Online Appendix to "Intertemporal Price Discrimination via Randomized Promotions": The Case of Multiple Customer Segments

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A. Randomized Pricing

The following notation is used throughout proofs for randomized pricing policies with n customer segments.

(i) $u_i = \underline{p}^i$, $\forall i \in \mathcal{W}$, and $u_i = v_i$, $\forall i \notin \mathcal{W}$, where $\underline{p}^i$ is defined in problem (6);

(ii) $\beta_1 = F(u_1)$, and $\beta_i = F(u_i) - F(u_{i-1})$, $\forall i > 1$;

(iii) $x_1 = \mathsf{E}[P|P \le u_1]$, and $x_i = \mathsf{E}[P|u_{i-1} < P \le u_i]$, $\forall i > 1$.

With this set of notation, problem (6) can be reformulated as follows, conditional on the set of customers who will wait.

Lemma OA.1. *Consider a set of randomized pricing policies, where the set of customers that will wait under any policy in the set remains the same, which is denoted by $\mathcal{W}$, and the rest of the customers will either purchase or leave immediately upon arrival. The optimal policy in this set can be derived by solving the following optimization problem.*

$$\begin{aligned}
\max_{\beta,\mathbf{x}} \quad & \sum_{i=1}^{n} \alpha_i \pi_i \\
s.t. \quad & \beta_1 v_1 - \beta_1 x_1 > c_1, \quad \text{if } 1 \in \mathcal{W}, \\
& \sum_{j=1}^{i} \beta_j x_i - \sum_{j=1}^{i} \beta_j x_j = c_i, \quad \forall i \in \mathcal{W} \setminus \{1\}, \\
& \sum_{j=1}^{i} \beta_j v_i - \sum_{j=1}^{i} \beta_j x_j \le c_i, \quad \forall i \notin \mathcal{W}, \\
& x_i < v_i, \quad \forall i \in \mathcal{W}, \\
& x_i \le v_i, \quad \forall i \notin \mathcal{W}, \\
& \sum_{i=1}^{n} \beta_i = 1,
\end{aligned} \tag{OA.1}$$

where $\pi_i = \sum_{j=1}^{i} \beta_j x_j \Big/ \sum_{j=1}^{i} \beta_j$, $\forall i \in \mathcal{W}$; *Otherwise,* $\pi_i = \sum_{j=1}^{i} \beta_j x_j$.

Lemma OA.2. *There exists a threshold on c_1, below which an optimal randomized pricing policy outperforms an optimal static pricing policy.*[OA.1]

Lemma OA.2 shows that when customers of the lowest valuation are sufficiently patient, an optimal randomized pricing policy will not reduce to a static pricing policy. The rationale is as follows. Recall that an optimal static pricing policy is to set the price at v_{i^*} such that $i^* = \arg\max_i \sum_{j=i}^{n} \alpha_j v_i$. So one simple way to construct a randomized pricing strategy is to charge v_{i^*} most of the time but run random promotions with a discount price lower than v_1 once in a while to extract surplus from customers of the lowest valuation. Because customers of valuation v_1 are sufficiently patient, it can be guaranteed that extra surplus from type-1 customers outweighs the profit loss from customers of higher valuations due to type-1 customers' purchases during a promotion that can be run sufficiently infrequently. Thus, a randomized pricing policy constructed as mentioned above outweighs an optimal static pricing policy, and thus so does an optimal randomized pricing policy.

B. Deterministic Cyclic Pricing

Proposition 5 ensures that we do not need to consider beyond cyclic pricing for optimal deterministic pricing. As shown in the proof of Proposition 5, the optimal customers' response under any cyclic pricing policy can be summarized as follows.

Corollary OA.1. (Optimal Customer Response under Cyclic Pricing) *The optimal customer response can be characterized as follows: there exists $k_c \in \{1, 2, \ldots, n\}$ and $k_0 \in \{1, 2, \ldots, k_c - 1\}$ such that,*

(i) any customer of valuation greater than or equal to v_{k_c} will either buy or leave immediately upon arrival;

(ii) any customer of valuation less than v_{k_0} will leave immediately upon arrival;

(iii) there exists τ_i for any type-i customer, $k_0 \le i < k_c$, such that she leaves immediately upon arrival if she arrives before the τ_i^{th} period within a cycle; Otherwise, she will wait to buy at the end of a cycle.

