □

Proof of Corollary 2. Under the optimal randomized pricing policy, the surplus of low-valuation customers is $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\leq \underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L$, according to Equation (OS.1), and the surplus of high-valuation customers is $\mathsf{E}[(v_H-P)^+]$. Due to Proposition 3, $\frac{F(\underline{p}^*)\mathsf{E}[v_L-P|P\leq \underline{p}^*]-c_L}{F(\underline{p}^*)}+c_L=v_L-\underline{p}^*-\frac{\bar{F}(\underline{p}^*)}{F(\underline{p}^*)}c_L$, and $\mathsf{E}[(v_H-P)^+]=F(\underline{p}^*)(v_H-\underline{p}^*)$. Therefore, the difference of the two is given by $(v_H-v_L)-\bar{F}(\underline{p}^*)\left(v_H-\underline{p}^*-\frac{c_L}{F(\underline{p}^*)}\right)$. Because $v_H>v_L\geq\underline{p}^*+\frac{c_L}{F(\underline{p}^*)}$, the difference in consumer surplus between high-valuation and low-valuation customers would be less than v_H-v_L. When $v_L\geq(1-\alpha)v_H$, the optimal static pricing policy is to charge a fixed price of v_L, and thus the difference in consumer surplus would be exactly v_H-v_L. We thus obtain the announced result. □

Proof of Proposition 5. We prove the result for the general case with n customer segments. A fraction α_i of customers are of type-i, who value the product at v_i and incur per-period waiting time

c_i. Without loss of generality, we assume that $c_1 < c_2 < \cdots < c_n$. This is an innocuous assumption as we do not require $v_i < v_j$ for any $i < j$. We first prove an auxiliary lemma.

LEMMA OS.3. *Without loss of generality, we assume that $c_1 < c_2 < \cdots < c_n$. Consider any deterministic pricing policy. If a customer with valuation v_i, for any $i \in \{1, \ldots, n\}$, arriving in period 1 purchases in period j, then*

(i) any customer with valuation v_i arriving in period $t \in \{2, \ldots, j\}$ will purchase in period j;

(ii) any customer with valuation $v_{i'}$, for any $i' < i$, arriving in period $t \in \{1, \ldots, j\}$ will not purchase earlier than period j, should she ever purchase.

Proof of Lemma OS.3. First, we prove Lemma OS.3(i). A customer with valuation v_i arriving in period 1 purchases in period j implies that $p_j + (j-1)c_i = \min_{t \geq 1}\{p_t + (t-1)c_i\}$ and $p_j + (j-1)c_i < v_i$. As a direct consequence, we have $p_j + (j-1)c_i = \min_{t \geq t'}\{p_t + (t-1)c_i\}$, for any $t' = 2, 3 \ldots, j$. Thus, all type-i customers arriving in period $t \in [2, j]$ will purchase in period j.

Next, we show that any customer with valuation v_i', for any $i' < i$, arriving in period $t \in [1, j]$ will only purchase in period j or afterwards, should she ever purchase. Because $p_j + (j-1)c_i \leq p_t + (t-1)c_i$ for any $t \leq j$, we have $p_j - p_t \leq (t-j)c_i \leq (t-j)c_{i'}$ for any $i' < i$, which implies that $p_j + (j-1)c_{i'} \leq p_t + (t-1)c_{i'}$. We thus obtained the announced result. $\square$

Note that under any deterministic pricing policy, a customer upon arrival would either leave immediately without purchasing or choose to purchase a unit of the product (either immediately or in a future period). We first show that under any deterministic pricing policy, there exists a cutoff period, say period T, such that all customers arriving before or during period T would have left by the end of period T. Without loss of generality, suppose type-$\underline{i}$ is the lowest type of customers who would make a purchase at some point under such a pricing policy. That is, any customer of type-i, $i < \underline{i}$, always leaves immediately. Then our preceding statement that any customer arriving before or during period T would have left by the end of the period holds automatically for customers of type-i, for any $i < \underline{i}$.

Consider a customer with valuation $v_{\underline{i}}$. Suppose that $t_{\underline{i}} \geq 1$ is the first period that a customer with valuation $v_{\underline{i}}$ would choose to purchase the product. That is, all customers with valuation $v_{\underline{i}}$ arriving before period $t_{\underline{i}}$ would leave immediately without purchasing. Denote the period in which the type-$\underline{i}$ customer arriving in period $t_{\underline{i}}$ makes the purchase as T. Based on Lemma OS.3(i), any customer with valuation $v_{\underline{i}}$ arriving in period $t \in \{t_{\underline{i}}+1,\ldots,T\}$ will make a purchase in period T. We first prove by contradiction that any customer with valuation v_i, $i \in \{\underline{i}+1,\ldots,n\}$, arriving in period $t \in \{t_{\underline{i}},\ldots,T\}$ would make a purchase no later than period T, should she ever purchase. Suppose on the contrary, a customer with valuation $v_{i'}$, $i' \in \{\underline{i}+1,\ldots,n\}$, arriving in period $t' \in \{t_{\underline{i}},\ldots,T\}$, would make a purchase in period $T' > T$. Then Lemma OS.3(ii) would imply that any customer with valuation $v_{\underline{i}}$ arriving in period t' will not purchase earlier than period T', which contradicts with the aforementioned result. Thus, any type-i customer, $i \geq \underline{i}$, arriving between period $t_{\underline{i}}$ and period T would have left (with or without purchasing) by the end of period T.

Next we show by contradiction that any customer with valuation v_i, $i \in \{\underline{i}+1,\ldots,n\}$, arriving in period $t \in \{1,\ldots,t_{\underline{i}}-1\}$ would make a purchase no later than period T, should she ever purchase. Suppose a customer with valuation $v_{i'}$, $i' \in \{\underline{i}+1,\ldots,n\}$, arriving in period $t' \in \{1,\ldots,t_{\underline{i}}-1\}$ would make a purchase in period $T' > T$. By Lemma OS.3(i), any customer with valuation $v_{i'}$ arriving in period $t \in \{t_{\underline{i}},\ldots,T\}$ would make a purchase in period T', which however contradicts with the result in the preceding paragraph. Thus, any type-i customer, $i \geq \underline{i}$, arriving between period 1 and period $t_{\underline{i}}-1$ would have left (with or without purchasing) by the end of period T.

Therefore, under any deterministic pricing policy, there exists a period T such that all customers arriving between period 1 and period T would have left by the end of period T. This statement also holds for, a fortiori, any optimal deterministic pricing policy, say $\mathbf{p}^* = \{p_t^*\}_{t\in\mathbb{N}}$. Following a similar approach, we can show that starting from period $T+1$, there exists a T' such that all customers arriving between period $T+1$ and period $T+T'$ would have left by the end of period $T+T'$. Due to the optimality of the pricing policy, we must have $T = T'$ and $p_t = p_{T+t}$, $t \in \{1,\ldots,T\}$, and thus we obtain the announced result. $\square$

Proof of Lemma 7. Consider the scenario when $c_L \geq c_H$. If low-valuation customers arriving in period t would make a purchase with price $p_{t'} \leq v_L$, then high-valuation customers arriving in the same period would also pay no more than $p_{t'}$ because $p_{t'} + (t'-t)c_H \leq p_{t'} + (t'-t)c_L \leq v_L < v_H$. In this case, $\pi_t(\mathbf{p}) \leq v_L$. On the other hand, if low-valuation customers in period t choose to leave without purchasing, then the firm's profit is bounded below by $(1-\alpha)v_H$, i.e., $\pi_t(\mathbf{p}) \le (1-\alpha)v_H$. Therefore, $\Pi(\mathbf{p}) = \frac{1}{T}\sum_{t=1}^{T}\pi_t(\mathbf{p}) \le \max\{v_L, (1-\alpha)v_H\}$. $\square$

Proof of Proposition 6. Suppose $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$ is an optimal cyclic pricing policy where some high-valuation customers would wait. Without loss of generality, we assume that high-valuation customers arriving in period 1 would wait and purchase in period $j > 1$. That is, $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$ and $p_j + (j-1)c_H < v_H$. As a direct consequence, we have $p_j + (j-t)c_H = \min\{p_t, p_{t+1} + c_H, \ldots, p_T + (T-t)c_H\}$, for any $t = 2, 3\ldots, j$. Next we show that any high-valuation customers arriving in period t, where $1 \le t \le j$, would always purchase in period j. Clearly, for a customer arriving in period t, purchasing in period j dominates the option of purchasing in any period between t and T. To this end, we need to show that the customer is worse off if she purchases in any period between $T+1$ and $T+t-1$. Due to $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$, we have $p_j + (j-1)c_H \le p_{t'} + (t'-1)c_H$, $t' = 1, 2, \ldots, t-1$. Coupling with the fact that $p_{t'} = p_{T+t'}$, we have $p_j + (j-t)c_H \le p_{t'} + (t'-t)c_H = p_{T+t'} + (t'-t)c_H < p_{T+t'} + (T+t'-t)c_H$. As a result, any high-valuation customer arriving in period $t \in [1, j]$ would make a purchase in period j.

Next we show that any low-valuation customer arriving in period $t \in [1, j]$ would also purchase in period j. We restrict our discussion to the case when $c_H > c_L$ because the optimal cyclic pricing policy degenerates into a static pricing policy when $c_H \le c_L$, based on Lemma 7. Because $p_j + (j-1)c_H \le p_t + (t-1)c_H$ for any $t \le j$, we have $p_j - p_t \le (t-j)c_H < (t-j)c_L$, which implies that $p_j + (j-1)c_L \le p_t + (t-1)c_L$. That is, low-valuation customers arriving in period $t \in [1, j]$ would make a purchase in period j and afterwards if they would ever purchase. Now we show that low-valuation customers arriving in period $t \in [1, j]$ will never purchase after period j by contradiction. Suppose low-valuation customers purchase in period $j' > j$. Let us denote $s = \arg\min_{1 \le i \le j-1}\{p_i + (i-1)c_H\}$. Recall that $p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$, then $p_j + (j-1)c_H \le p_s + (s-1)c_H$. However, if $p_j + (j-1)c_H = p_s + (s-1)c_H$, note $s < j$, with tie-breaking rule, the high-valuation customers arriving in period $t \in [1, s]$ would make a purchase in period s, which contradicts with the result that any high-valuation customer arriving in period $t \in [1, j]$ would make a purchase in period j from the previous paragraph. Therefore, we have $p_j + (j-1)c_H < p_s + (s-1)c_H$. Then, we can design a new pricing policy $\mathbf{p}'$, which is exactly the same as $\mathbf{p}$ except for p'_s. We let $p'_s + (s-1)c_H = p_j + (j-1)c_H$, and then $p'_s > p_j$ because $s < j$. Now we will show that the behavior of low-valuation customers under the new policy is exactly the same as before, however, the firm's profit from high-valuation customers would be higher with $\mathbf{p}'$. Consider the low-valuation customers' behavior, since low-valuation customers purchase in period $j' > j$ under policy $\mathbf{p}$, thus $p_{j'} + (j'-1)c_L = \min\{p_1, p_2 + c_L, \ldots, p_T + (T-1)c_L\}$. Then, note $s < j$, we have $p_{j'} + (j'-1)c_L \leq p_j + (j-1)c_L = p_j + (j-1)c_H - (j-1)(c_H - c_L) = p'_s + (s-1)c_H - (j-1)(c_H - c_L) < p'_s + (s-1)c_H - (s-1)(c_H - c_L) = p'_s + (s-1)c_L$. As a result, we have $p'_{j'} + (j'-1)c_L = \min\{p'_1, p'_2 + c_L, \ldots, p'_T + (T-1)c_L\}$ because $\mathbf{p}'$ is exactly the same as $\mathbf{p}$ except for p'_s. Then low-valuation customers arriving in period $t \in [1, j]$ will also purchase in period j' under the new policy. It is easy to see $p'_{j'+T} + (j'+T-t)c_L = \min\{p'_{T+1} + (T+1-t)c_L, p'_{T+2} + (T+2-t)c_L, \ldots, p'_{2T} + (2T-t)c_L\}$ for $t = j+1, j+2 \ldots, T$. Then low-valuation customers arriving in period $t \in [j+1, T]$ will purchase in a period between period t and T or in period $j'+T$ under policy $\mathbf{p}'$, which is as same as that under policy $\mathbf{p}$. Consider the high-valuation customers, recall that $p'_s + (s-1)c_H = p_j + (j-1)c_H = \min\{p_1, p_2 + c_H, \ldots, p_T + (T-1)c_H\}$, then the high-valuation customers arriving in period $t \in [1, s]$ purchase in period s with price $p'_s > p_j$ by tie-breaking rule under policy $\mathbf{p}'$. And the high-valuation customers arriving in period $t \in [s+1, j]$ still purchase in period j. It is easy to see $p'_{s+T} + (s+T-t)c_H = p'_{j+T} + (j+T-t)c_H = \min\{p'_{T+1} + (T+1-t)c_H, p'_{T+2} + (T+2-t)c_H, \ldots, p'_{2T} + (2T-t)c_H\}$ for $t = j+1, j+2 \ldots, T$. Then high-valuation customers arriving in period $t \in [j+1, T]$ will purchase in a period between period t and T or in period $s+T$ under policy $\mathbf{p}'$. For the first case, the behavior of high-valuation customer is as same as that under policy $\mathbf{p}$. For the second case, high-valuation customers arriving in period $t \in [j+1, T]$ will purchase in period $j+T$ with price $p_{j+T} = p_j < p'_s = p'_{s+T}$ under policy $\mathbf{p}$. Thus the firm's profit from high-valuation customers would be higher or equal to that under policy $\mathbf{p}$. The result contradicts with the assumption that $\mathbf{p}$ is an optimal cyclic pricing policy. Thus we conclude that low-valuation customers arriving in period $t \in [1, j]$ also purchase in period j.

Following the same approach, we can show that if high-valuation customers arriving in period $j+1$ choose to purchase in period $j+k$, then all customers arriving in period $t \in [j+1, j+k]$ will purchase in period $j+k$. That is, the cyclic policy $\mathbf{p}$ can be decomposed into many mini cycles. Without of loss generality, we assume that there are m mini cycles, where the length of each mini cycle is denoted by n_l, $l=1,\ldots,m$ and $\sum_{l=1}^{m} n_l = T$. Denote the profit from customers who made a purchase in period t by $\tilde{\pi}_t(\mathbf{p})$. Then, $\Pi(\mathbf{p}) = [n_1\tilde{\pi}_{n_1}(\mathbf{p}) + n_2\tilde{\pi}_{n_1+n_2}(\mathbf{p}) + \cdots + n_m\tilde{\pi}_{n_1+n_2+\cdots+n_m}(\mathbf{p})]/T$. Denote $\tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p}) = \max\{\tilde{\pi}_{n_1+n_2+\cdots+n_l}(\mathbf{p}), l=1,2,\ldots,m\}$. Then, we have $\Pi(\mathbf{p}) \le \tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p})$.

Last, we show that $\mathbf{p}$ cannot be an optimal cyclic pricing policy by contradiction. If $\mathbf{p}$ is an optimal cyclic pricing policy, then $\Pi(\mathbf{p}) \ge \max\{v_L, (1-\alpha)v_H\}$ because a static pricing policy is a special case of cyclic pricing policies. The analysis in preceding paragraphs shows that all customers arriving in period $t \in [1,j]$ would make a purchase in period j under policy $\mathbf{p}$. Thus, for any $t \le j$, if low-valuation customers purchase in period j, $\tilde{\pi}_t(\mathbf{p}) = p_j < v_L$; otherwise, only high-valuation customers would make a purchase in period j, and thus $\tilde{\pi}_t(\mathbf{p}) = (1-\alpha)p_j \le (1-\alpha)v_H$. Therefore, the average expected profit from this mini cycle cannot exceed $\max\{v_L, (1-\alpha)v_H\}$, and thus $\Pi(\mathbf{p}) \le \tilde{\pi}_{n_1+n_2+\cdots+n_{l^*}}(\mathbf{p}) < \max\{v_L, (1-\alpha)v_H\}$, which contradicts to the optimality of $\mathbf{p}$. $\square$

Proof of Proposition 7. Suppose $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$ is an optimal cyclic pricing policy. Let us denote by $p^L_{\min} = \min\{p_1, p_2 + c_L, \ldots, p_T + (T-1)c_L\}$, and $t^L_{\min}$ the earliest time period such that $p_{t^L_{\min}} + (t^L_{\min} - 1)c_L = p^L_{\min}$. Following the same approach as that in the proof of Proposition 6, we can show that any low-valuation customer arriving in period $t \in \{1, 2, \ldots, t^L_{\min}\}$ will purchase in period $t^L_{\min}$ with price $p_{t^L_{\min}}$. Thus, $p_{t^L_{\min}} < v_L - (t^L_{\min} - 1)c_L$ because any low-valuation customer arriving in period 1 purchases in period $t^L_{\min}$. On the other hand, high-valuation customers will not wait under an optimal cyclic pricing policy as shown in Proposition 6. Thus, $p_{t'} \le p_{t^L_{\min}} + (t^L_{\min} - t')c_H$, where $t' \in \{1, 2\ldots, t^L_{\min}\}$. Otherwise, high-valuation customers would be better off postponing purchase until period $t^L_{\min}$. Because $p_{t'} \le v_H$, we have

$$\begin{aligned} p_{t'} &\le \min\{v_H,\, p_{t^L_{\min}} + (t^L_{\min} - t')c_H\} < \min\{v_H,\, v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H\} \\ &= v_H - [v_H - v_L + (t^L_{\min} - 1)c_L - (t^L_{\min} - t')c_H]^+. \end{aligned}$$

Thus, the profit in the first $t^L_{\min}$ is bounded from above by

$$\begin{aligned} \sum_{t=1}^{t^L_{\min}} \pi_t(\mathbf{p}) &= (1-\alpha)(p_1 + p_2 + \cdots + p_{t^L_{\min}}) + \alpha t^L_{\min} p_{t^L_{\min}} \\ &< (1-\alpha)\sum_{i=1}^{t^L_{\min}} (v_H - [v_H - v_L + (t^L_{\min} - 1)c_L - (t^L_{\min} - t')c_H]^+) + \alpha t^L_{\min}(v_L - (t^L_{\min} - 1)c_L). \end{aligned}$$

