Online Supplemental Material for "Privacy Management in Service Systems" by Hu, Momot, Wang

Appendix A: Proofs of Technical Lemmas

A.1. Proofs of Lemma 2 and Lemma 3 (Page 2)

$\Xi(\gamma,\lambda)$ is linear in γ. After simplification:

$$\Xi(0,\lambda)=\sigma\left(\alpha^2\sigma^2-(\alpha-1)^2-\alpha(1-\alpha)(\sigma-1)\right)\mu-\lambda\alpha\left(\alpha\sigma^3+(\alpha-1)^2\sigma^2-(\alpha-1)(2\alpha-1)\sigma+(\alpha-1)^2\right).$$

The cubic function $\alpha\sigma^3+(\alpha-1)^2\sigma^2-(\alpha-1)(2\alpha-1)\sigma+(\alpha-1)^2$ is positive for $\alpha\in[0,1]$, because its discriminant $(\alpha-1)^3(\alpha+3)$ is negative for $\alpha\in[0,1]$, which means that it has only one negative real root. Thus, we have $\Xi(0,\lambda)\geq 0\Leftrightarrow\lambda\leq G_0(\alpha)$ where $G_0(\alpha)\equiv\frac{\mu\sigma\left(\alpha^2\sigma^2-(\alpha-1)^2-\alpha(1-\alpha)(\sigma-1)\right)}{\alpha\left(\alpha\sigma^3+(\alpha-1)^2\sigma^2-(\alpha-1)(2\alpha-1)\sigma+(\alpha-1)^2\right)}$. As part of the numerator of $G_0(\alpha)$, $\alpha^2\sigma^2-(\alpha-1)^2-\alpha(1-\alpha)(\sigma-1)$, a quadratic polynomial of α, has only one root $\alpha^*=\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$ in $(0,1)$; and $\alpha^2\sigma^2-(\alpha-1)^2-\alpha(1-\alpha)(\sigma-1)\leq 0\Leftrightarrow\alpha\leq\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$. From the fact that $\alpha^2\sigma^2-(\alpha-1)^2-\alpha(1-\alpha)(\sigma-1)\,|_{\alpha=1/1+\sigma}=\frac{\sigma(1-\sigma)}{(\sigma+1)^2}$, which is positive if $\sigma<1$ and negative if $\sigma>1$, we have

$$\begin{cases}\frac{1}{\sigma+1}>\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)} & \text{if } \sigma<1\\ \frac{1}{\sigma+1}<\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)} & \text{if } \sigma>1\end{cases}\tag{16}$$

From simple algebra, we have:

$$\Xi(1,\lambda)=\sigma\left(\left(\alpha^2\sigma^2-(\alpha-1)^2\right)\mu-\lambda\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)\right).$$

<u>**Case $\sigma>1$:**</u> we have $\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2>0$ for $\forall\alpha\in[0,1]$, from the fact that (i) $\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2$ increases in σ because $\partial\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)/\partial\sigma=(2\sigma-1/4)\alpha^2+(\alpha-2)^2/4>0$, and (ii) $\lim_{\sigma\searrow 1}\left(\alpha^2\sigma^2-\alpha\sigma+\sigma-(\alpha-1)^2\right)=\alpha\geq 0$. Then, we have $\Xi(1,\lambda)\geq 0\Leftrightarrow\lambda\leq G_1(\alpha)\equiv\frac{\mu\left(\alpha^2\sigma^2-(\alpha-1)^2\right)}{\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)}$. One can show that $\alpha^2\sigma^2-(\alpha-1)^2>0\Leftrightarrow\alpha>\frac{1}{\sigma+1}$. Thus, we have $G_1(\alpha)\leq 0\Leftrightarrow\alpha\leq\frac{1}{\sigma+1}$.

<u>**Case $\sigma<1$:**</u> we have $\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2$, as a quadratic function of α, has one root $\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}$ in $[0,1]$, value $\sigma-1<0$ at $\alpha=0$, and value $\sigma^2>0$ at $\alpha=1$. Hence, we have $\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2>0\Leftrightarrow\alpha>\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}$. One can show that $\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}<\frac{1}{\sigma+1}$ if $\sigma<1$, and $\alpha^2\sigma^2-(\alpha-1)^2>0\Leftrightarrow\alpha>\frac{1}{\sigma+1}$. To summarize, we have:

$$\begin{cases}\alpha^2\sigma^2-(\alpha-1)^2<0 \text{ and } \alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2<0 \text{ if } \alpha<\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}<\frac{1}{\sigma+1}\\ \alpha^2\sigma^2-(\alpha-1)^2<0 \text{ and } \alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2>0 \text{ if } \frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}<\alpha\leq\frac{1}{\sigma+1}\\ \alpha^2\sigma^2-(\alpha-1)^2>0 \text{ and } \alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2>0 \text{ if } \alpha>\frac{1}{\sigma+1}\end{cases}.$$

Next, we discuss the value of $\Xi(1,\lambda)$ by conditioning on α.

- If $\alpha\leq\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}<\frac{1}{\sigma+1}$, we have $\Xi(1,\lambda)<0\Leftrightarrow\lambda<G_1(\alpha)\equiv\frac{\mu\left(\alpha^2\sigma^2-(\alpha-1)^2\right)}{\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)}$ and $G_1(\alpha)>0$. Recall that the arrival rate is upper bounded, i.e., $\lambda<\bar{\mu}$. We have $G_1(\alpha)$ is above $\bar{\mu}$, because $G_1(\alpha)-\frac{\mu\sigma}{1+\alpha(\sigma-1)}=\frac{\mu(\alpha-1)^2\left(1-\alpha\left(1-\sigma^2\right)\right)}{-\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)(1+\alpha(\sigma-1))}\geq 0$ for $\alpha\leq\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}$. Thus, we have $\Xi(1,\lambda)<0$ for $\forall\lambda$.
- If $\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}<\alpha\leq\frac{1}{\sigma+1}$, we have $\Xi(1,\lambda)<0\Leftrightarrow\lambda>G_1(\alpha)$ and $G_1(\alpha)\leq 0$. We have $\Xi(1,\lambda)<0$ for $\forall\lambda$, because $\lambda>0$.
- If $\alpha>\frac{1}{\sigma+1}$, we have $\Xi(1,\lambda)<0\Leftrightarrow\lambda>G_1(\alpha)$ and $G_1(\alpha)>0$.