[OA.1] It boils down to Proposition 4 if $n = 2$.

Corollary OA.1(i) shows that there exists a cutoff on customer valuation, denoted by v_{k_c}, such that any customer with a valuation greater than or equal to v_{k_c} will never wait. This is a consequence of Assumption (O), which assumes that per-period waiting costs follow an ascending order. Thus, with the benefit of waiting the same for all types of customers (because they see the same deterministic price path), it is more costly for customers of a higher valuation to wait. In the other extreme, Corollary OA.1(ii) says that there may exist a segment of customers of low valuations who will leave immediately upon arrival. This segment includes those customers whose valuation is less than p_t, $\forall t$. Then, we also have a segment of customers of intermediate valuations who will wait if and only if she arrives sufficiently close to the end of a cycle. Otherwise, her utility from waiting becomes negative due to the cost of waiting, and thus she is better off leaving immediately upon arrival.

Given customers' optimal response as characterized in Corollary OA.1, the firm's long-run average expected profit with a cyclic pricing policy $\mathbf{p}$ is given by $\Pi(\mathbf{p}) = \lim_{T'\to\infty} \frac{1}{T'} \sum_{t=1}^{T'} \pi_t(\mathbf{p})$, where $\pi_t(\mathbf{p})$ is the profit from customers arriving in period t. We have $\pi_t(\mathbf{p}) = \pi_{t+T}(\mathbf{p})$ due to prices being cyclic. Our goal is to find a cyclic pricing policy $\mathbf{p}$ such that $\Pi(\mathbf{p})$ is maximized.

Next, we prove some key structural properties that an optimal cyclic pricing policy must satisfy. We derive these based on the following partial characterization of an optimal cyclic pricing policy.

Lemma OA.3. (Optimal Cyclic Pricing conditional on Cutoffs) *The optimal pricing schedule is in the following form, with k_c and k_0 defined in Corollary OA.1:*

(i) At the Tth period, $p_T = \min_{k_0 \le i < k_c} \{v_i - (T-\tau_i)c_i\} - \delta$, where $\delta > 0$ is sufficiently small;

(ii) At the tth period, $t = 1, \ldots, T-1$, we have

(a) *if $p_{t+1} + c_{k_c} \le v_{k_c}$, $p_t = p_{t+1} + c_{k_c}$;*

(b) *if $p_{t+1} + c_{k_c} > v_{k_c}$,*

$$p_t = \begin{cases} p_{t+1} + c_{k_t}, & \text{if } (\alpha_{k_t} + \cdots + \alpha_n)(p_{t+1} + c_{k_t}) > (\alpha_{l_t} + \cdots + \alpha_n) v_{l_t}, \\ v_{l_t}, & \text{if } (\alpha_{k_t} + \cdots + \alpha_n)(p_{t+1} + c_{k_t}) \le (\alpha_{l_t} + \cdots + \alpha_n) v_{l_t}, \\ v_l, & \text{if } k_t \text{ does not exist,} \end{cases}$$

where $k_t = \arg\min_{k_c \le i} \{p_{t+1} + c_i \le v_i\}$, $l_t = \arg\max_{k_c \le i < k_t} \{(\alpha_i + \cdots + \alpha_n) v_i\}$ and $l = \arg\max_{k_c \le i} \{(\alpha_i + \cdots + \alpha_n) v_i\}$.

Lemma OA.3 sheds light on the structure of an optimal cyclic pricing policy. The price at period T is simply the highest price under which customers' purchasing decisions are consistent with that characterized in Corollary OA.1. Then conditional on p_T, we can construct the optimal pricing schedule recursively backward in time. At time period $t < T$, p_t cannot be greater than $p_{t+1} + c_{k_t}$, where $k_t = \arg\min_{k_c \leq i}\{p_{t+1} + c_i \leq v_i\}$; otherwise, a customer of valuation v_{k_t} is better off waiting to buy in future periods, which contradicts to the optimal customer response as characterized in Corollary OA.1. Therefore, subject to this constraint, the optimal price at any time period $t < T$ is the one that maximizes the firm's profit in that focal period.