The upper bound can be approximated by $p_{t'} = v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t')c_H]^+$ for $t' = 1, 2 \ldots, t^L_{\min}$. It is easy to verify that $p_1 \geq p_2 \geq \cdots \geq p_{t^L_{\min}}$. Furthermore, we have $p_{t^L_{\min}} = v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t^L_{\min})c_H] = v_L - (t^L_{\min} - 1)c_L - \delta$ and thus $p_{t^L_{\min}} + (t^L_{\min} - 1)c_L = v_L - \delta$. For any $t' \in \{1, 2 \ldots, t^L_{\min} - 1\}$, $p_{t'}$ is equal to either v_H or $v_H - [v_H - v_L + (t^L_{\min} - 1)c_L + \delta - (t^L_{\min} - t')c_H] = v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H - \delta$. If $p_{t'} = v_H$, then $p_{t'} + (t' - 1)c_L = v_H + (t' - 1)c_L > v_L - \delta$. On the other hand, if $p_{t'} = v_L - (t^L_{\min} - 1)c_L + (t^L_{\min} - t')c_H - \delta$, then $p_{t'} + (t' - 1)c_L = v_L + (t^L_{\min} - t')(c_H - c_L) - \delta$, which is again greater than $v_L - \delta$ because $t' < t^L_{\min}$ and $c_H > c_L$. Thus all low type customers will wait until period $t^L_{\min}$ to purchase. We can then follow the same approach and show that optimal prices from period $t^L_{\min} + 1$ onward are simply replications of $\{p_1, p_2, \ldots, p_{t^L_{\min}}\}$. Thus, $T = t^L_{\min}$, and the optimal expected profit under policy $\mathbf{p}$ is given by $\Pi(\mathbf{p}) = \frac{1}{T} \sum_{t=1}^{T} \pi_t(\mathbf{p})$. $\square$

Proof of Corollary 3. (i) Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

$$\begin{aligned}
\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta) \\
&< (1-\alpha)\frac{\sum_{t'=1}^{T} [v_L - (T-1)c_L + (T-t')c_H]}{T} + \alpha(v_L - (T-1)c_L) \\
&= (1-\alpha)\frac{T-1}{2}c_H + (v_L - (T-1)c_L) \leq v_L \leq \max\{v_L, (1-\alpha)v_H\},
\end{aligned}$$

where the first inequality is due to $\delta > 0$ and $[v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+ \geq v_H - v_L + (T-1)c_L + \delta - (T-t')c_H$, and the second inequality is due to $c_H/c_L \leq 2/(1-\alpha)$. A static pricing policy can achieve a profit of $\max\{v_L, (1-\alpha)v_H\}$, with either the volume strategy of pricing at v_L or the margin strategy of pricing at v_H. Thus we obtain the announced result.

(ii) Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

$$\begin{aligned}
\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta) \\
&< (1-\alpha)\left[\frac{T-1}{T}v_H + \frac{1}{T}(v_L - (T-1)c_L)\right] + \alpha(v_L - (T-1)c_L) \equiv R(T),
\end{aligned}$$

where the first inequality is due to $\delta > 0$ and $v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+ \leq v_H$ for $t' \leq$

$T-1$. Taking the first order derivative with respect to T, we have $R'(T) = (1-\alpha)\frac{v_H - v_L - c_L}{T^2} - \alpha c_L$, which is no more than 0 when $v_H - v_L \le c_L$. Now we consider $c_L < v_H - v_L \le c_L/(1-\alpha)$, then

$$R'(T) = (1-\alpha)\frac{v_H - v_L - c_L}{T^2} - \alpha c_L \le (1-\alpha)(v_H - v_L - c_L) - \alpha c_L \le 0,$$

where the first inequality is due to $T \ge 1$, the second inequality is due to $v_H - v_L \le c_L/(1-\alpha)$. Hence, $R(T)$ is decreasing in T. Consequently, $\Pi(T) < R(T) \le R(1) = v_L \le \max\{v_L, (1-\alpha)v_H\}$. Thus we obtain the announced result. $\square$

Proof of Proposition 8. First, we show that if an optimal cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$, then it is always better than an optimal randomized pricing policy. Moreover, under the conditions that $v_H - v_L > c_L/(1-\alpha)$ and $v_H - v_L < c_H \Big/ \left(1 + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$, the optimal cyclic pricing policy is in this form. Suppose a cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$, by the proof of Corollary 3 (ii), then the profit of this policy is $\Pi(T) = R(T) - (1-\alpha)\frac{1}{T}\delta - \alpha\delta$. It is easy to verify that $R'(T)$ is decreasing in T when $v_H - v_L > c_L/(1-\alpha) > c_L$, and thus we conclude that $R(T)$ is concave, with its maximum achieved at $T_1 = \sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}}$. Since a cyclic length should be an integer, the maximum of $R(T)$ is thus given by $\max\{R(\lfloor T_1 \rfloor), R(\lfloor T_1 \rfloor + 1)\}$, where $\lfloor x \rfloor$ represents the greatest integer that is no more than x. Note that $\max\{R(\lfloor T_1 \rfloor), R(\lfloor T_1 \rfloor + 1)\}$ is an upper bound of the profit of an optimal cyclic pricing policy. Let $T = \lfloor T_1 \rfloor$ or $T = \lfloor T_1 \rfloor + 1$, then this cyclic pricing policy is asymptotically optimal. Note $R(T)$ is concave and decreasing when $T \ge T_1$, then $R(\lfloor T_1 \rfloor + 1) \ge R(T_1 + 1)$ since $T_1 + 1 \ge \lfloor T_1 \rfloor + 1$. Consequently, $\max\{R(\lfloor T_1 \rfloor), R(\lfloor T_1 \rfloor + 1)\} \ge R(T_1 + 1)$, i.e., $R(T_1 + 1)$ is a lower bound of the profit of an optimal cyclic pricing policy. Hence, we have

$$\begin{aligned}
R(T_1+1) &= (1-\alpha)\left[\frac{T_1}{T_1+1}v_H + \frac{1}{T_1+1}(v_L - T_1 c_L)\right] + \alpha(v_L - T_1 c_L) \\
&= \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \alpha T_1 c_L - (1-\alpha)\frac{1}{T_1+1}(v_H - v_L - c_L) \\
&> \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \alpha T_1 c_L - (1-\alpha)\frac{1}{T_1}(v_H - v_L - c_L) \\
&= \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - 2\sqrt{\alpha(1-\alpha)c_L(v_H - v_L - c_L)} \\
&> \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - 2\sqrt{\alpha(1-\alpha)c_L(v_H - v_L)} = U_1,
\end{aligned}$$

where the first inequality is due to $T_1 > 0$ and $v_H - v_L > c_L/(1-\alpha) > c_L$, the second inequality is due to $c_L > 0$, the second equality is due to $T_1 = \sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}}$. Based on Lemma 6, the expected profit of an optimal randomized pricing policy is either U_1 or U_2. Note that $U_1 \geq U_2$. Now we will prove $v_H - v_L + (T-1)c_L + \delta - (T-t')c_H \leq 0$ for $t' = 1, 2, \ldots, T-1$ when $T = \lfloor T_1 \rfloor$ or $T = \lfloor T_1 \rfloor + 1$ under the condition $v_H - v_L < c_H \Big/ \left(1 + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$, which induces the cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$. Thus, we have

$$v_H - v_L + (T-1)c_L + \delta - (T-t')c_H < v_H - v_L + T_1 c_L - c_H \leq v_H - v_L + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}(v_H - v_L) - c_H < 0,$$

where the first inequality is due to $\delta > 0$, $T = \lfloor T_1 \rfloor$ or $T = \lfloor T_1 \rfloor + 1$, $t' \leq T-1$, the second inequality is due to $T_1 c_L = \sqrt{\frac{(1-\alpha)c_L(v_H - v_L - c_L)}{\alpha}}$ is maximized at $c_L = (v_H - v_L)/2$, the third inequality is due to $v_H - v_L < c_H \Big/ \left(1 + \frac{1}{2}\sqrt{\frac{1-\alpha}{\alpha}}\right)$. Hence, we get the announced result of Proposition 8(i).

Next, we show under the conditions that $v_H - v_L > c_L/(1-\alpha)$ and $v_H - v_L$ is higher than a threshold, an optimal randomized pricing policy is always better. Since the optimal cyclic pricing policy is in the form of the prices staying constant at v_H for some time, dropping by a size no more than c_H and then dropping by a size of exactly c_H to the end-of-cycle price $v_L - (T-1)c_L - \delta$ by Proposition 7. By the previous paragraph, we know that if an optimal cyclic pricing policy is in the form of the first $T-1$ periods priced at v_H and the last period priced at $v_L - (T-1)c_L - \delta$, then it is always better than an optimal randomized pricing policy. Therefore, if an optimal randomized pricing policy is always better, then the optimal cyclic pricing policy should be in the general form of a series of markdowns as illustrated in Figure 2(a). By Corollary 3(i), we just need to consider the case $c_H/c_L > 2/(1-\alpha)$. Based on Proposition 7, the profit from an optimal cyclic pricing policy is given by

$$
\begin{aligned}
\Pi(T) &= (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L + \delta - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L - \delta) \\
&< (1-\alpha)\frac{\sum_{t'=1}^{T} v_H - [v_H - v_L + (T-1)c_L - (T-t')c_H]^+}{T} + \alpha(v_L - (T-1)c_L) \\
&< (1-\alpha)\frac{\int_0^T v_H - [v_H - v_L + (T-1)c_L - (T-t)c_H]^+ \mathrm{d}t}{T} + \alpha(v_L - (T-1)c_L) \\
&= (1-\alpha)v_H + \alpha(v_L - (T-1)c_L) - (1-\alpha)\frac{[v_H - v_L + (T-1)c_L]^2}{2Tc_H} \equiv H(T),
\end{aligned}
$$

where the first inequality is due to $\delta > 0$, and the second is due to $v_H - [v_H - v_L + (T-1)c_L - (T-t)c_H]^+$ being monotonically decreasing in t. Taking the first order derivative with respect to T, we have $H'(T) = -\alpha c_L + (1-\alpha)\frac{(v_H - v_L - c_L)^2}{2c_H T^2} - (1-\alpha)\frac{c_L^2}{2c_H}$. It is easy to verify that $H'(T)$ is decreasing in T, and thus we conclude that $H(T)$ is concave, with its maximum achieved at $T_2 = \sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)^2}{2\alpha c_L c_H + (1-\alpha)c_L^2}}$. The upper bound for the profit of a cyclic pricing policy is thus given by

$$
H(T_2) = \alpha v_L + (1-\alpha)v_H - \alpha(T_2 - 1)c_L - \alpha T_2 c_L - (1-\alpha)\frac{T_2 c_L^2}{c_H} - (1-\alpha)\frac{c_L}{c_H}(v_H - v_L - c_L).
$$

Based on Lemma 6, the expected profit of an optimal randomized pricing policy is either U_1 or U_2. Note that $U_1 \geq U_2$ and $U_2 = \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_H - \alpha c_L \frac{v_H - v_L}{c_H - c_L}$. Now we show $H(T_2) < U_2$ if $v_H - v_L$ is higher than a threshold. First, we show $T_2 > \frac{v_H - v_L}{c_H - c_L}$ if $v_H - v_L$ is higher than a threshold. Since $T_2 - \frac{v_H - v_L}{c_H - c_L} = \sqrt{\frac{1-\alpha}{2\alpha c_L c_H + (1-\alpha)c_L^2}}(v_H - v_L) - \frac{v_H - v_L}{c_H - c_L} - \sqrt{\frac{1-\alpha}{2\alpha c_L c_H + (1-\alpha)c_L^2}}c_L$, it is sufficiently to show $T_2 - \frac{v_H - v_L}{c_H - c_L}$ is increasing in $v_H - v_L$. In other words, $\frac{(1-\alpha)}{2\alpha c_L c_H + (1-\alpha)c_L^2} > \frac{1}{(c_H - c_L)^2}$, which is equivalent to $(1-\alpha)(c_H^2 - 2c_L c_H) > 2\alpha c_L c_H$. It is easy to verify $(1-\alpha)(c_H^2 - 2c_L c_H) > 2\alpha c_L c_H$ when $c_H/c_L > 2/(1-\alpha)$. Consequently, we have $-\alpha T_2 c_L < -\alpha c_L \frac{v_H - v_L}{c_H - c_L}$. Hence, a sufficient condition for $H(T_2) < U_2$ is given by

$$
-\alpha(T_2 - 1)c_L - (1-\alpha)\frac{T_2 c_L^2}{c_H} - (1-\alpha)\frac{c_L}{c_H}(v_H - v_L - c_L) < -(1-\alpha)c_H.
$$

Note the left hand of the inequality is decreasing in $v_H - v_L$ when c_H and c_L are given. In other words, there is a threshold on $v_H - v_L$ above which the inequality holds. We thus obtain the announced result of Proposition 8(ii). $\square$

Proof of Proposition 9. We first prove an auxiliary lemma below, which gives an upper bound for Problem (4).

LEMMA OS.4. *Let* $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}}$.

(i) When $(1-\alpha)(v_H - v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$*, the optimal expected profit from* (4) *is no more than that from an optimal static pricing policy;*

(ii) When $(1-\alpha)(v_H - v_L) > \alpha(1-\gamma)v_L + \alpha\gamma c_L$*, i.e.,* $\tilde{\Delta}_R < 1$*,*

(a) if $\tilde{\Delta}_R < \frac{c_H - c_L}{v_H - v_L}$*, the optimal expected profit from* (4) *is no more than* $\tilde{U}_1 \equiv \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - 2[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R$*;*

(b) otherwise, the optimal expected profit from (4) *is no more than* $\tilde{U}_2 \equiv \alpha\gamma v_L + (1-\alpha)v_H - \alpha(1-\gamma)c_L - (1-\alpha)c_H - \alpha\gamma c_L \frac{v_H - v_L}{c_H - c_L} + \alpha(1-\gamma)v_L \frac{c_H - c_L}{v_H - v_L}$.

Proof of Lemma OS.4. Denote $\beta = F(v_L)$ and $\tilde{U}(\beta) = \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\beta} - (1-\alpha)\beta(v_H - v_L) + \alpha(1-\gamma)\beta v_L$. We first show that $\tilde{U}(\beta)$ is an upper bound for the expected profit from the optimization problem (4). Suppose $F(p)$ is an optimal solution for (4). Then, the expected profit under $F(p)$ satisfies

$$\begin{aligned}
&\alpha\gamma \mathsf{E}[P|P \le \underline{p}^L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
\le&\alpha\gamma \mathsf{E}[P|P \le v_L] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
=&\alpha\gamma \mathsf{E}[P|P \le v_L] + (1-\alpha)[F(v_L)\mathsf{E}[P|P \le v_L] + (1-F(v_L))v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L] \\
<&\alpha\gamma v_L - \frac{\alpha\gamma c_L}{F(v_L)} + (1-\alpha)[v_L F(v_L) - c_L + (1-F(v_L))v_H] + \alpha(1-\gamma)[v_L F(v_L) - c_L] \\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\beta} - (1-\alpha)\beta(v_H - v_L) + \alpha(1-\gamma)\beta v_L,
\end{aligned}$$

where $\beta = F(v_L)$. The first inequality is due to $\underline{p}^L \le v_L$, the second inequality is due to $c_L < \mathsf{E}[(v_L - P)^+] = v_L F(v_L) - F(v_L)\mathsf{E}[P|P \le v_L]$, and the first equality is due to Lemma 5, which also holds for problem (4) by following a similar proof. Taking the derivative of $\tilde{U}(\beta)$ with respect to β, we have $\frac{\partial \tilde{U}(\beta)}{\partial \beta} = \frac{\alpha\gamma c_L}{\beta^2} - (1-\alpha)(v_H - v_L) + \alpha(1-\gamma)v_L$.

(i) If $(1-\alpha)(v_H - v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$, $\frac{\partial \tilde{U}(\beta)}{\partial \beta}$ is guaranteed to be greater than or equal to 0 because $\beta \le 1$. That is, $\tilde{U}(\beta)$ is increasing in β when $(1-\alpha)(v_H - v_L) \le \alpha(1-\gamma)v_L + \alpha\gamma c_L$. Consequently, $\tilde{U}(\beta) \le \tilde{U}(1) = \alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \alpha\gamma c_L - (1-\alpha)(v_H - v_L) + \alpha(1-\gamma)v_L = v_L - c_L \le \max\{v_L, (1-\alpha)v_H\}$.

(ii) If $(1-\alpha)(v_H - v_L) > \alpha(1-\gamma)v_L + \alpha\gamma c_L$, $\tilde{U}(\beta)$ is concave in β. $\frac{\partial \tilde{U}(\beta)}{\partial \beta} = 0$ is realized at $\beta = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}}$. Recall that any feasible solution to problem (4) satisfies $\beta <$

$\frac{c_H-c_L}{v_H-v_L}$. Consequently, if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$, the maximum of $\tilde{U}(\beta)$ is realized at $\tilde{\beta}^* = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}}$, and thus $\tilde{U}(\beta) \le \tilde{U}_1 = U(\tilde{\beta}^*)$. Otherwise, $\tilde{U}(\beta) \le \tilde{U}(\frac{c_H-c_L}{v_H-v_L} - \delta) = \alpha\gamma v_L + (1-\alpha)v_H - \alpha(1-\gamma)c_L - (1-\alpha)c_H - \alpha\gamma c_L\frac{v_H-v_L}{c_H-c_L} + \alpha(1-\gamma)v_L\frac{c_H-c_L}{v_H-v_L} - \eta(\delta)$, where $\lim_{\delta\searrow 0}\eta(\delta) = 0$. Thus, we obtain the announced results. $\square$

Lemma OS.4 shows that the optimal profit given by problem (4) is bounded below by $\tilde{U}_1$ if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$, or by $\tilde{U}_2$ if $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} \ge \frac{c_H-c_L}{v_H-v_L}$. Next we show that the two-point distribution shown in the theorem is a feasible solution to problem (4), and the corresponding profit converges to the upper bounds when δ converges to 0.

For the feasibility, we notice that $\mathsf{E}[(v_L-P)^+] = \tilde{\beta}^*(v_L - \underline{\tilde{p}}^*) = c_L + \tilde{\beta}^*\eta(\delta) > c_L$, and $\mathsf{E}[(v_L-P)^+] = \tilde{\beta}^*(v_H - \underline{\tilde{p}}^*) = \tilde{\beta}^*(v_H - v_L) + c_L + \tilde{\beta}^*\eta(\delta) \le c_H$, which is due to $\tilde{\beta}^* < \frac{c_H-c_L}{v_H-v_L}$ and $\eta(\delta) \searrow 0$. Thus the two-point distribution is a feasible solution to problem (4).

Next we prove the optimality of the two-point distribution. Consider first when $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} < \frac{c_H-c_L}{v_H-v_L}$. The expected profit is given by

$$\begin{aligned}
&\alpha\gamma\mathsf{E}[P|P \le \underline{\tilde{p}}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L]\\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\tilde{\beta}^*} - (1-\alpha)\tilde{\beta}^* v_H + (1-\alpha\gamma)\tilde{\beta}^* v_L - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta)\\
=&\tilde{U}_1 - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta).
\end{aligned}$$

Next when $\sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H-v_L)-\alpha(1-\gamma)v_L}} \ge \frac{c_H-c_L}{v_H-v_L}$, the expected profit is given by

$$\begin{aligned}
&\alpha\gamma\mathsf{E}[P|P \le \underline{\tilde{p}}^*] + (1-\alpha)F(v_H)\mathsf{E}[P|P \le v_H] + \alpha(1-\gamma)F(v_L)\mathsf{E}[P|P \le v_L]\\
=&\alpha\gamma v_L + (1-\alpha)v_H - (1-\alpha\gamma)c_L - \frac{\alpha\gamma c_L}{\tilde{\beta}^*} - (1-\alpha)\tilde{\beta}^* v_H + (1-\alpha\gamma)\tilde{\beta}^* v_L - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta)\\
=&\tilde{U}_2 - \left[\alpha\gamma v_L\frac{v_H-v_L}{c_H-c_L}\frac{1}{\frac{c_H-c_L}{v_H-v_L} - \delta} - (1-\alpha)(v_H-v_L) + \alpha(1-\gamma)\right]\delta - [\alpha\gamma + (1-\alpha\gamma)\tilde{\beta}^*]\eta(\delta).
\end{aligned}$$

We thus obtain the announced result. $\square$

Proof of Corollary 4. We first consider the monotonicity of $\tilde{\beta}^*$ and $\underline{\tilde{p}}^*$ with respect to γ. Note that

$$\frac{\partial\tilde{\Delta}_R^2}{\partial\gamma} = \frac{\alpha c_L[(1-\alpha)v_H - v_L]}{[(1-\alpha)(v_H-v_L) - \alpha(1-\gamma)v_L]^2}.$$

Thus, $\tilde{\Delta}_R$ is increasing in γ if $v_L \le (1-\alpha)v_H$, and decreasing in γ otherwise. As a direct consequence, both $\tilde{\beta}^* = \min\left\{\tilde{\Delta}_R, \frac{c_H-c_L}{v_H-v_L} - \delta\right\}$ and $\underline{\tilde{p}}^* = v_L - \frac{c_L}{\tilde{\beta}^*} - \eta(\delta)$ are increasing in γ if $v_L \le (1-\alpha)v_H$, and decreasing in γ otherwise.

We next consider the monotonicity of the expected profit from the optimal randomized pricing policy with respect to γ. Based on Lemma OS.4, we know that the optimal expected profit is bounded above by either $\tilde{U}_1$ or $\tilde{U}_2$. First taking the derivative of $\tilde{U}_1$ with respect to γ, we have

$$\frac{\partial \tilde{U}_1}{\partial \gamma} = \alpha v_L + \alpha c_L - \frac{\alpha c_L[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L] + \alpha\gamma c_L \alpha v_L}{\sqrt{\alpha\gamma c_L[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]}} = \alpha v_L + \alpha c_L - \frac{\alpha c_L}{\tilde{\Delta}_R} - \alpha v_L \tilde{\Delta}_R.$$

As $\frac{\alpha c_L}{\tilde{\Delta}_R} + \alpha v_L \tilde{\Delta}_R$ is convex in $\tilde{\Delta}_R$, we have $\frac{\alpha c_L}{\tilde{\Delta}_R} + \alpha v_L \tilde{\Delta}_R < \alpha v_L + \alpha c_L$ for any $c_L/v_L < \tilde{\Delta}_R < 1$. Next we show that $\tilde{\Delta}_R > c_L/v_L$. If $(1-\alpha)v_H < v_L$, $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}} > \sqrt{\frac{\alpha\gamma c_L}{\alpha v_L - \alpha(1-\gamma)v_L}} = \sqrt{\frac{c_L}{v_L}} > \frac{c_L}{v_L}$. If $(1-\alpha)v_H \geq v_L$, the profit of the optimal static pricing policy is given by $(1-\alpha)v_H$. Thus, we have $\tilde{U}_1 \geq (1-\alpha)v_H$, which implies $\alpha\gamma v_L > (1-\alpha\gamma)c_L + 2[(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R > [(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L]\tilde{\Delta}_R$. This is equivalent to $\sqrt{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L} < \frac{\alpha\gamma v_L}{\sqrt{\alpha\gamma c_L}}$. Consequently, $\tilde{\Delta}_R = \sqrt{\frac{\alpha\gamma c_L}{(1-\alpha)(v_H - v_L) - \alpha(1-\gamma)v_L}} > \frac{\alpha\gamma c_L}{\alpha\gamma v_L} = \frac{c_L}{v_L}$, and thus we conclude $\frac{\partial \tilde{U}_1}{\partial \gamma} > 0$. Next we show that $\tilde{U}_2$ increases in γ. Recall that $\tilde{U} = \tilde{U}_2$ when $\frac{c_H - c_L}{v_H - v_L} < \tilde{\Delta}_R < 1$. As shown in Lemma OS.1, $\frac{v_L}{c_L} > \frac{v_H}{c_H}$ is a necessary condition for low-valuation customers to wait, and for high-valuation customers to either purchase or leave immediately under any randomized pricing policy. It is easy to verify $\frac{c_L}{v_L} < \frac{c_H - c_L}{v_H - v_L}$ due to $\frac{v_L}{c_L} > \frac{v_H}{c_H}$, and thus

$$\frac{\partial \tilde{U}_2}{\partial \gamma} = \alpha v_L + \alpha c_L - \alpha c_L \frac{v_H - v_L}{c_H - c_L} - \alpha v_L \frac{c_H - c_L}{v_H - v_L} = \alpha \left(1 - \frac{c_H - c_L}{v_H - v_L}\right)\left(v_L - c_L \frac{v_H - v_L}{c_H - c_L}\right) > 0.$$

The expected profit from the optimal randomized pricing policy differs from the upper bounds $\tilde{U}_1$ and $\tilde{U}_2$ only by an infinitesimal term, and therefore we obtain the announced results. $\square$

Proof of Lemma 8. Consider the low-valuation customers first. Based on the transition matrix $\mathbf{M}$ and Equation (5), we have $\mathsf{E}[V(P)] = q_H(v_L - p_d) + (1-q_H)[-c_L + \mathsf{E}[V(P)]]$, and thus $\mathsf{E}[V(P)] = v_L - p_d - c_L \frac{1-q_H}{q_H}$. Low valuation customers would wait for a price of p_d if and only if $-c_L + \mathsf{E}[V(P)] > 0$, i.e., $q_H(v_L - p_d) > c_L$. Similarly, we can show that high-valuation customers always purchase immediately if and only if $q_H(v_H - p_d) \leq c_H$. $\square$

Proof of Proposition 10. Denote by π_L and π_H the steady state probabilities for p_d and v_H, respectively. Solving the equations $(\pi_L, \pi_H)\mathbf{M} = (\pi_L, \pi_H)$ and $\pi_L + \pi_H = 1$, we have $\pi_L = \frac{q_H}{1-q_L+q_H}$, and $\pi_H = \frac{1-q_L}{1-q_L+q_H}$. Under an optimal Markovian pricing policy, low-valuation customers always purchase with price p_d, while high-valuation customers purchase immediately upon arrival. Thus, the average expected profit under an optimal Markovian pricing policy is given by

$$\begin{aligned}\Pi(\mathbf{M}) =& \alpha p_d + (1-\alpha)(\pi_L p_d + (1-\pi_L)v_H) \\ \leq& \alpha p_d + (1-\alpha)\left(\frac{q_H}{1+q_H}p_d + \frac{1}{1+q_H}v_H\right) \\ <& \alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}v_H\right] \equiv U^M(q_H).\end{aligned}$$

The first inequality holds because $\pi_L = \frac{q_H}{1-q_L+q_H}$ increases in q_L and $\Pi(\mathbf{M})$ decreases in π_L, and thus its maximum is realized when $q_L = 0$. The second inequality is due to $q_H(v_L - p_d) > c_L$. Let $q_L = 0$ and $q_d = v_L - \frac{c_L}{q_H} - \eta(\delta)$ for an arbitrarily small $\delta > 0$ and $\lim_{\delta \searrow 0} \eta(\delta) = 0$. Then $\Pi(\mathbf{M})$ converges to the upper bound $U^M(q_H)$ when δ converges to 0. Therefore, the optimal Markovian pricing policy is letting $q_L^* = 0$ and $q_d^* = v_L - \frac{c_L}{q_H} - \eta(\delta)$. Now we turn to the optimal q_H. Taking the derivative of $U^M(q_H)$ with respect to q_H, we have

$$\frac{\partial U^M(q_H)}{\partial q_H} = \alpha\frac{c_L}{q_H^2} - (1-\alpha)\frac{v_H - v_L - c_L}{(1+q_H)^2} = \frac{\alpha c_L}{(1+q_H)^2}\left[\left(\frac{1+q_H}{q_H}\right)^2 - \frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}\right].$$

(i) If $v_H - v_L \leq c_L$ or $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \geq 1$.

When $v_H - v_L \leq c_L$, we have

$$\begin{aligned}U^M(q_H) =& \alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}v_H\right] \\ \leq& \alpha(v_L - \frac{c_L}{q_H}) + (1-\alpha)\left[\frac{q_H}{1+q_H}(v_L - \frac{c_L}{q_H}) + \frac{1}{1+q_H}(v_L + c_L)\right] \\ =& v_L - \alpha\frac{c_L}{q_H} < v_L \leq \max\{v_L, (1-\alpha)v_H\}.\end{aligned}$$

Therefore, when $v_H - v_L \leq c_L$, a Markovian pricing policy is strictly dominated by an optimal static pricing policy. Consider $v_H - v_L > c_L$, then Δ_M is well-defined as a real number. Because $0 \leq q_H \leq 1$, we have $(\frac{1+q_H}{q_H})^2 = (1 + \frac{1}{q_H})^2 \geq 4$. Thus if $\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} \leq 4$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \geq \frac{1}{2}$, then $\frac{\partial U^M(q_H)}{\partial q_H} \geq 0$. Coupling with $q_H < \frac{c_H - c_L}{v_H - v_L}$ as shown in Lemma 8, $U^M(q_H)$ is maximized at $q_H = \min\left\{1, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$. When $0 < \frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} \leq 1$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} \geq 1$, we have $c_L \geq (1-\alpha)(v_H - v_L)$. Consequently,

$$\begin{aligned}U^M(q_H) \leq U^M(1) = \alpha(v_L - c_L) + (1-\alpha)(\frac{1}{2}(v_L - c_L) + \frac{1}{2}v_H) = v_L - c_L + \frac{1}{2}(1-\alpha)(v_H - v_L + c_L) \\ \leq v_L - c_L + \frac{1}{2}(1-\alpha)c_L + \frac{1}{2}c_L = v_L - \frac{1}{2}\alpha c_L \leq \max\{v_L, (1-\alpha)v_H\}.\end{aligned}$$

That is, if $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} \geq 1$, an optimal Markovian pricing policy reduces to a static pricing policy.

Next we show that an optimal randomized pricing policy also reduces to a static pricing policy when $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} \geq 1$ or when Δ_M is not well-defined as a real number. Note that the preceding conditions is equivalent to $c_L \geq (1-\alpha)(v_H - v_L)$. Lemma OS.2 shows that the necessary and sufficient conditions for $U_1 > v_L$ and $U_2 > v_L$ are given by $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ and $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} < 1$, respectively. Now we show that $U_1 \leq v_L$ or $U_2 \leq v_L$ when $c_L \geq (1-\alpha)(v_H-v_L)$. When $\Delta_R < 1$ and $\Delta_R < \frac{c_H-c_L}{v_H-v_L}$, Lemma 6 shows that the expected profit from an optimal randomized pricing policy is no more than U_1. In this case, as $c_L \geq (1-\alpha)(v_H-v_L)$, we have $\sqrt{\frac{c_L}{v_H-v_L}} \geq \sqrt{1-\alpha} > \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$ since $0<\alpha<1$, and thus the expected profit from an optimal randomized pricing policy cannot be greater than v_L. On the other hand, when $\frac{c_H-c_L}{v_H-v_L} \leq \Delta_R < 1$, Lemma 6 shows that the expected profit from an optimal randomized pricing policy is no more than U_2. In this case, we have $\frac{c_H}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} > \frac{c_L}{v_H-v_L} + \frac{\alpha c_L}{(1-\alpha)(c_H-c_L)} \geq 1-\alpha+\alpha\frac{v_H-v_L}{c_H-c_L} > 1$, where the second inequality is due to the condition $c_L \geq (1-\alpha)(v_H-v_L)$, and the last inequality is due to $0 < \frac{c_H-c_L}{v_H-v_L} < 1$. Therefore, an optimal randomized pricing policy also reduces to a static pricing policy when $\frac{c_H-c_L}{v_H-v_L} \leq \Delta_R < 1$.

(ii) If $1/2 \leq \Delta_M < 1$ and $\frac{c_H-c_L}{v_H-v_L} > 1$.

As shown in the previous paragraph, $U^M(q_H)$ is maximized at $q_H = \min\left\{1, \frac{c_H-c_L}{v_H-v_L} - \delta\right\}$ when $1/2 \leq \Delta_M < 1$. Coupling with $\frac{c_H-c_L}{v_H-v_L} > 1$, we have $q_H^* = 1$ and $q_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta) = v_L - c_L - \eta(\delta)$. Since we have shown $q_L^* = 0$, then $\pi_L^* = \frac{q_H^*}{1-q_L^*+q_H^*} = 1/2$ and $\pi_H = \frac{1-q_L^*}{1-q_L^*+q_H^*} = 1/2$. Then the Markovian pricing policy reduces to a cyclic pricing policy with a cyclic length of 2. Next we show that $U^M(q_H^*)$ is greater than $U(\beta^*)$, which is the expected profit from an optimal randomized pricing policy. We have

$$\begin{aligned}
U^M(q_H^*) =& \alpha\left(v_L - \frac{c_L}{q_H^*}\right) + (1-\alpha)\left[\frac{q_H^*}{1+q_H^*}\left(v_L - \frac{c_L}{q_H^*}\right) + \frac{1}{1+q_H^*}v_H\right] \\
=& \alpha v_L + (1-\alpha)v_H - (1-\alpha)\frac{c_L}{1+q_H^*} - \alpha\frac{c_L}{q_H^*} - (1-\alpha)\frac{q_H^*}{1+q_H^*}(v_H - v_L) \\
\geq& \alpha v_L + (1-\alpha)v_H - (1-\alpha)\frac{c_L}{1+\beta^*} - \alpha\frac{c_L}{\beta^*} - (1-\alpha)\frac{\beta^*}{1+\beta^*}(v_H - v_L) \\
>& \alpha v_L + (1-\alpha)v_H - (1-\alpha)c_L - \alpha\frac{c_L}{\beta^*} - (1-\alpha)\beta^*(v_H - v_L) = U(\beta^*),
\end{aligned}$$

which the first inequality is due to $U^M(q_H^*) \geq U^M(q_H)$, for any $q_H < \frac{c_H - c_L}{v_H - v_L}$ and $\beta^* < \frac{c_H - c_L}{v_H - v_L}$, and the second inequality is due to $\beta^* > 0$.

(iii) Otherwise.

Consider the case that $1/2 \leq \Delta_M < 1$ and $\frac{c_H - c_L}{v_H - v_L} \leq 1$. From the analysis in the previous paragraph, note $\frac{\Delta_M}{1-\Delta_M} \geq 1$, we know that $q_H^* = \min\left\{1, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \frac{c_H - c_L}{v_H - v_L} - \delta = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$. Consider the case when $\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L} > 4$, i.e., $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} < \frac{1}{2}$. Solving $\frac{\partial U^M(q_H)}{\partial q_H} = 0$, we have $q_H = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} = \frac{\Delta_M}{1-\Delta_M}$. It is easy to verify that $\frac{\partial U^M(q_H)}{\partial q_H} \geq 0$ for any $q_H \leq \frac{\Delta_M}{1-\Delta_M}$; Otherwise $\frac{\partial U^M(q_H)}{\partial q_H} \leq 0$. Coupling with $q_H < \frac{c_H - c_L}{v_H - v_L}$ as shown in Lemma 8, the maximum of $U^M(q_H)$ is realized when $q_H = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$ in this case. At the optimality, $q_L^* = 0$, and $p_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta)$. With the same approach in the proof of case (ii), we have the expected profit from this optimal Markovian pricing policy is greater than that from an optimal randomized pricing policy. □

Proof of Corollary 5. According to Lemma 6(i) and Proposition 10(i), if $\Delta_R \geq 1$ or $\Delta_M \geq 1$, randomized or Markovian pricing policy is dominated by the optimal static pricing policy. Note $\Delta_R < \Delta_M$, then we only consider the case $\Delta_R < \Delta_M < 1$. Based on Proposition 3, $\beta^* = \min\left\{\Delta_R, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \min\left\{\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} < 1$, and $\underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$. Because $\left(\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1\right)\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} = \sqrt{\frac{v_H - v_L - c_L}{v_H - v_L}} - \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \sqrt{\frac{v_H - v_L - c_L}{v_H - v_L}} < 1$, we have $\frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} > \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$ when $\Delta_M = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}} < 1$. Based on Proposition 10, $q_H^* = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\} = \min\left\{\frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}, \frac{c_H - c_L}{v_H - v_L} - \delta\right\}$ or $q_H^* = 1$. As a result, $q_H^* \geq \beta^*$, and thus $p_d^* = v_L - \frac{c_L}{q_H^*} - \eta(\delta) \geq \underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$.