With Lemma OA.3 in hand, we are able to characterize some structural properties of an optimal cyclic pricing policy. First, we show that an optimal cyclic pricing policy follows a markdown pattern, which is summarized formally in the lemma below.

Lemma OA.4. (Markdown Pattern) *An optimal cyclic pricing policy follows a (weakly) markdown pattern.*

This result is not surprising in the sense that, given that customers of higher valuations would either buy or leave immediately upon arrival, and customers of lower valuations would wait, the firm has an incentive to charge higher prices early on in a cycle to extract as much surplus from customers of higher valuations as possible, and then clear the market of customers of lower valuations when it approaches the end of a cycle. Having said that, this strategy is only viable when it is incentive compatible for customers of higher valuations to buy immediately, which implies that the price reduction in any two consecutive periods shall be less than or equal to the waiting cost of those customers of higher valuations.

Lemma OA.3 also implies that, under any optimal cyclic pricing policy, the firm will not charge a price higher than v_n, the highest valuation from customers. With this, we are able to identify two sufficient conditions under which an optimal cyclic pricing policy reduces to a static pricing policy.

Lemma OA.5. (Degeneration of Optimal Cyclic Pricing to Static Pricing) *An optimal cyclic pricing policy reduces to a static pricing policy if either of the following conditions is satisfied:*

(i) $v_1/c_1 \le v_2/c_2 \le \cdots \le v_n/c_n$;

(ii) $c_{i+1}/c_i \le \sum_{j=i}^{n} \alpha_j \Big/ \sum_{j=i+1}^{n} \alpha_j$, $\forall\, i < n$.

In particular, under condition (i), we have $\tau_{i+1} \le \tau_i$*, for any* $k_0 \le i < k_c - 1$*, where* τ_i *is defined in Corollary OA.1.*

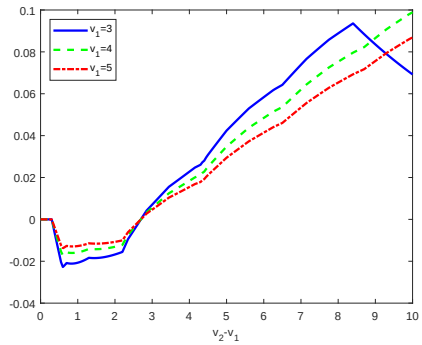
As the counterpart of Lemma 9, Lemma OA.5(i) illustrates the impact of customers' patience levels on the structure of the optimal cyclic pricing policy. When customers of a higher valuation are more patient than customers of a lower valuation, i.e., $v_i/c_i \le v_j/c_j$, $\forall\, i \le j$, cyclic pricing policies cannot outperform the optimal static pricing policy. In particular, under the condition in Lemma OA.5(i), among customers with valuation v_i, $k_0 \le i < k_c - 1$ who might wait by Corollary OA.1, customers of a higher valuation are more patient than customers of a lower valuation in the sense that they will start to wait earlier within a cycle, i.e., $\tau_{i+1} \le \tau_i$, for any $k_0 \le i < k_c - 1$. This contributes negatively to the firm's profit due to: (i) the firm is not able to effectively price discriminate among customers who will wait as customers of a higher valuation can effectively mimic the behavior of customers of a lower valuation; (2) the firm needs to charge a lower price at the end of a cycle to compensate for the waiting of customers of a higher valuation, who also have a higher per-period waiting cost.

An alternative way to understand Lemma OA.5(i) is as follows. We can rewrite the condition in Lemma OA.5(i) as $c_{i+1}/c_i \le v_{i+1}/v_i$, $\forall\, i < n$. That is, if the ratio of per-period waiting costs of two customer segments is less than the ratio of their valuations, i.e., the discrepancy of per-period waiting costs of any two neighboring customer types is sufficiently small, any cyclic pricing policy cannot effectively price discriminate among the two customers. Lemma OA.5(ii) shows an alternative threshold for the ratio of per-period waiting costs, below which any cyclic pricing policy cannot outperform an optimal static pricing policy.