If $\frac{c_H - c_L}{v_H - v_L} \leq \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, then $\beta^* = \frac{c_H - c_L}{v_H - v_L} - \delta$. Note $\frac{c_H - c_L}{v_H - v_L} - \delta \leq \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < 1$, by Proposition 10(iii), we have $q_H^* = \frac{c_H - c_L}{v_H - v_L} - \delta$. Consequently, we have $\pi_L^* = \frac{q_H^*}{1 - q_L^* + q_H^*} = \frac{q_H^*}{1 + q_H^*} < q_H^* = \beta^*$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{c_H - c_L}{v_H - v_L} \leq \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$ and $\frac{c_H - c_L}{v_H - v_L} > 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$, and $q_H^* = 1$, based on Proposition 10(ii). As a result, we have $\pi_L^* = \frac{q_H^*}{1 - q_L^* + q_H^*} = \frac{q_H^*}{1 + q_H^*} = \frac{1}{2}$. Because we restrict our discussion to the case when an optimal randomized pricing policy dominates an optimal static pricing policy, all conditions in Lemma OS.2 shall be satisfied. That is, when $v_L \geq (1 -$

$\alpha)v_H$, we have $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$ and $\sqrt{\frac{c_L}{v_H-v_L}} < \frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}}$. Consequently, $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\frac{1-\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} < \frac{1}{2} = \pi_L^*$. On the other hand, when $v_L < (1-\alpha)v_H$, we have $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L}$ and $\sqrt{\frac{c_L}{v_H-v_L}} < \sqrt{\frac{\alpha}{1-\alpha}}\left(\sqrt{\frac{v_H}{v_H-v_L}}-1\right)$. Consequently, $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{c_L}{v_H-v_L}} < \frac{\alpha}{1-\alpha}\left(\sqrt{\frac{1}{\alpha}}-1\right) = \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} < \frac{1}{2} = \pi_L^*$, where the first inequality is due to $v_L < (1-\alpha)v_H$, i.e., $\frac{v_H}{v_H-v_L} < \frac{1}{\alpha}$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} \le 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta$, based on Proposition 10(iii). Recall that $\pi_H^* = \frac{q_H^*}{1+q_H^*}$ is increasing in q_H^*, then $\frac{\beta^*}{1+\beta^*} < \pi_H^* < \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}$, which is due to $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < q_H^* = \frac{c_H-c_L}{v_H-v_L} - \delta < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$. It is easy to see $\frac{\beta^*}{1+\beta^*} < \beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}$. Therefore, we can't say $\pi_L^* \ge \beta^*$ or $\pi_L^* \le \beta^*$ in this case.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1} < \frac{c_H-c_L}{v_H-v_L}$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ or 1. If $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, we have $\pi_L^* = \frac{q_H^*}{1+q_H^*} = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}} = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} \ge \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} = \beta^*$. If $q_H^* = 1$, we have shown in the preceding paragraph that $\pi_L^* \ge \beta^*$ in this case.

To sum up, comparing an optimal Markovian pricing policy with an optimal randomized pricing policy, we have

(i) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} < 1$, then $q_H^* \ge \beta^*$ and $p_d^* \ge \underline{p}^*$, where β^* and $\underline{p}^*$ are given by Proposition 3;

(ii) If $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, then $\pi_L^* < \beta^*$;

(iii) If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} > 1$, or $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1} < \frac{c_H-c_L}{v_H-v_L}$, then $\pi_L^* \ge \beta^*$.

When c_L is sufficiently small, the sets of conditions (i) and (iii) are satisfied, and we thus obtain the desired result.

Lastly, we show $L_M^* = 1/\pi_L^*$ and $L_F^* = 1/\beta^*$. For an optimal Markovian pricing policy, when the price of current period i is p_d^*, then the probability of next period's price is also p_d^* is q_L^*. Moreover, for $j \ge 2$, the discount price appears in period $i+j$ for the first time, then its probability is $(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$. Thus $L_M^* = q_L^* + \sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$. Let

$$S=\sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^* = 2(1-q_L^*)q_H^* + \sum_{j=3}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-2}q_H^*$$
$$= 2(1-q_L^*)q_H^* + \sum_{j=2}^{\infty} (j+1)(1-q_L^*)(1-q_H^*)^{j-1}q_H^*,$$

then $(1-q_H^*)S = \sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-1}q_H^*$. Therefore,

$$q_H^*S = 2(1-q_L^*)q_H^* + \sum_{j=2}^{\infty}(j+1)(1-q_L^*)(1-q_H^*)^{j-1}q_H^* - \sum_{j=2}^{\infty} j(1-q_L^*)(1-q_H^*)^{j-1}q_H^*$$
$$= (1-q_L^*)q_H^*\left[2+\sum_{j=2}^{\infty}(1-q_H^*)^{j-1}\right] = (1-q_L^*)q_H^*\left[2+\lim_{n\to\infty}(1-q_H^*)\frac{1-(1-q_H^*)^n}{1-(1-q_H^*)}\right] = (1-q_L^*)(1+q_H^*).$$

Note $q_L^*=0$ and $\pi_L^* = \frac{q_H^*}{1-q_L^*+q_H^*} = \frac{q_H^*}{1+q_H^*}$, then $S=(1-q_L^*)(1+1/q_H^*)$, and $L_M^* = q_L^* + S = 1+1/q_H^* = 1/\pi_L^*$. For an optimal randomized pricing policy, when the price of current period i is $\underline{p}^*$, then the probability of discount price $\underline{p}^*$ appears in period $i+j$ for the first time is $\beta^*(1-\beta^*)^{j-1}$. Then $L_F^* = \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j-1}$. Consequently,

$$\beta^*L_F^* = L_F^* - (1-\beta^*)L_F^* = \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j-1} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \beta^* + \sum_{j=2}^{\infty} j\beta^*(1-\beta^*)^{j-1} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \beta^* + \sum_{j=1}^{\infty} (j+1)\beta^*(1-\beta^*)^{j} - \sum_{j=1}^{\infty} j\beta^*(1-\beta^*)^{j}$$
$$= \sum_{j=1}^{\infty} \beta^*(1-\beta^*)^{j-1} = \beta^* \lim_{n\to\infty}\frac{1-(1-\beta^*)^n}{1-(1-\beta^*)} = 1.$$

Therefore, $L_F^* = 1/\beta^*$. $\square$

Proof of Corollary 6. First we show it is true for low-valuation customers. By Corollary 5, we know $p_d^* \geq \underline{p}^*$. Recall the low-valuation customers always buy with the price p_d^* and $\underline{p}^*$ under an optimal Markovian and randomized pricing policy, respectively.

Next we consider high-valuation customers. Since high-valuation customers will buy immediately under both polices, we have

$$U_H^M(q_H^*) = \pi_L^* p_d^* + (1-\pi_L^*)v_H, \quad U_H(\beta^*) = \beta^*\underline{p}^* + (1-\beta^*)v_H,$$

where $U_H^M(q_H^*)$ and $U_H(\beta^*)$ are the expected profit earned from high-valuation customers under an optimal Markovian and randomized pricing policy, respectively. Now we will show $U_H^M(q_H^*) > U_H(\beta^*)$. According to Proposition 3, $\beta^* = \min\left\{\Delta_R, \frac{c_H-c_L}{v_H-v_L}-\delta\right\} = \min\left\{\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}, \frac{c_H-c_L}{v_H-v_L}-\delta\right\} < 1$, and $\underline{p}^* = v_L - \frac{c_L}{\beta^*} - \eta(\delta)$. Based on Proposition 10, $q_H^* = \min\left\{\frac{\Delta_M}{1-\Delta_M}, \frac{c_H-c_L}{v_H-v_L}-\delta\right\} = \min\left\{\frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}, \frac{c_H-c_L}{v_H-v_L}-\delta\right\}$ or $q_H^*=1$. Note $\pi_L^* = \frac{q_H^*}{1-q_L^*+q_H^*} = \frac{q_H^*}{1+q_H^*}$, thus,

$$U_H^M(q_H^*) = \pi_L^* p_d^* + (1-\pi_L^*)v_H = v_H - c_L - \frac{q_H^*}{1+q_H^*}\eta(\delta) - \frac{q_H^*}{1+q_H^*}(v_H-v_L-c_L),$$

$$U_H(\beta^*) = \pi_L^* p_d^* + (1-\pi_L^*)v_H = v_H - c_L - \beta^*\eta(\delta) - \beta^*(v_H-v_L).$$

Note $\lim_{\delta\searrow 0}\eta(\delta)=0$. Therefore, if we want to show $U_H^M(q_H^*) > U_H(\beta^*)$, it is equivalent to verify $\frac{q_H^*}{1+q_H^*}(v_H-v_L-c_L) < \beta^*(v_H-v_L)$.

If $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, then $\beta^* = \frac{c_H-c_L}{v_H-v_L}-\delta$. Note $\frac{c_H-c_L}{v_H-v_L} \le \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < 1$, by Proposition 10(iii), we have $q_H^* = \frac{c_H-c_L}{v_H-v_L}-\delta$. Hence, we have $\frac{q_H^*}{1+q_H^*}(v_H-v_L-c_L) < q_H^*(v_H-v_L-c_L) < \beta^*(v_H-v_L)$, where the first inequality is due to $0<q_H^*\le 1$, and the second is due to $c_L>0$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} > 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$. Note $1 < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$, i.e., $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}} > \frac{1}{2}$, then $q_H^*=1$, based on Proposition 10(ii). Hence, we have

$$\begin{aligned}\frac{q_H^*}{1+q_H^*}(v_H-v_L-c_L) &= \frac{1}{2}(v_H-v_L-c_L) < \frac{1}{2}\sqrt{v_H-v_L-c_L}\sqrt{v_H-v_L}\\ &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}\sqrt{v_H-v_L-c_L}\sqrt{v_H-v_L}\\ &= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}(v_H-v_L) = \beta^*(v_H-v_L).\end{aligned}$$

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}} < \frac{c_H-c_L}{v_H-v_L} \le \frac{1}{\sqrt{\frac{(1-\alpha)(v_H-v_L-c_L)}{\alpha c_L}}-1}$ and $\frac{c_H-c_L}{v_H-v_L} \le 1$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}$, and $q_H^* = \frac{c_H-c_L}{v_H-v_L}-\delta$, based on Proposition 10(iii). Hence, we have

$$\begin{aligned}\frac{q_H^*}{1+q_H^*}(v_H-v_L-c_L) &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}(v_H-v_L-c_L)\\ &< \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L-c_L)}}\sqrt{v_H-v_L-c_L}\sqrt{v_H-v_L}\\ &= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H-v_L)}}(v_H-v_L) = \beta^*(v_H-v_L),\end{aligned}$$

where the first inequality is due to $\frac{q_H^*}{1+q_H^*}$ is increasing in q_H^* and $q_H^* = \frac{c_H - c_L}{v_H - v_L} - \delta < \frac{c_H - c_L}{v_H - v_L} \leq \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, and the second inequality is due to $c_L > 0$.

If $\sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}} < \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1} < \frac{c_H - c_L}{v_H - v_L}$, then $\beta^* = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}$ and $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$ or 1. If $q_H^* = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}} - 1}$, we have $\frac{q_H^*}{1+q_H^*} = \frac{1}{\sqrt{\frac{(1-\alpha)(v_H - v_L - c_L)}{\alpha c_L}}} = \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L - c_L)}}$. Hence,

$$\begin{aligned}
\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) &= \sqrt{\frac{\alpha c_L}{1-\alpha}}\sqrt{v_H - v_L - c_L} < \sqrt{\frac{\alpha c_L}{1-\alpha}}\sqrt{v_H - v_L} \\
&= \sqrt{\frac{\alpha c_L}{(1-\alpha)(v_H - v_L)}}(v_H - v_L) = \beta^*(v_H - v_L).
\end{aligned}$$

If $q_H^* = 1$, we have shown in the preceding paragraph that $\frac{q_H^*}{1+q_H^*}(v_H - v_L - c_L) < \beta^*(v_H - v_L)$ in this case. $\square$

Proof of Lemma 9. In order to show the result, we first prove an auxiliary lemma.

LEMMA OS.5. $\frac{c_{i_1}}{v_{i_1}} < \frac{c_{i_2}}{v_{i_2}}$ *is a necessary condition that type-i_1 customers would wait and type-i_2 customers would either purchase or leave immediately, for any $i_1 < i_2$.*

Proof of Lemma OS.5. According to Lemma OA.1, type-i_1 customers wait and type-i_2 customers either purchase or leave immediately imply that $\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j > \sum_{j=1}^{i_1} \beta_j x_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j = c_{i_1}$ and $\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j \leq c_{i_2}$, respectively. Thus, we have

$$c_{i_2} \geq \sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j > \sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j,$$

and

$$c_{i_1} < \sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j.$$

Rearranging the above two inequalities, we have

$$\frac{c_{i_2}}{c_{i_1}} > \frac{\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j}{\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j} > \frac{v_{i_2}}{v_{i_1}},$$

where the second inequality is due to $\left(\sum_{j=1}^{i_2} \beta_j v_{i_2} - \sum_{j=1}^{i_2} \beta_j x_j\right) v_{i_1} - \left(\sum_{j=1}^{i_1} \beta_j v_{i_1} - \sum_{j=1}^{i_1} \beta_j x_j\right) v_{i_2} = v_{i_1}\left(\sum_{j=i_1+1}^{i_2} \beta_j v_{i2} - \sum_{j=i_1+1}^{i_2} \beta_j x_j\right) + (v_{i_2} - v_{i_1}) \sum_{j=1}^{i_1} \beta_j x_j > 0$. We thus obtain the announced result. $\square$

Lemma OS.5 and the condition $\frac{c_1}{v_1} \geq \cdots \geq \frac{c_n}{v_n}$ imply that there exists a k such that any customer with valuation greater than v_k will wait, and any customer with valuation less than or equal to v_k will either purchase or leave immediately upon arrival, under any randomized pricing policy. Thus, based on Lemma OA.1, its optimal expected profit can be derived by solving the following problem.

$$\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \sum_{i=1}^{n} \alpha_i \pi_i \\
\text{s.t.} \quad & \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \leq c_i, \quad \forall\, i \leq k, \\
& \sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i, \quad \forall\, i > k, \\
& x_i \leq v_i, \forall\, i \leq k, \text{ and } x_i < v_i, \forall\, i > k, \text{ and } \sum_{i=1}^{n} \beta_i = 1,
\end{aligned} \tag{OS.3}$$

where $\pi_i = \sum_{j=1}^{i} \beta_j x_j$, $\forall\, i \leq k$, and $\pi_i = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$ otherwise.

First consider the case when $k = n-1$ or n. In this case, we have

$$\sum_{i=1}^{n} \alpha_i \pi_i = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{i} \beta_j x_j = \beta_1 x_1 + \sum_{i=2}^{n} \alpha_i \beta_2 x_2 + \cdots + \alpha_n \beta_n x_n \leq \max_j \left\{ \sum_{i=j}^{n} \alpha_i x_j \right\} \leq \max_j \left\{ \sum_{i=j}^{n} \alpha_i v_j \right\},$$

where the last term indicates the expected profit under an optimal static pricing policy.

Next consider the case when $k = 0$. We will prove the dominance of static pricing by induction. When $n = 2$, the profit function is given by $\alpha_1 x_1 + \alpha_2[\beta_1 x_1 + (1-\beta_1)x_2]$, and constraints are $\beta_1(v_1 - x_1) > c_1$ and $\beta_1(x_2 - x_1) = c_2$. That is, $x_1 < v_1 - \frac{c_1}{\beta_1}$ and $x_2 = x_1 + \frac{c_2}{\beta_1}$. Thus, we have

$$\alpha_1 x_1 + \alpha_2[\beta_1 x_1 + (1-\beta_1)x_2] = x_1 + \alpha_2 \left(\frac{1}{\beta_1} - 1 \right) c_2 < v_1 - \frac{c_1}{\beta_1} + \alpha_2 \left(\frac{1}{\beta_1} - 1 \right) c_2 \equiv h(\beta_1),$$

where the equality is due to $x_2 = x_1 + \frac{c_2}{\beta_1}$ and $\alpha_1 + \alpha_2 = 1$, and the inequality is due to $x_1 < v_1 - \frac{c_1}{\beta_1}$. Taking the derivative of $h(\beta_1)$ with respect to β_1, we have $\frac{\partial h(\beta_1)}{\partial \beta_1} = \frac{c_1 - \alpha_2 c_2}{\beta_1^2}$. If $c_1 \geq \alpha_2 c_2$, $h(\beta_1)$ is increasing in β_1 and thus $h(\beta_1) \leq h(1) = v_1 - c_1$. If $c_1 < \alpha_2 c_2$, $h(\beta_1)$ is decreasing in β_1 and thus $h(\beta_1) \leq h(c_1/v_1) = \alpha_2 \left(\frac{v_1}{c_1} - 1 \right) c_2 < \alpha_2 \frac{v_1}{c_1} c_2 \leq \alpha_2 v_2$, where the first inequality is due to $v_1 - \frac{c_1}{\beta_1} > 0$, and the last inequality is due to $\frac{c_1}{v_1} \geq \frac{c_2}{v_2}$. So the expected profit from any randomized pricing policy is less than or equal to the expected profit from the optimal static pricing policy, which is given by $\max\{v_1, \alpha_2 v_2\}$.

Now suppose that, under the case when $k=0$, an optimal randomized pricing policy reduces to the optimal static pricing policy for $n-1$, $\forall\, n\geq 3$. We next show that this statement also holds for n. Based on Problem (OS.3), we can rewrite the expected profit function as:

$$\sum_{i=1}^{n}\alpha_i\pi_i=\alpha_1x_1+\cdots+\alpha_{n-1}\left(x_{n-1}-\frac{c_{n-1}}{\beta_1+\cdots+\beta_{n-1}}\right)+\alpha_n\left(x_n-c_n\right),$$

where $x_n=x_{n-1}+\frac{c_n-c_{n-1}}{\beta_1+\cdots+\beta_{n-1}}$. Taking the derivative of $\sum_{i=1}^{n}\alpha_i\pi_i$ with respect to $(\beta_1+\cdots+\beta_{n-1})$, we have $\frac{\partial\left(\sum_{i=1}^{n}\alpha_i\pi_i\right)}{\partial(\beta_1+\cdots+\beta_{n-1})}=\frac{(\alpha_{n-1}+\alpha_n)c_{n-1}-\alpha_nc_n}{(\beta_1+\cdots+\beta_{n-1})^2}$. If $(\alpha_{n-1}+\alpha_n)c_{n-1}\geq\alpha_nc_n$, $\sum_{i=1}^{n}\alpha_i\pi_i$ is increasing in $\beta_1+\cdots+\beta_{n-1}$. Thus, it is optimal to let $\beta_1+\cdots+\beta_{n-1}=1$ and $\beta_n=0$, which reduces to the case with $n-1$ types of customers. On the other hand, if $(\alpha_{n-1}+\alpha_n)c_{n-1}<\alpha_nc_n$, $\sum_{i=1}^{n}\alpha_i\pi_i$ is decreasing in $\beta_1+\cdots+\beta_{n-1}$. In this case, it is optimal to let $\beta_1+\cdots+\beta_{n-1}=\beta_1+\cdots+\beta_{n-2}$, i.e., $\beta_{n-1}=0$, which again reduces to the case with $n-1$ types of customers. We thus establish the result for n.

Lastly, consider the case when $1\leq k\leq n-2$. Again, we prove the dominance of static pricing by induction. When $n=3$ and $k=1$, the profit function is given by $\alpha_1\beta_1x_1+\alpha_2\frac{\beta_1x_1+\beta_2x_2}{\beta_1+\beta_2}+\alpha_3[\beta_1x_1+\beta_2x_2+(1-\beta_1-\beta_2)x_3]$, and constraints are $\beta_1(v_1-x_1)\leq c_1$, $\beta_1(x_2-x_1)=c_2$, and $(\beta_1+\beta_2)x_3-\beta_1x_1-\beta_2x_2=c_3$. Thus, we have $x_3=x_2+\frac{c_3-c_2}{\beta_1+\beta_2}$, and we can rewrite the profit function as:

$$\sum_{i=1}^{3}\alpha_i\pi_i=\alpha_1\beta_1x_1+\alpha_2\left(x_2-\frac{c_2}{\beta_1+\beta_2}\right)+\alpha_3\left(x_2+\frac{c_3-c_2}{\beta_1+\beta_2}-c_3\right).$$

Taking the derivative of the profit function with respect to $(\beta_1+\beta_2)$, we have $\frac{\partial\left(\sum_{i=1}^{3}\alpha_i\pi_i\right)}{\partial(\beta_1+\beta_2)}=\frac{(\alpha_2+\alpha_3)c_2-\alpha_3c_3}{(\beta_1+\beta_2)^2}$. If $(\alpha_2+\alpha_3)c_2\geq\alpha_3c_3$, profit is maximized by letting $\beta_1+\beta_2=1$. Plugging $\beta_1+\beta_2=1$ into the profit function, we have

$$\begin{aligned}\sum_{i=1}^{3}\alpha_i\pi_i&=\alpha_1\beta_1x_1+(\alpha_2+\alpha_3)(x_2-c_2)=\alpha_1\beta_1x_1+(\alpha_2+\alpha_3)(\beta_1x_1+\beta_2x_2)\\&\leq\max\{x_1,(\alpha_2+\alpha_3)x_2\}\leq\max\{v_1,(\alpha_2+\alpha_3)v_2\},\end{aligned}$$

where the second equality is due to $\beta_1(x_2-x_1)=c_2$. On the other hand, if $(\alpha_2+\alpha_3)c_2<\alpha_3c_3$, profit is maximized by letting $\beta_2=0$. Plugging $\beta_2=0$ into the profit function, we have

$$\sum_{i=1}^{3}\alpha_i\pi_i=\alpha_1\beta_1x_1+\alpha_2\left(x_2-\frac{c_2}{\beta_1}\right)+\alpha_3\left(x_2+\frac{c_3-c_2}{\beta_1}-c_3\right)=\alpha_1\beta_1x_1+\alpha_2x_1+\alpha_3\left[x_1+\left(\frac{1}{\beta_1}-1\right)c_3\right],$$

where the second equality is due to $x_2 = x_1 + \frac{c_2}{\beta_1}$. Taking the derivative of the profit function with respect to β_1, we have $\frac{\partial\left(\sum_{i=1}^{3} \alpha_i \pi_i\right)}{\partial \beta_1} = \alpha_1 x_1 - \frac{\alpha_3 c_3}{\beta_1^2}$, which is increasing in β_1. Thus, the profit function $\sum_{i=1}^{3} \alpha_i \pi_i$ is convex in β_1. Here we need to consider two cases separately: $\beta_1 \leq \frac{c_2}{v_2 - v_1}$ and $\beta_1 > \frac{c_2}{v_2 - v_1}$. When $\beta_1 > \frac{c_2}{v_2 - v_1}$, we have $x_2 = x_1 + \frac{c_2}{\beta_1} < v_2$ due to $x_1 \leq v_1$. On the other hand, when $\beta_1 \leq \frac{c_2}{v_2 - v_1}$, $x_1 = x_2 - \frac{c_2}{\beta_1} < v_1$ due to $x_2 \leq v_2$.

Consider first when $\beta_1 \leq \frac{c_2}{v_2 - v_1}$. In this case, the optimal x_2 is given by $x_2 = v_2 - \delta$, for any small $\delta > 0$. Combining with the condition $x_1 = x_2 - \frac{c_2}{\beta_1} \geq 0$, β_1 must be greater than $\frac{c_2}{v_2}$. Thus, in order to show the dominance of static pricing policy, we only need to prove that the expected profit evaluated at $\beta_1 = \frac{c_2}{v_2}$ and $\beta_1 = \frac{c_2}{v_2 - v_1}$ are less than the expected profit from an optimal static pricing policy. Plugging $\beta_1 = \frac{c_2}{v_2}$ and $x_2 = v_2 - \delta$ into the profit function, we have

$$\sum_{i=1}^{3} \alpha_i \pi_i = -(\alpha_1 \beta_1 + \alpha_2 + \alpha_3)\delta + \alpha_3 \left(\frac{v_2}{c_2} - 1\right) c_3 < \alpha_3 \left(\frac{v_2}{c_2} - 1\right) c_3 \leq \alpha_3 \left(\frac{v_3}{c_3} - 1\right) c_3 < \alpha_3 v_3,$$

where the first equality is due to $x_1 = x_2 - \frac{c_2}{\beta_1} = -\delta$, and the second inequality is due to $\frac{c_2}{v_2} \geq \frac{c_3}{v_3}$. Next plugging $\beta_1 = \frac{c_2}{v_2 - v_1}$ and $x_2 = v_2 - \delta$ into the profit function, we have

$$\sum_{i=1}^{3} \alpha_i \pi_i < \alpha_1 \frac{c_2}{v_2 - v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2 - v_1}{c_2} - 1\right) c_3\right].$$

Consider three sub-cases.

(a) $v_1 \geq \alpha_3 v_3$. Because $\frac{c_1}{v_1} \geq \frac{c_3}{v_3} = \frac{\alpha_3 c_3}{\alpha_3 v_3}$, we have $c_1 \geq \alpha_3 c_3$. Furthermore, because $(\alpha_2 + \alpha_3) c_2 < \alpha_3 c_3$ and $\frac{(\alpha_2 + \alpha_3) c_2}{(\alpha_2 + \alpha_3) v_2} = \frac{c_2}{v_2} \geq \frac{c_3}{v_3} = \frac{\alpha_3 c_3}{\alpha_3 v_3}$, we have $\alpha_3 v_3 > (\alpha_2 + \alpha_3) v_2$. Combining the preceding conditions, we have $v_1 > (\alpha_2 + \alpha_3) v_2$. That is, $\frac{\alpha_1 v_2}{v_2 - v_1} > 1$. Therefore,

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2 - v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2 - v_1}{c_2} - 1\right) c_3\right] \\
=&v_1 - \alpha_1 \left(1 - \frac{c_2}{v_2 - v_1}\right) v_1 + \alpha_3 \left(\frac{v_2 - v_1}{c_2} - 1\right) c_3 \\
=&v_1 - \left(\frac{v_2 - v_1}{c_2} - 1\right)\left(\alpha_1 \frac{c_2}{v_2 - v_1} v_1 - \alpha_3 c_3\right) < v_1 - \left(\frac{v_2 - v_1}{c_2} - 1\right)\left(\frac{v_1}{v_2} c_2 - c_1\right) \leq v_1,
\end{aligned}$$

where the first inequality is due to $\frac{\alpha_1 v_2}{v_2 - v_1} > 1$ and $c_1 \geq \alpha_3 c_3$, and the second inequality is due to $\frac{c_1}{v_1} \geq \frac{c_2}{v_2}$.

(b) $v_1 \leq (\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$, i.e., $\frac{\alpha_1 v_2}{v_2-v_1} \leq 1$. In this case, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
\leq &\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) \frac{v_3}{v_2} c_2 \right] \\
= &\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 v_3 - \alpha_3 v_3 \left(\frac{v_1}{v_2} + \frac{c_2}{v_2} \right) \\
= &\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 v_3 \frac{c_2}{v_2} + \alpha_3 v_3 + (\alpha_2+\alpha_3) v_2 \frac{v_1}{v_2} - \alpha_3 v_3 \frac{v_1}{v_2} \\
< &\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 v_3 \frac{c_2}{v_2} + \alpha_3 v_3 < \frac{v_1}{v_2} c_2 - v_1 \frac{c_2}{v_2} + \alpha_3 v_3 = \alpha_3 v_3,
\end{aligned}$$

where the second inequality is due to $(\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$, and the third inequality is due to $\frac{\alpha_1 v_2}{v_2-v_1} \leq 1$ and $v_1 < \alpha_3 v_3$.

(c) $(\alpha_2+\alpha_3)v_2 \leq v_1 < \alpha_3 v_3$. If $\alpha_1 \frac{c_2}{v_2-v_1} v_1 \geq \alpha_3 c_3$, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
= &v_1 - \left(\frac{v_2-v_1}{c_2} - 1 \right) \left(\alpha_1 \frac{c_2}{v_2-v_1} v_1 - \alpha_3 c_3 \right) \leq v_1.
\end{aligned}$$

On the other hand, if $\alpha_1 \frac{c_2}{v_2-v_1} v_1 < \alpha_3 c_3$, we have

$$\begin{aligned}
&\alpha_1 \frac{c_2}{v_2-v_1} v_1 + \alpha_2 v_1 + \alpha_3 \left[v_1 + \left(\frac{v_2-v_1}{c_2} - 1 \right) c_3 \right] \\
< &\alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 c_3 \frac{v_2-v_1}{c_2} \\
< &\alpha_2 v_1 + \alpha_3 v_1 + \alpha_3 v_3 \frac{v_2-v_1}{v_2} < \alpha_3 v_3,
\end{aligned}$$

where the second inequality is due to $c_2 \leq v_2$, and the third inequality is due to $(\alpha_2+\alpha_3)v_2 < \alpha_3 v_3$. Now we prove that under the condition $\beta_1 \leq \frac{c_2}{v_2-v_1}$, an optimal randomized pricing policy reduces to the optimal static pricing policy. The case when $\beta_1 > \frac{c_2}{v_2-v_1}$ is trivial, as, when plugging in $\beta_1 = 1$, the profit function reduces to x_1, which is less than or equal to v_1.

Now suppose that, under the case when $1 \leq k \leq n-2$, an optimal randomized pricing policy reduces to an optimal static pricing policy for $n-1$, $\forall n \geq 3$. We next show that this statement also holds for n. Based on Problem (OS.3), we can rewrite the expected profit function as:

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i \pi_i = &\alpha_1 \beta_1 x_1 + \cdots + \alpha_k (\beta_1 x_1 + \cdots + \beta_k x_k) + \alpha_{k+1} \left(x_{k+1} - \frac{c_{k+1}}{\beta_1 + \cdots + \beta_{k+1}} \right) \\
&+ \cdots + \alpha_{n-1} \left(x_{n-1} - \frac{c_{n-1}}{\beta_1 + \cdots + \beta_{n-1}} \right) + \alpha_n \left(x_n - c_n \right),
\end{aligned}$$

where $x_n = x_{n-1} + \frac{c_n - c_{n-1}}{\beta_1+\cdots+\beta_{n-1}}$. Taking the derivative of $\sum_{i=1}^{n} \alpha_i \pi_i$ with respect to $(\beta_1 + \cdots + \beta_{n-1})$, we have $\frac{\partial\left(\sum_{i=1}^{n}\alpha_i\pi_i\right)}{\partial(\beta_1+\cdots+\beta_{n-1})} = \frac{(\alpha_{n-1}+\alpha_n)c_{n-1}-\alpha_n c_n}{(\beta_1+\cdots+\beta_{n-1})^2}$. If $(\alpha_{n-1}+\alpha_n)c_{n-1} \geq \alpha_n c_n$, $\sum_{i=1}^{n}\alpha_i\pi_i$ is increasing in $\beta_1+\cdots+\beta_{n-1}$. Thus, it is optimal to let $\beta_1+\cdots+\beta_{n-1} = 1$ and $\beta_n = 0$, which reduces to the case with $n-1$ types of customers. On the other hand, if $(\alpha_{n-1}+\alpha_n)c_{n-1} < \alpha_n c_n$, $\sum_{i=1}^{n}\alpha_i\pi_i$ is decreasing in $\beta_1+\cdots+\beta_{n-1}$. In this case, it is optimal to let $\beta_1+\cdots+\beta_{n-1} = \beta_1+\cdots+\beta_{n-2}$, i.e., $\beta_{n-1} = 0$, which again reduces to the case with $n-1$ types of customers. We thus obtain the announced result. □

Proof of Proposition 11. Consider any randomized pricing policy. First we prove by contradiction that, under the condition $v_1 - c_1 \geq v_2 - c_2 \geq \cdots \geq v_n - c_n$, a customer with valuation v_{i_2} will not wait, if type-i_1 customers do not wait, $\forall i_1 < i_2$. Suppose there exists $i_2 > i_1$, where type-i_1 customers do not wait but type-i_2 customers wait. By Proposition 1, we have

$$\sum_{i=1}^{i_1}\beta_i v_{i_1} - \sum_{i=1}^{i_1}\beta_i x_i \leq c_{i_1}, \text{ and } \sum_{i=1}^{i_2}\beta_i v_{i_2} - \sum_{i=1}^{i_2}\beta_i x_i > c_{i_2}.$$

Because $x_i > v_{i_1}$, $\forall i > i_1$, we have $\sum_{i=1}^{i_2}\beta_i v_{i_1} - \sum_{i=1}^{i_2}\beta_i x_i < \sum_{i=1}^{i_1}\beta_i v_{i_1} - \sum_{i=1}^{i_1}\beta_i x_i \leq c_{i_1}$. Combining with the inequality $\sum_{i=1}^{i_2}\beta_i v_{i_2} - \sum_{i=1}^{i_2}\beta_i x_i > c_{i_2}$, we have

$$\sum_{i=1}^{i_2}\beta_i(v_{i_2} - v_{i_1}) > c_{i_2} - c_{i_1} \geq v_{i_2} - v_{i_1},$$

where the second inequality is due to $v_{i_1} - c_{i_1} \geq v_{i_2} - c_{i_2}$. However, the above inequality contradicts to $\sum_{i=1}^{i_2}\beta_i \leq 1$. Thus, we obtain the announced result.