C. Performance Comparison: Numerical Results

Analytically, we are able to characterize the optimal randomized pricing policy under certain conditions with multiple customer segments, as shown in Section 4.3. In this section, we seek to complement our theoretical analysis with comprehensive numerical studies.

Figure OA.1 Relative Profit Gap between Optimal Randomized Pricing and Optimal Cyclic Pricing

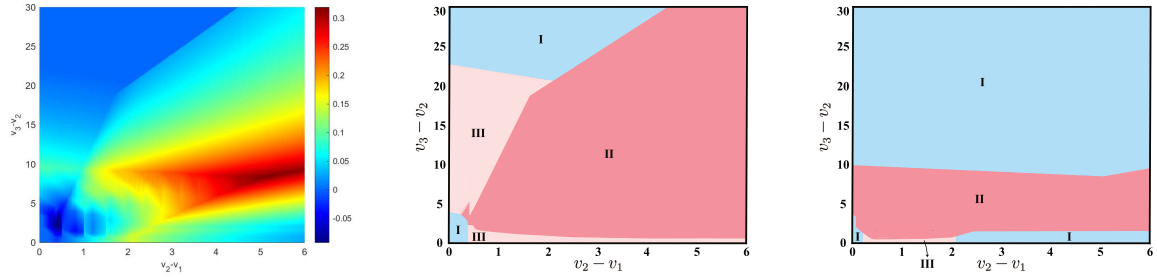


Note. Consider the case with two customer segments. Half of them value the product at v_1 with a per-period waiting cost of $c_1 = 0.1$, and the other half value the product at v_2 with a per-period waiting cost of $c_2 = 2.5$.

Our first study focuses on the case when there are two segments of customers. The parameters of the study are specified as follows: one-half of customers value the product at v_1 with a per-period waiting cost of $c_1 = 0.1$, and the other half of customers value the product at v_2 with a per-period waiting cost of $c_2 = 2.5$. We consider 3 scenarios, where v_1 takes a value of 3, 4 and 5, respectively. In each scenario, $v_2 - v_1$ ranges from 0 to 10 with an incremental value of 0.05. Figure OA.1 summarizes the relative profit differences between an optimal randomized pricing policy and an optimal cyclic pricing policy. When the difference between the valuations is marginal, neither policy can effectively price discriminate the two customer segments, and thus both of them reduce to the optimal static pricing policy, leading to a zero performance gap. As $v_2 - v_1$ increases, an optimal cyclic pricing policy first outperforms an optimal randomized pricing policy, but this relationship is reversed when $v_2 - v_1$ becomes sufficiently large. This result corroborates our theoretical finding in Proposition 13. In particular, the turning points are identical across the three scenarios, which suggests that whether randomized pricing can outperform cyclic pricing solely depends on the difference in valuations rather than their absolute values. This result has empirical implications, as in many circumstances, it is easier to estimate the valuation difference than the valuation itself.

Next, we move on to the case with multiple customer segments. In particular, we will focus on the case when there are three segments of customers, which proves to be a sufficiently sophisticated context that allows us to investigate the nuances properly. The setup of the study is as follows: 1/3 of customers have a low valuation of $v_1 = 3$ with a per-period waiting cost of $c_1 = 0.1$, 1/3 of