The implication of this result is that there always exists $1 \leq k_r \leq n$, such that any customer with valuation greater than or equal to v_{k_r} does not wait, and any customer with valuation smaller than v_{k_r} will wait. Hence, we get Proposition 11(i). Consequently, based on Lemma OA.1, the optimal policy with such induced customer behavior can be derived by solving:

$$\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \sum_{j=1}^{n} \alpha_j \pi_j \\
\text{s.t.} \quad & \sum_{i=1}^{j} \beta_i x_j - \sum_{i=1}^{j} \beta_i x_i = c_j, \quad \forall\, j < k_r, \\
& \sum_{i=1}^{j} \beta_i v_j - \sum_{i=1}^{j} \beta_i x_i \le c_j, \quad \forall\, j \ge k_r, \\
& x_j < v_j, j < k_r, \text{ and } x_j \le v_j, j \ge k_r, \text{ and } \sum_{j=1}^{n} \beta_j = 1,
\end{aligned} \tag{OS.4}$$

where $\pi_j = \sum\limits_{i=1}^{j} \beta_i x_i \Big/ \sum\limits_{i=1}^{j} \beta_i$, if $j < k_r$; Otherwise, $\pi_j = \sum\limits_{i=1}^{j} \beta_i x_i$. The set of conditions $\sum\limits_{i=1}^{j} \beta_i x_j - \sum\limits_{i=1}^{j} \beta_i x_i = c_j, \forall\, j < k_r$ imply that

$$\begin{aligned}
& x_1 = x_2 - c_2/\beta_1 = x_3 - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \\
& \quad = \cdots = x_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}).
\end{aligned}$$

Due to the assumption $c_1 < c_2 < \cdots < c_n$, we have

$$\begin{aligned}
& v_1 - c_1/\beta_1 > v_2 - c_2/\beta_1 > v_3 - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) > \cdots \\
& \quad > v_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}) > x_1,
\end{aligned}$$

where the last inequality is due to $x_{k_r-1} \le v_{k_r-1}$. As the profit function is increasing in x_j, $\forall\, j$, the optimal x_1^* is given by $x_1^* = v_{k_r-1} - c_2/\beta_1 - (c_3 - c_2)/(\beta_1 + \beta_2) \cdots - (c_{k_r-1} - c_{k_r-2})/(\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}) - \delta$. The optimal x_j^*, $\forall j \in \{2, \ldots, k_r\}$ can be derived by solving the system of equations $\sum\limits_{i=1}^{j} \beta_i x_j^* - \sum\limits_{i=1}^{j} \beta_i x_i^* = c_j, \forall\, j < k_r$, and the optimal x_j^*, $\forall\, j \ge k_r$ is given by $x_j^* = v_j$. Plugging the optimal x_i^* into the objective function, and ignoring the sufficiently small δ, we have

$$\begin{aligned}
\sum_{j=1}^{n} \alpha_j \pi_j = & \alpha_1 \left(v_{k_r-1} - \frac{c_2}{\beta_1} - \frac{c_3 - c_2}{\beta_1 + \beta_2} \cdots - \frac{c_{k_r-1} - c_{k_r}}{\beta_1 + \beta_2 + \cdots + \beta_{k_r-2}} \right) + \alpha_2 \left(v_{k_r-1} - \frac{c_{k_r-1}}{\beta_1 + \beta_2} \right) + \cdots \\
& + \alpha_{k_r-1} \left(v_{k_r-1} - \frac{c_{k_r-1}}{\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}} \right) + \alpha_{k_r} \left((\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}) v_{k_r-1} - c_{k_r-1} + \beta_{k_r} v_{k_r} \right) + \cdots \\
& + \alpha_n \left((\beta_1 + \beta_2 + \cdots + \beta_{k_r-1}) v_{k_r-1} - c_{k_r-1} + \beta_{k_r} v_{k_r} + \cdots + \beta_n v_n \right).
\end{aligned}$$

It is easy to verify that the profit function is increasing in β_1, $(\beta_1 + \beta_2)$, $\ldots$, $(\beta_1 + \beta_2 + \cdots + \beta_{k_r-1})$. Thus, the profit is maximized when $\beta_2 = \beta_3 = \cdots = \beta_{k_r-2} = \beta_{k_r-1} = 0$. Next consider β_j, $\forall\, j \ge k_r$. The partial derivative of the profit function with respect to β_j is given by $\frac{\partial \left(\sum_{i=1}^{n} \alpha_i \pi_i \right)}{\partial \beta_j} = \sum\limits_{i=j}^{n} \alpha_i v_j - \alpha_n v_n$. Denote $h_r = \arg\max\limits_{k_r \le j \le n} \sum\limits_{i=j}^{n} (\alpha_i v_j)$. The profit is maximized when $\beta_{h_r} = 1 - \beta_1$, and the rest are all equal to 0. We thus obtain the announced result. $\square$

Proof of Proposition 12. The existence of k_c is guaranteed by Corollary OA.1. Next we show that $\tau_{i+1} \geq \tau_i$, for any $1 \leq i < k_c - 1$, and $\tau_1 = 1$. Because $v_{i+1} - c_{i+1} \leq v_i - c_i$, we have $v_{i+1} - (T - \tau_{i+1})c_{i+1} \leq v_i - (T - \tau_{i+1})c_i$. Thus, we know that $\tau_i \leq \tau_{i+1}$ based on the definition of τ_i, for any $1 \leq i < k_c - 1$. We show $\tau_1 = 1$ by contradiction. Suppose $\tau_1 > 1$. Then, the optimal expected profit per period from period 1 to period $\tau_1 - 1$ is given by $\sum\limits_{j=k_c}^{n} \alpha_j v_{k_c}$, which is less than the expected profit from an optimal static pricing policy. Thus, we can increase profit by getting rid of these time periods, which contradicts to the optimality of the policy. The optimal pricing schedule can be constructed according to Lemma OA.3. In particular, $p_T = \min\limits_{1 \leq i < k_c} \{v_i - (T - \tau_i)c_i\} - \delta$, and the price in the t^{th} period of a cycle is given by $p_t = \min\{v_{k_c}, p_T + (T-t)c_{k_c}\}$. $\square$

Proof of Proposition 13. Proposition 13(i) can be shown as follows. By Proposition 11, under the condition $v_1 - c_1 \geq v_2 - c_2 \geq \cdots \geq v_n - c_n$, an optimal randomized pricing policy follows a two-point price distribution, with the lower price being less than v_1 and the higher price equal to v_{h_r}. Under the condition $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$, $h_r = k_r$. Then again by Proposition 11, customers of types from 1 to $k_r - 1$ behave the same as type $k_r - 1$ and customers of types from k_r to n behave the same as type k_r. Hence, the problem with n customer segments is equivalent to the case with two customer segments in which the $\sum_{j=1}^{k_r-1} \alpha_j$ fraction of customers have valuation v_{k_r-1} and per-period waiting cost c_{k_r-1} and the rest fraction of customers have valuation v_{k_r} and per-period waiting cost c_{k_r}.

To prove Proposition 13(ii), we first prove an auxiliary lemma.

Lemma OS.6. *Under the conditions $v_1 - c_1 \geq v_2 - c_2 \geq \cdots > v_{n-1} - c_{n-1} \geq v_n - c_n$ and $v_1 \geq (\alpha_2 + \cdots + \alpha_n)v_2 \geq \cdots \geq \alpha_n v_n$, and consider a cyclic pricing policy with a price of v_k in the first $T-1$ periods, a price of $v_{k-1} - (T-1)c_{k-1} - \delta$ in the last period. If $v_k - v_{k-1} > c_{k-1}$ and $v_k - v_{k-1}$ is below the threshold $c_k \Big/ \left(1 + \frac{1}{2}\sqrt{\sum\limits_{j=k}^{n} \alpha_j \Big/ \sum\limits_{j=1}^{k-1} \alpha_j}\right)$,*

(i) The optimal cycle length for this policy is either $\lfloor T_1 \rfloor$ or $\lfloor T_1 \rfloor + 1$, where $T_1 = \sqrt{\sum\limits_{j=k}^{n} \alpha_j (v_k - v_{k-1} - c_{k-1}) \Big/ \sum\limits_{j=1}^{k-1} \alpha_j c_{k-1}}$ and $\lfloor x \rfloor$ represents the greatest integer that is no more than x;

(ii) Any customer with valuation greater than or equal to v_k will buy immediately upon arrival, and customer with valuation smaller than v_k will wait to purchase in the last period of a cycle.

Proof of Lemma OS.6. Consider a cyclic pricing policy where the firms charges v_k in the first $T-1$ periods, and $v_{k-1}-(T-1)c_{k-1}-\delta$ in the T^{th} period within a cycle. We first derive the optimal cycle length assuming that any customer with valuation greater than or equal to v_k will buy immediately upon arrival, and any customer with valuation smaller than v_k will wait to purchase in period T. Based on Proposition 12, the expected profit from this policy is given by $\Pi(T)=\sum_{j=1}^{k-1}\alpha_j(v_{k-1}-(T-1)c_{k-1}-\delta)+\sum_{j=k}^{n}\alpha_j\left[(T-1)v_k+v_{k-1}-(T-1)c_{k-1}-\delta\right]/T$. Denote $R(T)=\sum_{j=1}^{k-1}\alpha_j(v_{k-1}-(T-1)c_{k-1})+\sum_{j=k}^{n}\alpha_j\left[(T-1)v_k+v_{k-1}-(T-1)c_{k-1}\right]/T$ by ignoring the small δ in $\Pi(T)$. Taking the derivative of $R(T)$ with respect to T, we have

$$R'(T)=\sum_{j=k}^{n}\alpha_j\frac{v_k-v_{k-1}-c_{k-1}}{T^2}-\sum_{j=1}^{k-1}\alpha_j c_{k-1},$$

which is decreasing in T under the condition $v_k-v_{k-1}>c_{k-1}$. Thus, we conclude that $R(T)$ is concave, and it is maximized with $T_1=\sqrt{\sum_{j=k}^{n}\alpha_j(v_k-v_{k-1}-c_{k-1})\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}}$. Thus, the optimal cycle length T^* is given by either $\lfloor T_1\rfloor$ or $\lfloor T_1\rfloor+1$, where $\lfloor x\rfloor$ represents the greatest integer that is less than or equal to x. Note that $\max\{R(\lfloor T_1\rfloor),R(\lfloor T_1\rfloor+1)\}$ is an upper bound of the expected profit from this cyclic pricing policy.

Next we show that under the conditions specified in the Lemma and the cycle length T^*, any customer with valuation greater than or equal to v_k will buy immediately upon arrival, and any customer with valuation smaller than v_k will wait to purchase in period T^*. Because $v_{i+1}-v_i< c_{i+1}-c_i\le(T^*-1)(c_{i+1}-c_i)$, we have $v_{k-1}-(T^*-1)c_{k-1}=\min_{1\le i\le k-1}(v_i-(T^*-1)c_i)<v_1$. Thus, in order to show that any customer with valuation smaller than or equal to v_{k-1} will wait to purchase in period T^*, it suffices to prove $v_{k-1}-(T^*-1)c_{k-1}\ge 0$. Note that T^* is equal to either $\lfloor T_1\rfloor$ or $\lfloor T_1\rfloor+1$. Thus, one sufficient condition is given by $\frac{v_{k-1}}{c_{k-1}}\ge T_1$. This condition holds as:

$$\frac{v_{k-1}^2}{c_{k-1}^2}>\frac{v_{k-1}}{c_{k-1}}=\sum_{j=1}^{k-1}\alpha_j v_{k-1}\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}\ge\sum_{j=k}^{n}\alpha_j v_k\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1}>\sum_{j=k}^{n}\alpha_j(v_k-v_{k-1}-c_{k-1})\Big/\sum_{j=1}^{k-1}\alpha_j c_{k-1},$$

where the first inequality is due to $v_{k-1}>c_{k-1}$ and the second inequality is due to $\sum_{j=1}^{k-1}\alpha_j v_{k-1}\ge\sum_{j=k}^{n}\alpha_j v_k$.

Lastly, we show that any customer with valuation greater than v_{k-1} will buy immediately upon arrival. As $v_i - c_i \geq v_{i+1} - c_{i+1}$, for any $k \leq i \leq n-1$, it suffices to show that $p_{T^*} + c_k = v_{k-1} - (T^* - 1)c_{k-1} - \delta + c_k \geq v_k$. We have

$$
\begin{aligned}
& v_k - v_{k-1} + (T^* - 1)c_{k-1} + \delta - c_k \\
< & v_k - v_{k-1} + T_1 c_{k-1} - c_k \\
\leq & v_k - v_{k-1} + \frac{(v_k - v_{k-1})}{2} \sqrt{\sum_{j=k}^{n} \alpha_j \Big/ \sum_{j=1}^{k-1} \alpha_j} - c_k < 0,
\end{aligned}
$$

where the first inequality is due to $T^* = \lfloor T_1 \rfloor$ or $\lfloor T_1 \rfloor + 1$, the second inequality is due to $T_1 c_{k-1} = \sqrt{\sum_{j=k}^{n} \alpha_j c_{k-1}(v_k - v_{k-1} - c_{k-1}) \Big/ \sum_{j=1}^{k-1} \alpha_j}$ is maximized with $c_{k-1} = (v_k - v_{k-1})/2$, and the third inequality is due to $v_k - v_{k-1} < c_k \Big/ \left(1 + \frac{1}{2}\sqrt{\sum_{j=k}^{n} \alpha_j \Big/ \sum_{j=1}^{k-1} \alpha_j}\right)$. We thus obtain the announced result. $\square$

Now we are ready to prove Proposition 13(ii). Based on Proposition 11 and conditions specified in the proposition, we know that an optimal randomized pricing policy follows a two-point price distribution. In particular, under the optimal two-point price distribution, there exists k_r such that any customer with valuation less than v_{k_r} would wait to purchase, while any customer with valuation greater than or equal to v_{k_r} would buy immediately upon arrival. Based on the proof of Proposition 11, the expected profit from an optimal randomized pricing policy is given by $\sum_{j=1}^{k_r-1} \alpha_j (v_{k_r-1} - c_{k_r-1}/\beta_1) + \sum_{j=k_r}^{n} \alpha_j (\beta_1 v_{k_r-1} - c_{k_r-1} + (1-\beta_1) v_{k_r})$. The function is maximized when $\beta_1 = \sqrt{\sum_{j=1}^{k_r-1} \alpha_j c_{k_r-1} \Big/ \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - v_{k_r-1})}$, with a value of $\sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j\right) \cdot \left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1})\right)}$.

We next show that this optimal randomized pricing policy is dominated by a cyclic pricing policy. Consider a cyclic pricing policy where the price is equal to v_{k_r} in the first $T-1$ periods, and $v_{k_r-1} - (T-1)c_{k_r-1} - \delta$ in the last period. Lemma OS.6 shows that any customer with valuation greater than or equal to v_{k_r} buy immediately upon arrival, while any customer with valuation less than v_{k_r} would wait to purchase at period T. Based on the proof of Lemma OS.6, its optimal profit is given by $\max\{R(\lfloor T_1 \rfloor), R(\lfloor T_1 \rfloor + 1)\}$. Also, we know from the proof of Lemma OS.6 that $R(T)$

is concave and decreasing, for any $T \geq T_1$, and thus we have $R(\lfloor T_1 \rfloor + 1) \geq R(T_1+1)$. That is, the expected profit from this cyclic pricing is bounded below by $R(T_1+1)$. We have

$$
\begin{aligned}
R(T_1+1) &= \sum_{j=k_r}^{n} \alpha_j \left[\frac{T_1}{T_1+1} v_{k_r} + \frac{1}{T_1+1}(v_{k_r-1} - T_1 c_{k_r-1}) \right] + \sum_{j=1}^{k_r-1} \alpha_j (v_{k_r-1} - T_1 c_{k_r-1}) \\
&= \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - \sum_{j=1}^{k_r-1} \alpha_j T_1 c_{k_r-1} - \sum_{j=k_r}^{n} \alpha_j \frac{1}{T_1+1}(v_{k_r} - v_{k_r-1} - c_{k_r-1}) \\
&> \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - \sum_{j=1}^{k_r-1} \alpha_j T_1 c_{k_r-1} - \sum_{j=k_r}^{n} \alpha_j \frac{1}{T_1}(v_{k_r} - v_{k_r-1} - c_{k_r-1}) \\
&= \sum_{j=1}^{k_r-1} \alpha_j v_k + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j\right)\left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1} - c_{k_r-1})\right)} \\
&> \sum_{j=1}^{k_r-1} \alpha_j v_{k_r-1} + \sum_{j=k_r}^{n} \alpha_j (v_{k_r} - c_{k_r-1}) - 2\sqrt{\left(\sum_{j=1}^{k_r-1} \alpha_j\right)\left(\sum_{j=k_r}^{n} \alpha_j c_{k_r-1}(v_{k_r} - v_{k_r-1})\right)},
\end{aligned}
$$

where the first inequality is due to $v_{k_r} - v_{k_r-1} > c_{k_r-1}$, and the second equality is due to $T_1 = \sqrt{\sum_{j=k_r}^{n} \alpha_j (v_{k_r} - v_{k_r-1} - c_{k_r-1}) \Big/ \sum_{j=1}^{k_r-1} \alpha_j c_{k_r-1}}$. We thus obtain Proposition 13(ii).

Next we prove Proposition 13(iii). Based on Proposition 12, we know that, under an optimal cyclic pricing policy, there exists a $k_c \in \{1, \ldots, n\}$ such that any customer with valuation greater than or equal to v_{k_c} would either purchase or leave immediately upon arrival. On the other hand, a type-i, $1 \leq i < k_c$, customer would wait upon arrival if she arrives no earlier than the τ_i^{th} period with a cycle. In particular, $\tau_1 = 1 \leq \tau_2 \leq \ldots \leq \tau_{k_c-1}$. Thus, the expected profit from an optimal cyclic pricing policy is:

$$
\begin{aligned}
\Pi(T) &\leq \sum_{j=k_c}^{n} \alpha_i \frac{\sum_{t=1}^{T} v_{k_c} - [v_{k_c} - p_T - (T-t)c_{k_c}]^+}{T} + \sum_{j=1}^{k_c-1} \alpha_j p_T \\
&< \sum_{j=k_c}^{n} \alpha_j \frac{\sum_{t=1}^{T} v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+}{T} + \sum_{j=1}^{k_c-1} \alpha_j (v_1 - (T-1)c_1) \\
&< \sum_{j=k_c}^{n} \alpha_j \frac{\int_0^T v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+ \mathrm{d}t}{T} + \sum_{j=1}^{k_c-1} \alpha_j (v_1 - (T-1)c_1) \\
&= \sum_{j=k_c}^{n} \alpha_j v_{k_c} + \sum_{j=1}^{k_c-1} \alpha_i (v_1 - (T-1)c_1) - \sum_{j=k_c}^{n} \alpha_j \frac{[v_{k_c} - v_1 + (T-1)c_1]^2}{2T c_{k_c}} \equiv H(T),
\end{aligned}
$$

where the second inequality is due to $p_T < v_1 - (T-1)c_1$, and the third inequality is due to $v_{k_c} - [v_{k_c} - v_1 + (T-1)c_1 - (T-t)c_{k_c}]^+$ being monotonically decreasing in t. Taking the first order derivative of $H(T)$ with respect to T, we have $H'(T) = -\sum_{j=1}^{k_c-1}\alpha_j c_1 + \sum_{j=k_c}^{n}\alpha_j \frac{(v_{k_c}-v_1-c_1)^2}{2c_{k_c}T^2} - \sum_{j=k_c}^{n}\alpha_j \frac{c_1^2}{2c_{k_c}}$, which is decreasing in T. Thus, we conclude that $H(T)$ is concave in T, and its maximum is realized when $T = T_2 \equiv \sqrt{\frac{\sum_{j=k_c}^{n}\alpha_j(v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1}\alpha_j c_1 c_{k_c} + \sum_{j=k_c}^{n}\alpha_j c_1^2}}$. Consequently, the expected profit from an optimal cyclic pricing policy is bounded above by

$$\begin{aligned}
H(T_2) = &\sum_{j=1}^{k_c-1}\alpha_j v_1 + \sum_{j=k_c}^{n}\alpha_j v_{k_c} - \sum_{j=1}^{k_c-1}\alpha_j(T_2-1)c_1 - \sum_{j=k_c}^{n}\alpha_j \frac{(v_{k_c}-v_1-c_1)^2}{2T_2 c_{k_c}} \\
&- \sum_{j=k_c}^{n}\alpha_j \frac{T_2 c_1^2}{2c_{k_c}} - \sum_{j=k_c}^{n}\alpha_j \frac{c_1}{c_{k_c}}(v_{k_c}-v_1-c_1) \\
< &\sum_{j=1}^{k_c-1}\alpha_j v_1 + \sum_{j=k_c}^{n}\alpha_j v_{k_c} - \sum_{j=1}^{k_c-1}\alpha_j(T_2-1)c_1 - \sum_{j=1}^{k_c-1}\alpha_j T_2 c_1 - \sum_{j=k_c}^{n}\alpha_j \frac{T_2 c_1^2}{2c_{k_c}} \\
&- \sum_{j=k_c}^{n}\alpha_j \frac{c_1}{c_{k_c}}(v_{k_c}-v_1-c_1),
\end{aligned}$$

where the inequality is due to $T_2 = \sqrt{\frac{\sum_{j=k_c}^{n}\alpha_j(v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1}\alpha_j c_1 c_{k_c} + \sum_{j=k_c}^{n}\alpha_j c_1^2}} > \sqrt{\frac{\sum_{j=k_c}^{n}\alpha_j(v_{k_c}-v_1-c_1)^2}{2\sum_{j=1}^{k_c-1}\alpha_j c_1 c_{k_c}}}$.

Lastly, based on the proof of Proposition 11, we know that the expected profit from a randomized pricing policy can be derived by maximizing $\sum_{j=1}^{k_c-1}\alpha_j(v_{k_c-1}-c_{k_c-1}/\beta_1) + \sum_{j=k_c}^{n}\alpha_j(\beta_1 v_{k_c-1} - c_{k_c-1} + (1-\beta_1)v_{k_c})$ over $\beta_1 \in [0,1]$. Plugging $\beta_1 = c_{k_c-1}/[v_{k_c-1}-v_1+(T_2-1)c_1]$ (in which case, $v_{k_c-1}-c_{k_c-1}/\beta_1 = v_1-(T_2-1)c_1$) into the profit function, we have

$$\sum_{j=1}^{k_c-1}\alpha_j v_1 + \sum_{j=k_c}^{n}\alpha_j v_{k_c} - \sum_{j=1}^{k_c-1}\alpha_j(T_2-1)c_1 - \sum_{j=k_c}^{n}\alpha_j \frac{v_{k_c}-v_{k_c-1}}{v_{k_c-1}-v_1+(T_2-1)c_1}c_{k_c-1} - \sum_{j=1}^{k_c-1}\alpha_j c_k \equiv U.$$

So in order to show that an optimal randomized pricing policy outperforms an optimal cyclic pricing policy, it suffices to show $H(T_2) < U$. It is easy to verify that $T_2(T_2-1)/(v_{k_c}-v_{k_c-1})$ is increasing in $v_{k_c}-v_{k_c-1}$ when $v_{k_c}-v_{k_c-1}$ is sufficiently large. Consequently, we have $-\sum_{j=k_c}^{n}\alpha_j \frac{T_2 c_1^2}{2c_{k_c}} \leq -\sum_{j=k_c}^{n}\alpha_j \frac{v_{k_c}-v_{k_c-1}}{(T_2-1)c_1}c_{k_c-1} < -\sum_{j=k_c}^{n}\alpha_j \frac{v_{k_c}-v_{k_c-1}}{v_{k_c-1}-v_1+(T_2-1)c_1}c_{k_c-1}$. Comparing against the upper bound of $H(T_2)$, we know that one sufficient condition for $H(T_2) < U$ is given by

$$-\sum_{j=1}^{k_c-1}\alpha_j T_2 c_1 - \sum_{j=k_c}^{n}\alpha_j \frac{c_{k_c-1}}{c_{k_c}}(v_{k_c}-v_{k_c-1}-c_{k_c-1}) < -\sum_{j=k_c}^{n}\alpha_j c_{k_c-1}.$$

Because the left hand of the inequality is decreasing in $v_{k_c}-v_{k_c-1}$, and thus the inequality is guaranteed when $v_{k_c}-v_{k_c-1}$ is sufficiently large. We thus obtain Proposition 13(iii). $\square$

B. Proofs of Results in the Online Appendix

Proof of Lemma OA.1. Consider a randomized pricing policy, where the set of types of customers who will wait is denoted by $\mathcal{W}$. First we prove that β_i and x_i, $\forall i$, are properly defined by showing that $u_{i-1} < u_i$. If $i \notin \mathcal{W}$, we have $u_i = v_i > v_{i-1} \geq u_{i-1}$. If $i \in \mathcal{W}$ and $i-1 \in \mathcal{W}$, then we know that $u_i = \underline{p}^i > u_{i-1} = \underline{p}^{i-1}$ because $\underline{p}$ increases in c. The only non-trivial case is when $i \in \mathcal{W}$ and $i-1 \notin \mathcal{W}$. In this case, we have $\mathsf{E}[(v_{i-1} - P)^+] \leq c_{i-1} < c_i$. So based on the definition of $\underline{p}^i = \max\{v' \mid \mathsf{E}[(v' - P)^+] \leq c\}$, we have $u_i = \underline{p}^i > v_{i-1} = u_{i-1}$. Based on the definition of x_i, it is easy to verify that $x_i \leq \underline{p}^i < v_i$, $\forall i \in \mathcal{W}$, and $x_i \leq v_i$, $\forall i \notin \mathcal{W}$. Furthermore, the support of the price distribution of an optimal randomized pricing policy will never include any price greater than v_n, as no customers will purchase at those price points. Thus, an optimal policy must satisfy $\sum_{i=1}^{n} \beta_i = 1$.

For any $i \in \mathcal{W}$, we can rewrite the condition as $\mathsf{E}[(v_i - P)^+] = \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$, $\forall i \notin \mathcal{W}$, based on the definitions of β_i and x_i. Similarly, for any $i \in \mathcal{W}$, we have $F(\underline{p}^i)\mathsf{E}[\underline{p}^i - P | P \leq \underline{p}^i] = \sum_{j=1}^{i} \beta_j \underline{p}^i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$. Because $x_i \leq \underline{p}^i$, $\forall i \in \mathcal{W}$, we have $\sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j \leq \sum_{j=1}^{i} \beta_j \underline{p}^i - \sum_{j=1}^{i} \beta_j x_j \leq c_i$. Notice that the profit function $\sum_{i=1}^{n} \alpha_i \pi_i$ is increasing in x_i because both $\mathsf{E}[P | P \leq \underline{p}^i] = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$ and $F(v_i)\mathsf{E}[P | P \leq v_i] = \sum_{j=1}^{i} \beta_j x_j$ increase in x_i, $\forall i$. Thus, the following condition $\sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i$ must be satisfied for an optimal policy; Otherwise, we can always improve profit by increasing x_i. Note that when $i = 1$, this constraint is guaranteed to be satisfied as the left hand side of the inequality is zero. So we resort to the initial condition derived in Proposition 1, i.e., $\beta_1 v_1 - \beta_1 x_1 > c_1$, should type-1 customers wait. Consequently, we can find the optimal policy, in the space of all policies where type-i customers, $\forall i \in \mathcal{W}$, wait and the rest purchase or leave immediately, by solving Problem (OA.1). $\square$

Proof of Lemma OA.2. Consider a set of randomized pricing policies, which includes all policies under which type-1 customers would wait, and the rest of customers either purchase or leave immediately upon arrival. Based on Lemma OA.1, the optimal policy in the set can be derived by solving the following optimization problem:

$$
\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j \sum_{i=1}^{j} \beta_i x_i \\
\text{s.t.} \quad & \beta_1 v_1 - \beta_1 x_1 > c_1, \\
& \sum_{i=1}^{j} \beta_i v_j - \sum_{i=1}^{j} \beta_i x_i \le c_j, \quad j = 2,3,\ldots,n, \\
& x_1 < v_1, \text{ and } x_j \le v_j, j = 2,3,\ldots,n, \text{ and } \sum_{j=1}^{n} \beta_j = 1.
\end{aligned}
\tag{OS.5}
$$

Next we show that, when c_1 is sufficiently small, the optimal value of the objective function in Problem (OS.5) is greater than the expected profit from an optimal static pricing policy, which is given by $\max_{j\le n}\left\{\sum_{i=j}^{n}\alpha_i v_j\right\}$, by induction. The case when $n=2$ is proved in Proposition 4. Now suppose that the result holds when $n=k$, and we next show that it also holds when $n=k+1$.

Consider first the trivial case when $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \max_{j\le k}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\}$. In this case, we can simply let $\beta_{k+1}=0$, and adopt the optimal randomized pricing policy when $n=k$. Its expected profit is guaranteed to be greater than $\max_{j\le k}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\}$.

Next consider the case when $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \alpha_{k+1}v_{k+1}$. Conditional on β_j, $\forall j\in\{1,\ldots,k+1\}$, the objective function of Problem (OS.5) is increasing in x_j, $\forall j\in\{1,\ldots,k+1\}$. Therefore, Problem (OS.5) is maximized when $x_1 = v_1 - \frac{c_1}{\beta_1} - \delta$, for any sufficiently small δ, and $x_j = v_j$, $\forall j\in\{2,\ldots,k+1\}$. Thus, we can reformulate Problem (OS.5) as:

$$
\begin{aligned}
\max_{\beta} \quad & \alpha_1\left(v_1 - \frac{c_1}{\beta_1} - \delta\right) + \sum_{j=2}^{k+1}\alpha_j\left(\sum_{i=1}^{j}\beta_i v_i - c_1 - \beta_1\delta\right) \\
\text{s.t.} \quad & \sum_{i=1}^{j}\beta_i(v_j - v_i) < c_j - c_1, \forall j\in\{2,\ldots,k+1\}, \text{ and } \sum_{j=1}^{k+1}\beta_j = 1.
\end{aligned}
\tag{OS.6}
$$

Denote $U(\beta) = \alpha_1\left(v_1 - \frac{c_1}{\beta_1}\right) + \sum_{j=2}^{k+1}\alpha_j\left(\sum_{i=1}^{j}\beta_i v_i - c_1\right)$. Thus, the derivatives of $U(\beta)$ with respect to β_j, $j=2,\ldots,k$, are given by

$$
\frac{\partial U(\beta)}{\partial \beta_1} = \frac{\alpha_1 c_1}{\beta_1^2} + \sum_{i=2}^{k+1}\alpha_i v_1 - \alpha_{k+1}v_{k+1}, \text{ and } \frac{\partial U(\beta)}{\partial \beta_j} = \sum_{i=j}^{k+1}\alpha_i v_j - \alpha_{k+1}v_{k+1}, \quad j=2,3,\ldots,k.
$$

Recall that $\max_{j\le k+1}\left\{\sum_{i=j}^{k+1}\alpha_i v_j\right\} = \alpha_{k+1}v_{k+1} \ge \sum_{i=j}^{k+1}\alpha_i v_j$. Thus, we can conclude that $U(\beta)$ is decreasing in β_j and the profit is maximized when $\beta_j = 0$, $\forall j\in\{2,\ldots,k\}$. Ignoring δ, Problem (OS.6) can thus be simplified as a single variable optimization problem below.

$$\begin{aligned} \max_{\beta_1} \quad & \alpha_1\left(v_1 - \frac{c_1}{\beta_1}\right) + \sum_{j=2}^{k+1} \alpha_j(\beta_1 v_1 - c_1) + (1-\beta_1)\alpha_{k+1}v_{k+1} \\ \text{s.t.} \quad & \beta_1(v_j - v_1) < c_j - c_1, \quad \forall j \in \{2,\dots,k+1\}. \end{aligned} \tag{OS.7}$$

The solution to the first order condition $\frac{\partial U(\beta)}{\partial \beta_1} = \frac{\alpha_1 c_1}{\beta_1^2} + \sum_{i=2}^{k+1} \alpha_i v_1 - \alpha_{k+1}v_{k+1} = 0$ is given by $\beta_1^* = \sqrt{\frac{\alpha_1 c_1}{\alpha_{k+1}v_{k+1} - \sum_{i=2}^{k+1}\alpha_i v_1}}$. When $\beta_1^*(v_j - v_i) < c_j - c_1, \forall j \in \{2,\dots,k+1\}$, which holds when c_1 is sufficiently small, the expected profit with $\beta_1 = \beta_1^*$ is given by:

$$\begin{aligned} & \alpha_1(v_1 - \frac{c_1}{\beta_1}) + \sum_{j=2}^{k+1} \alpha_j(\beta_1 v_1 - c_1) + (1-\beta_1)\alpha_{k+1}v_{k+1} \\ =& \alpha_1 v_1 + \alpha_{k+1}v_{k+1} - 2\sqrt{\alpha_1 c_1\left(\alpha_{k+1}v_{k+1} - \sum_{i=2}^{k+1}\alpha_i v_1\right)} - \sum_{j=2}^{k+1}\alpha_j c_1, \end{aligned}$$

which is greater than $\alpha_{k+1}v_{k+1}$ when c_1 is sufficiently small. This completes the proof. □

Proof of Lemma OA.3. We first show that consumer behavior is consistent with the definition of k_0 and k_c under the price schedule as constructed in Lemma OA.3. For any $k_0 \le i < k_c$, it suffices to show that either $p_T + (T-t)c_i < p_t$ or $p_t > v_i$, for any $t \ge \tau_i$. If $p_s + c_{k_c} > v_{k_c}$, for any $t < s \le T$, then we have $p_t \ge v_{k_c} > v_i$. On the other hand, if $p_s + c_{k_c} \le v_{k_c}$, for all $t < s \le T$, then we have $p_t = p_{t+1} + c_{k_c} = p_{t+2} + 2c_{k_c} = \cdots = p_T + (T-t)c_{k_c} > p_T + (T-t)c_i$ due to $c_i < c_{k_c}$ for any $k_0 \le i < k_c$. Next consider any $i \ge k_c$. In this case, we need to show that either $p_t \le p_{t+1} + c_i$ or $p_{t+1} + c_i > v_i$ for any $t < T$. If $p_{t+1} + c_{k_c} \le v_{k_c}$, we have $p_t = p_{t+1} + c_{k_c} \le p_{t+1} + c_i$, for any $i \ge k_c$. On the other hand, if $p_{t+1} + c_{k_c} > v_{k_c}$, we have $p_{t+1} > v_i - c_i$ for any $i < k_t$, and $p_t \le p_{t+1} + c_{k_t} \le p_{t+1} + c_i$ for any $i \ge k_t$, if k_t exists. If k_t does not exist, simply we have $p_{t+1} > v_i - c_i$ for any $i \ge k_c$.

We next show that this price policy is optimal. First consider period T. Based on the definition of τ_i, we have $v_i - (T-\tau_i)c_i \ge p_T$, for any $k_0 \le i < k_c$. Thus, $p_T \le \min_{k_0 \le i < k_c}\{v_i - (T-\tau_i)c_i\}$ and the optimal price for period T is given by $p_T = \min_{k_0 \le i < k_c}\{v_i - (T-\tau_i)c_i\} - \delta$. Next we work backwards and consider period $t < T$. Consider first the case when $p_{t+1} + c_{k_c} \le v_{k_c}$. In this case, we prove $p_t \le p_{t+1} + c_{k_c}$ by contradiction. Suppose $p_t > p_{t+1} + c_{k_c}$. Then, we have $v_{k_c} - p_t < v_{k_c} - p_{t+1} - c_{k_c}$. Combining with the condition $p_{t+1} + c_{k_c} \le v_{k_c}$, we have $v_{k_c} - p_{t+1} - c_{k_c} \ge \max\{0, v_{k_c} - p_t\}$. That is, a customer with valuation v_{k_c} is better off purchasing at period $t+1$, which contradicts to the fact that a customer with valuation v_{k_c} will not wait. Consequently, the optimal price is $p_t = p_{t+1} + c_{k_c}$

if $p_{t+1}+c_{k_c} \le v_{k_c}$. Last consider the case when $p_{t+1}+c_{k_c} > v_{k_c}$. Following a similar approach as described above, we can show that $p_t \le p_{t+1}+c_{k_t}$, if k_t exists. Thus, if $(\alpha_{k_t}+\cdots+\alpha_n)(p_{t+1}+c_{k_t}) > \max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, the profit maximizing price is given by $p_t = p_{t+1}+c_{k_t}$. On the other hand, if $(\alpha_{k_t}+\cdots+\alpha_n)(p_{t+1}+c_{k_t}) \le \max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, the profit maximizing price is given by v_{l_t}, where $l_t = \arg\max_{k_c \le i < k_t}\{(\alpha_i+\cdots+\alpha_n)v_i\}$. If k_t does not exist, $p_{t+1}+c_i > v_i$ for $k_c \le i$, then the firm could charge the optimal price $p_t = v_l$, without worrying about customers' strategic waiting behavior. We thus obtain the announced result. $\square$

Proof of Lemma OA.4. We prove the result by showing that any policy as characterised in Lemma OA.3 follows a (weakly) markdown pattern. For any given cyclic pricing policy $\mathbf{p} = \{p_1, p_2, \ldots, p_T\}$, we show $p_t \ge p_{t+1}$, $\forall\, t \le T-1$ by induction. Because a customer with valuation v_i, for any $k_0 \le i < k_c$, will wait to purchase in period T, we have $p_T < p_t$, for any $t \le T-1$. Hence, we have $p_T > p_{T-1}$. Now suppose that $p_{t+1} \ge p_{t+2}$, and next we prove $p_t \ge p_{t+1}$. Let us consider three cases.

(a) $p_{t+2}+c_{k_c} \le p_{t+1}+c_{k_c} \le v_{k_c}$. Based on Lemma OA.3, we have $p_t = p_{t+1}+c_{k_c} > p_{t+2}+c_{k_c} = p_{t+1}$.

(b) $p_{t+2}+c_{k_c} \le v_{k_c} < p_{t+1}+c_{k_c}$. Based on Lemma OA.3, we have $p_{t+1} = p_{t+2}+c_{k_c} < v_{k_c}$, and $p_t > v_{k_c}$. Hence, we can conclude that $p_t > p_{t+1}$.