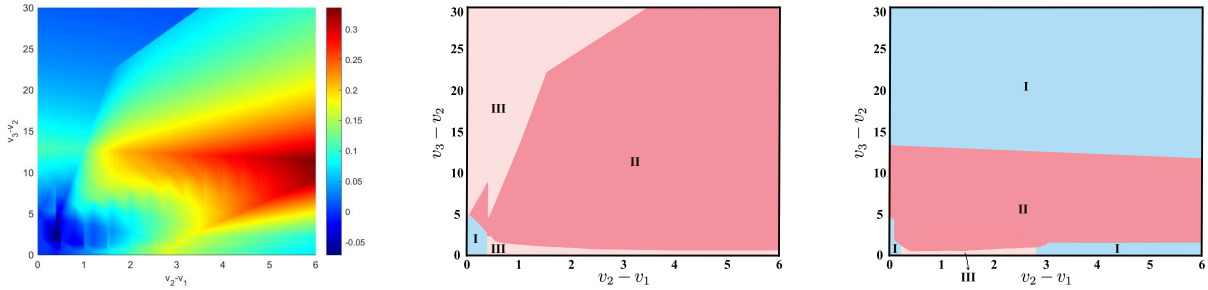
Figure OA.2 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 3$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. In Figures OA.2(b) and OA.2(c), in those regions of type (I), the corresponding pricing policy reduces to a static pricing policy; in those regions of type (II), customers of valuations v_1 and v_2 would wait, and customers of valuation v_3 would buy immediately upon arrival under the corresponding pricing policy; in those regions of type (III), customers of valuation v_1 would wait, customers of valuation v_2 would either purchase or leave immediately upon arrival, and customers of valuation v_3 would buy immediately upon arrival under the corresponding pricing policy.

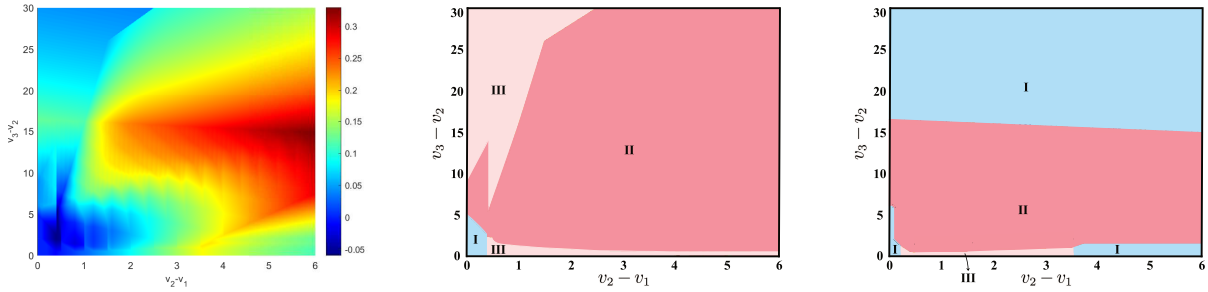
Figure OA.3 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 4$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. Parameters are the same as those in Figure OA.2, except for $v_1 = 4$.

Figure OA.4 Optimal Randomized Pricing vs. Optimal Cyclic Pricing when $n = 3$ and $v_1 = 5$



(a) Relative Profit Gap (b) Optimal Randomized Pricing (c) Optimal Cyclic Pricing

Note. Parameters are the same as those in Figure OA.2, except for $v_1 = 5$.

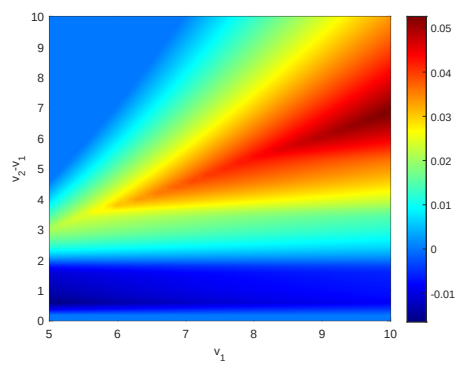
customers have a medium valuation of v_2 with a per-period waiting cost of $c_2 = 0.5$, and the rest 1/3 of customers have a high valuation of v_3 with a per-period waiting cost of $c_3 = 2.5$. In this study, $v_2 - v_1$ ranges from 0 to 6, and $v_3 - v_2$ ranges from 0 to 30. Figure OA.2(a) summarizes the relative profit gap between optimal randomized pricing policies and optimal cyclic pricing policies. To facilitate the understanding of Figure OA.2(a), we also plot the customers' responses under the optimal randomized pricing and optimal cyclic pricing policies in Figures OA.2(b) and OA.2(c), respectively. We test the robustness of these observations with other parameters, and observe similar patterns, see Figures OA.3 and OA.4.