(c) $v_{k_c} < p_{t+2}+c_{k_c} < p_{t+1}+c_{k_c}$. It is easy to verify that $k_t \ge k_{t+1}$ due to $p_{t+1} \ge p_{t+2}$. To complete the proof, we need to further consider three sub-cases.

(c1) k_t and k_{t+1} do not exist. In this case, we have $p_t = p_{t+1} = v_l$.

(c2) k_t does not exist but k_{t+1} exists. If $p_{t+1} = v_{l_{t+1}}$, then we have $p_t = v_l \ge v_{l_{t+1}} = p_{t+1}$ due to the definitions of l and l_t. If $p_{t+1} = p_{t+2}+c_{k_{t+1}}$, which happens under the condition that $(\alpha_{k_{t+1}}+\cdots+\alpha_n)(p_{t+2}+c_{k_{t+1}}) > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, then we have $\max_{k_c \le i}\{(\alpha_i+\cdots+\alpha_n)v_i\} \ge (\alpha_{k_{t+1}}+\cdots+\alpha_n)v_{k_{t+1}} \ge (\alpha_{k_{t+1}}+\cdots+\alpha_n)(p_{t+2}+c_{k_{t+1}}) > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$, where the second inequality is due to $p_{t+2}+c_{k_{t+1}} \le v_{k_{t+1}}$.The condition $\max_{k_c \le i}\{(\alpha_i+\cdots+\alpha_n)v_i\} > \max_{k_c \le i < k_{t+1}}\{(\alpha_i+\cdots+\alpha_n)v_i\}$ implies that $l \ge k_{t+1}$. Thus, we have $p_t = v_l \ge v_{k_{t+1}} \ge p_{t+2}+c_{k_{t+1}} = p_{t+1}$.

(c3) k_t and k_{t+1} exist. If $p_{t+1} = v_{l_{t+1}}$, then either $p_t = p_{t+1}+c_{k_t}$, which is greater than p_{t+1}, or $p_t = v_{l_t}$, which again is greater than $p_{t+1} = v_{l_{t+1}}$ due to $k_t \ge k_{t+1}$. Next consider the case when $p_{t+1} = p_{t+2}+c_{k_{t+1}}$. In this case, either $p_t = p_{t+1}+c_{k_t}$, which is greater than p_{t+1}, or $p_t = v_{l_t}$. To prove $p_t \geq p_{t+1}$ when $p_t = v_{l_t}$, we first prove k_t must be greater than k_{t+1} by contradiction. Suppose $k_t = k_{t+1}$, then we have $v_{l_t} = v_{l_{t+1}}$. Combining with the condition that $p_{t+1} = p_{t+2} + c_{k_{t+1}}$, we have $(\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+2} + c_{k_{t+1}}) > (\alpha_{l_{t+1}} + \cdots + \alpha_n)v_{l_{t+1}} = \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+1} + c_{k_{t+1}})$, which contradicts to $p_{t+1} \geq p_{t+2}$. Thus, we have $k_t > k_{t+1}$. Under this condition, we have $\max_{k_c \leq i < k_t} \{(\alpha_i + \cdots + \alpha_n)v_i\} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)v_{k_{t+1}} \geq (\alpha_{k_{t+1}} + \cdots + \alpha_n)(p_{t+2} + c_{k_{t+1}}) > \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\}$, where the second inequality is due to $p_{t+2} + c_{k_{t+1}} \leq v_{k_{t+1}}$ and the last inequality is due to $p_{t+1} = p_{t+2} + c_{k_{t+1}}$. The condition $\max_{k_c \leq i < k_t} \{(\alpha_i + \cdots + \alpha_n)v_i\} > \max_{k_c \leq i < k_{t+1}} \{(\alpha_i + \cdots + \alpha_n)v_i\}$ implies that $l_t > k_{t+1}$, leading to $p_t = v_{l_t} > v_{k_{t+1}} \geq p_{t+2} + c_{k_{t+1}} = p_{t+1}$. We thus obtain the announced results. $\square$

Proof of Lemma OA.5. We first prove Lemma OA.5(i). With a bit abuse of notation, we denote the total profit generated by customers with valuation greater than or equal to v_{k_c} in period t by π_t. We first show that, for any t, there exists $k_c \leq i_t$ such that $\pi_t \leq \sum_{i=i_t}^{n} \alpha_i(p_T + (T-t)c_{i_t})$. Consider first when $p_{t+1} + c_{k_c} \leq v_{k_c}$. In this case, we have $p_t = p_{t+1} + c_{k_c} \leq v_{k_c}$ according to Lemma OA.3. As any customer with valuation greater than or equal to v_{k_c} will not wait, the incentive compatibility constraint is given by $p_t \leq p_T + (T-t)c_{k_c}$. Consequently, we have $\pi_t = \sum_{i=k_c}^{n} \alpha_i p_t \leq \sum_{i=k_c}^{n} \alpha_i(p_T + (T-t)c_{k_c})$. Second, we consider the case when $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = p_{t+1} + c_{k_t} \leq v_{k_t}$. In this case, any customer with valuation greater than or equal to v_{k_t} will buy immediately in period t, and thus it must satisfy that $p_t \leq p_T + (T-t)c_{k_t}$. Consequently, we have $\pi_t = \sum_{i=k_t}^{n} \alpha_i p_t \leq \sum_{i=k_t}^{n} \alpha_i(p_T + (T-t)c_{k_t})$. The third case is $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = v_{l_t}$. In this case, any customer with valuation greater than or equal to v_{l_t} will buy immediately in period t, and thus it must satisfy that $p_t = v_{l_t} \leq p_T + (T-t)c_{l_t}$. Consequently, we have $\pi_t = \sum_{i=l_t}^{n} \alpha_i p_t = \sum_{i=l_t}^{n} \alpha_i v_{l_t} \leq \sum_{i=l_t}^{n} \alpha_i(p_T + (T-t)c_{l_t})$. Last consider the case $p_{t+1} + c_{k_c} > v_{k_c}$ and $p_t = v_l$. In this case, any customer with valuation greater than or equal to v_l will buy immediately in period t, and thus it must satisfy that $p_t = v_l \leq p_T + (T-t)c_l$. Consequently, we have $\pi_t = \sum_{i=l}^{n} \alpha_i p_t = \sum_{i=l}^{n} \alpha_i v_l \leq \sum_{i=l}^{n} \alpha_i(p_T + (T-t)c_l)$.

Recall that τ_i is defined as follows. Any type-i customer will leave immediately upon arrival if she arrives before the τ_i^{th} period within a cycle; Otherwise, she will wait till the end of the cycle to purchase. Next we show that $\tau_{i+1} \leq \tau_i$, for any $k_0 \leq i < k_c - 1$. Due to $c_1/v_1 \geq c_2/v_2 \geq \cdots \geq c_n/v_n$ and $v_i - (T - \tau_i)c_i \geq 0$, it is easy to verify that $\frac{v_{i+1} - v_i}{c_{i+1} - c_i} \geq \frac{v_{i+1}}{c_{i+1}} \geq \frac{v_i}{c_i} = T - \tau_i$, which implies that

$v_{i+1} - (T-\tau_i)c_{i+1} \geq v_i - (T-\tau_i)c_i \geq p_T$. That is, any customer with valuation v_{i+1} arriving in period τ_i will wait to purchase in period T, and thus we have $\tau_{i+1} \leq \tau_i$, $\forall\, i < k_c - 1$.

Now we are ready to show that the expected profit from an optimal cyclic pricing is bounded above by the expected profit from an optimal static pricing policy. We first consider the case when $\tau_{k_0} = 1$. As $\tau_{i+1} \leq \tau_i$, we have $\tau_i = 1$, for any $k_0 \leq i < k_c$. Hence, the expected profit from an optimal cyclic pricing policy is bounded above by:

$$\begin{aligned}
\Pi(T) &= \sum_{t=1}^{T} \pi_t/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\leq \sum_{t=1}^{T}\sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\leq \sum_{t=1}^{T}\sum_{i=i_t}^{n} \alpha_i \left[(T-t)\frac{c_{k_0}}{v_{k_0}} v_{i_t}\right]/T + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\leq \max_{1\leq k\leq n}\left\{\sum_{i=k}^{n} \alpha_i v_k\right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\leq \max_{1\leq k\leq n}\left\{\sum_{i=k}^{n} \alpha_i v_k\right\} \frac{c_{k_0}}{v_{k_0}} \frac{T-1}{2} + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}),
\end{aligned}$$

where the second inequality is due to $i_t \geq k_c$ and $\frac{c_{i_t}}{v_{i_t}} \leq \frac{c_{k_0}}{v_{k_0}}$, the third inequality is due to $i_t \geq k_c$ and $\sum_{i=i_t}^{n} \alpha_i v_{i_t} \leq \max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}$, and the last inequality is due to $p_T \leq v_{k_0} - (T-1)c_{k_0}$. If $\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}\frac{c_{k_0}}{2v_{k_0}} \leq \sum_{i=k_0}^{n}\alpha_i c_{k_0}$, then $\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}\frac{c_{k_0}}{v_{k_0}}\frac{T-1}{2} + \sum_{i=k_0}^{n}\alpha_i(v_{k_0} - (T-1)c_{k_0}) \leq \sum_{i=k_0}^{n}\alpha_i v_{k_0} \leq \max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}$; Otherwise, it is easy to verify that $\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}\frac{c_{k_0}}{v_{k_0}}\frac{T-1}{2} + \sum_{i=k_0}^{n}\alpha_i(v_{k_0} - (T-1)c_{k_0})$ is increasing in $T-1$. Because of $T-1 \leq \frac{v_{k_0}}{c_{k_0}}$, we have

$$\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}\frac{c_{k_0}}{v_{k_0}}\frac{T-1}{2} + \sum_{i=k_0}^{n}\alpha_i(v_{k_0} - (T-1)c_{k_0}) \leq \frac{1}{2}\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\} < \max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}.$$

Next consider the case when $\tau_{k_0} > 1$. In this case, each cycle can be decomposed into a couple of separable mini cycles, namely, $[1, \tau_{k_c-1} - 1]$, $[\tau_{k_c-1}, \tau_{k_c-2} - 1]$, $\cdots$, $[\tau_{k_0}, T]$. We prove an optimal cyclic pricing policy reducing to a static pricing policy by showing that the average profit per period within each mini cycle is no more than the expected profit from an optimal static pricing policy, i.e., $\max_{1\leq k\leq n}\left\{\sum_{i=k}^{n}\alpha_i v_k\right\}$. Consider the first mini cycle $[1, \tau_{k_c-1}]$. No customers arriving in these time periods will wait. Customers with valuation greater than or equal to v_{k_c} will purchase or leave immediately upon arrival, and thus the average profit per period is no more than $\max_{k_c \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, which is less than or equal to $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$. Next consider the last mini cycle $[\tau_{k_0}, T]$. Within this mini cycle, it is as if $\tau_{k_0} = 1$, and the cycle length is $T - \tau_{k_0} + 1$. Based on our preceding discussion, we know that the expected profit per period is bounded above by the expected profit from the optimal static pricing policy.

To that end, we only need to consider the mini cycles $[\tau_{j+1}, \tau_j - 1]$, for any $k_0 \le j \le k_c - 2$. We show below that the average profit per period within any mini cycle $[\tau_{j+1}, \tau_j - 1]$ is bounded above by:

$$
\begin{aligned}
&\sum_{t=\tau_{j+1}}^{\tau_j - 1} \pi_t / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{k_c - 1} \alpha_i p_T \\
&\le \sum_{t=\tau_{j+1}}^{\tau_j - 1} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t) c_{i_t}) / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{k_c - 1} \alpha_i p_T \\
&\le \sum_{t=\tau_{j+1}}^{\tau_j - 1} \sum_{i=i_t}^{n} \alpha_i \left[(T-t) \frac{c_{j+1}}{v_{j+1}} v_{i_t} \right] / (\tau_j - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i p_T \\
&\le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \frac{2T - \tau_j - \tau_{j+1} + 1}{2} + \sum_{i=j+1}^{n} \alpha_i p_T \\
&< \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \frac{2T - \tau_j - \tau_{j+1} + 1}{2} + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}) \\
&< \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}),
\end{aligned}
$$

where the second inequality is due to $i_t \ge k_c$ and $\frac{c_{i_t}}{v_{i_t}} \le \frac{c_{j+1}}{v_{j+1}}$, the third inequality is due to $i_t \ge k_c > j+1$ and $\sum_{i=i_t}^{n} \alpha_i v_{i_t} \le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}$, the fourth inequality is due to $v_{j+1} - (T - \tau_{j+1}) c_{j+1} \ge p_T$, and the last inequality is due to $\tau_{j+1} < \tau_j - 1$. Because $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} \ge \sum_{i=j+1}^{n} \alpha_i v_{j+1} \frac{c_{j+1}}{v_{j+1}} = \sum_{i=j+1}^{n} \alpha_i c_{j+1}$, we know that $\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1})$ is increasing in $T - \tau_{j+1}$. Combining with the condition that $T - \tau_{j+1} \le \frac{v_{j+1}}{c_{j+1}}$, we have

$$
\max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\} \frac{c_{j+1}}{v_{j+1}} (T - \tau_{j+1}) + \sum_{i=j+1}^{n} \alpha_i (v_{j+1} - (T - \tau_{j+1}) c_{j+1}) \le \max_{1 \le k \le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\}.
$$

Thus, we obtain Lemma OA.5(i).

Next we prove Lemma OA.5(ii). Consider first the case when $\tau_{k_0} = 1$. Following a similar approach as that of the proof of Lemma OA.5(i), we show that the expected profit from an optimal cyclic pricing policy is bounded above by:

$$
\begin{aligned}
\Pi(T) &\le \sum_{t=1}^{T} \pi_t/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/T + \sum_{i=k_0}^{k_c-1} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/T + \sum_{i=k_0}^{n} \alpha_i p_T \\
&\le \sum_{t=1}^{T} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/T + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \\
&\le 2\sum_{t=1}^{T} \sum_{i=k_0}^{n} \alpha_i ((T-t)c_{k_0})/T + \sum_{i=k_0}^{n} \alpha_i (v_{k_0} - (T-1)c_{k_0}) \\
&= \sum_{i=k_0}^{n} \alpha_i v_{k_0} \le \max_{1\le k\le n} \left\{ \sum_{i=k}^{n} \alpha_i v_k \right\},
\end{aligned}
$$

where the third inequality is due to $i_t \ge k_c$, the fourth inequality is due to $p_T \le v_{k_0} - (T-1)c_{k_0}$, and the fifth inequality is due to $c_j/c_i \le \sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k \le 2\sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k$, for any $1 \le i < j \le n$. Thus, an optimal cyclic pricing policy reduces to a static pricing policy in this case.

Next consider the case when $\tau_{k_0} > 1$. Let $l_1 = \underset{k_0<i<k_c}{\arg\min}\{\tau_i < \tau_{k_0}\}$, and we denote recursively $l_{m+1} = \underset{l_m<i<k_c}{\arg\min}\{\tau_i < \tau_{l_m}\}$. The largest properly defined m is denoted by M. As a result, any cycle can be decomposed into many separable mini cycles, namely, $[1, \tau_{l_M} - 1]$, $[\tau_{l_M}, \tau_{l_{M-1}} - 1]$, $\cdots$, $[\tau_{l_1}, \tau_{k_0} - 1]$, $[\tau_{k_0}, T]$. We prove an optimal cyclic pricing policy reducing to a static pricing policy by showing that the average profit per period within each mini cycle is no more than the expected profit per period from an optimal static pricing, namely, $\max_{1\le k\le n}\left\{\sum_{i=k}^{n} \alpha_i v_k\right\}$. Consider the first mini cycle $[1, \tau_{l_M} - 1]$. Because only customers with valuation greater than or equal to v_{k_c} will either purchase or leave immediately upon arrival, and no customers arriving in these periods will wait, the average profit per period is thus no more than $\max_{k_c\le k\le n}\left\{\sum_{i=k}^{n} \alpha_i v_k\right\}$, which is no more than $\max_{1\le k\le n}\left\{\sum_{i=k}^{n} \alpha_i v_k\right\}$. Next consider the last mini cycle $[\tau_{k_0}, T]$. This mini cycle effectively reduces to the case as if $\tau_{k_0} = 1$ with a cycle length of $T - \tau_{k_0} + 1$, and thus we know that the expected profit per period is bounded above by the expected profit from an optimal static pricing policy based on the preceding discussions.

Thus, we just need to consider $[\tau_{l_{m+1}}, \tau_{l_m} - 1]$ for any $k_0 < l_{m+1} < l_M$. The average profit per period within this mini cycle is bounded above by:

$$
\begin{aligned}
&\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \pi_t/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{k_c-1} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i (p_T + (T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{k_c-1} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i p_T \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} \sum_{i=i_t}^{n} \alpha_i ((T-t)c_{i_t})/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T-\tau_{l_{m+1}})c_{l_{m+1}}) \\
\leq &\sum_{t=\tau_{l_{m+1}}}^{\tau_{l_m}-1} (T-t) \sum_{i=l_{m+1}}^{n} \alpha_i c_{l_{m+1}}/(\tau_{l_m} - \tau_{l_{m+1}}) + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T-\tau_{l_{m+1}})c_{l_{m+1}}) \\
= &\frac{(2T - \tau_{l_{m+1}} - \tau_{l_m} + 1)}{2} \sum_{i=l_{m+1}}^{n} \alpha_i c_{l_{m+1}} + \sum_{i=l_{m+1}}^{n} \alpha_i (v_{l_{m+1}} - (T-\tau_{l_{m+1}})c_{l_{m+1}}) < \sum_{i=l_{m+1}}^{n} \alpha_i v_{l_{m+1}},
\end{aligned}
$$

where the second inequality is due to $i_t \geq k_c$, the third inequality is due to $p_T \leq v_{l_{m+1}} - (T - \tau_{l_{m+1}})c_{l_{m+1}}$, the fourth inequality is due to $c_j/c_i \leq \sum_{k=i}^{n} \alpha_k \Big/ \sum_{k=j}^{n} \alpha_k$, for any $1 \leq i < j \leq n$, and the last inequality is due to $\tau_{l_{m+1}} < \tau_{l_m} - 1$. Thus, we obtain the announced result. $\square$