We make some observations from Figure OA.2. First, when the differences of valuations (i.e., both $v_2 - v_1$ and $v_3 - v_2$) are sufficiently small, neither randomized pricing nor cyclic pricing can effectively price discriminate customers, and thus both of them reduce to an optimal static pricing policy, leading to zero performance gap. As $v_2 - v_1$ or $v_3 - v_2$ increases, cyclic pricing first becomes effective in price-discriminating customers whereas optimal randomized pricing still reduces to optimal static pricing, as suggested by the smaller region of type (I) in the bottom left corner of Figure OA.2(c) than that in OA.2(b). However, as $v_2 - v_1$ and $v_3 - v_2$ continue to increase, this relationship is reversed where optimal randomized pricing consistently outperforms optimal cyclic pricing. This observation is consistent with our analytical result in Proposition 13, even though the condition $v_1 - c_1 \geq v_2 - c_2 \geq v_3 - c_3$ is not satisfied in the majority of the parameter space in this study.

Second, the relative performance gap between optimal randomized pricing and cyclic pricing is nonlinear, which reaches its maximum when both $v_2 - v_1$ and $v_3 - v_2$ are of medium values. The rationale is as follows. When both $v_2 - v_1$ and $v_3 - v_2$ are relatively small, performance gaps are small, with cyclic pricing dominating randomized pricing under certain circumstances, as discussed above. As $v_2 - v_1$ and/or $v_3 - v_2$ increase, the effectiveness of randomized pricing becomes more salient, as it is able to extract most of the surpluses from customers of higher valuations while keeping customers of lower valuations waiting in the system, whereas optimal cyclic pricing policies quickly reduce to optimal static pricing policies, as suggested by the large area of type (I) in Figure OA.2(c). This is because when the differences in customers' valuations are large, an optimal cyclic

pricing policy needs to follow a sequence of markdowns to prevent the high-valuation customers from waiting, which leaves too much surplus to those customers, rendering its degeneration to a static pricing policy. On the other end of the spectrum, when the differences in valuations are extremely large, especially when the highest valuation is significantly higher than the rest, capturing the surplus from customers of the highest valuation becomes the utmost priority, and thus an optimal static pricing policy becomes more favorable, compared to those price discrimination mechanisms. Thus, the performance gap becomes smaller as a result. In fact, when v_3 dominates, no price discrimination policy would be effective, and thus both randomized pricing and cyclic pricing policies reduce to an optimal static pricing policy. This is reflected by the area of type (I) in the top left corner in both Figures OA.2(b) and OA.2(c).

Figure OA.5 Relative Profit Gap between Optimal Randomized Pricing and Optimal Cyclic Pricing



Note. Parameters of this study are consistent with estimates in Moon et al. (2017). In particular, we consider two customer segments with $\alpha_1 = 0.18$, $\alpha_2 = 0.82$, $c_1 = 0.2$, and $c_2 = 2.5$.

To validate the generality of our results, we run another numerical study by adopting parameters from the empirical context of Moon et al. (2017). We consider the case with two customer segments, where 18% of them have a low valuation of v_1 with a per-period waiting cost of $c_1 = 0.2$ and 82% of customers have a high valuation of v_2 with a per-period waiting cost of $c_2 = 2.5$.[OA.2] In this study, v_1 ranges from 5 to 10 (i.e., equivalent to \$50 to \$100 in Moon et al. 2017), and $v_2 - v_1$ varies from 0

[OA.2] Under a stationary randomized pricing policy, the monitoring cost estimated in Moon et al. (2017) is equivalent to customers' per-period waiting cost in our setting, as both measure the cost for customers to sample another price point. We also scale the estimated valuations and costs in Moon et al. (2017) down by a factor of 10 to make results comparable to the rest of our numerical analysis.

to 10. Figure OA.5 summarizes the relative profit gap between optimal randomized pricing policies and optimal cyclic pricing policies. Consistent with our findings in Figure OA.1, we observe that an optimal cyclic pricing policy outperforms an optimal randomized pricing policy when $v_2 - v_1$ is relatively small, whereas this dominance is reversed when $v_2 - v_1$ becomes sufficiently large. Moreover, we also see the benefit of randomized pricing generally reaches its maximum when the differences in customer valuations, $v_2 - v_1$, are of medium values.