Next, we investigate

$$G_0(\alpha)-G_1(\alpha)=\frac{\mu\left(\alpha\sigma^2+1-\alpha\right)(1-\alpha)\cdot\left((1-\alpha)^2-\alpha^2\sigma^3\right)}{\alpha\left(\alpha\sigma^3+(\alpha-1)^2\sigma^2-(\alpha-1)(2\alpha-1)\sigma+(\alpha-1)^2\right)\left(\alpha^2\sigma^2-\alpha\sigma+\sigma-(\alpha-1)^2\right)},$$

multiplier of $(1-\alpha)^2-\alpha^2\sigma^3$ is clearly non-negative for $\alpha\in\left[\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)},1\right]$ from our above results. Thus, we have $G_0(\alpha)-G_1(\alpha)\le 0\Leftrightarrow(1-\alpha)^2-\alpha^2\sigma^3\le 0\Leftrightarrow\alpha\ge\frac{1}{\sigma^{3/2}+1}$ and $G_0(\alpha)-G_1(\alpha)>0\Leftrightarrow\frac{2-\sigma-\sigma\sqrt{5-4\sigma}}{2(1-\sigma^2)}\le\alpha<\frac{1}{\sigma^{3/2}+1}$.

At last, one can verify that:

$$\begin{cases}\frac{1}{\sigma^{3/2}+1}>\frac{1}{\sigma+1} & \text{if } \sigma<1\\ \frac{1}{\sigma^{3/2}+1}<\frac{1}{\sigma+1} & \text{if } \sigma>1\end{cases}\quad\text{and}\quad\begin{cases}\frac{1}{\sigma^{3/2}+1}>\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)} & \text{if } \sigma<1\\ \frac{1}{\sigma^{3/2}+1}<\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)} & \text{if } \sigma>1\end{cases}. \tag{17}$$

The last result can be derived by combining (16) and (17).

A.2. Proof of Lemma 4 (Page 3)

Using U_d and U_w from (2), we derive expression for $\text{CS}=\lambda\left(\gamma U_d(\gamma,\lambda)+(1-\gamma)U_w(\gamma,\lambda)\right)$:

$$\text{CS}=\lambda R-\lambda c\left(\frac{\sigma(\sigma-1)\mu+\lambda\left(\alpha\sigma^3-\alpha\sigma^2+\alpha\sigma+1-\alpha\right)}{\sigma\lambda\left(\sigma\mu-\lambda(\alpha\sigma+1-\alpha)\right)}+\frac{(1-\sigma)\left(\mu-\lambda\alpha(1-\sigma)\right)\left(1-\alpha(1+\sigma)\right)^2}{\sigma\lambda\left((\mu-\alpha\lambda)\left(\alpha^2\sigma^2+(\alpha-1)^2\right)-\alpha(1-\alpha)\Gamma(\gamma)\right)}\right),$$

where $\Gamma(\gamma)\equiv\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)+\frac{\sigma^2(\mu-\alpha\lambda)^2}{\gamma\lambda(1-\alpha-\alpha\sigma)+(\sigma\mu-(1-\alpha)\lambda)}$. We next prove that $\Gamma(\gamma)$ decreases in γ by showing that its first derivative

$$\frac{\mathrm{d}\Gamma(\gamma)}{\mathrm{d}\gamma}=\lambda\left(1-\alpha(1+\sigma)\right)\left(1-\frac{\sigma^2(\mu-\alpha\lambda)^2}{\left(\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)\right)^2}\right)<0.$$

<u>Case</u> $\alpha<\frac{1}{1+\sigma}$**:** we have $1-\alpha(1+\sigma)>0$. From $\lambda<\bar{\mu}=\frac{\sigma\mu}{1+\alpha(\sigma-1)}$, we have $\left(\sigma\mu-(1-\alpha)\lambda\right)>\sigma\alpha\lambda>0$. Thus, $\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)>0$. Then, we have:

$$\sigma(\mu-\alpha\lambda)-\left(\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)\right)=\lambda(1-\gamma)\left(1-\alpha(1+\sigma)\right)>0$$

Hence, $1-\frac{\sigma^2(\mu-\alpha\lambda)^2}{(\gamma\lambda(1-\alpha(1+\sigma))+(\sigma\mu-(1-\alpha)\lambda))^2}<0\Rightarrow\Gamma'(\gamma)<0$.

<u>Case</u> $\alpha>\frac{1}{1+\sigma}$**:** we have $1-\alpha(1+\sigma)<0$. First, if $\gamma<\frac{\sigma\mu-(1-\alpha)\lambda}{\lambda(\alpha(1+\sigma)-1)}$, we have $\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)>0$. Then, we have:

$$\sigma(\mu-\alpha\lambda)-\left(\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)\right)=\lambda(1-\gamma)\left(1-\alpha(1+\sigma)\right)<0$$

Hence, $1-\frac{\sigma^2(\mu-\alpha\lambda)^2}{(\gamma\lambda(1-\alpha(1+\sigma))+(\sigma\mu-(1-\alpha)\lambda))^2}>0\Rightarrow\Gamma'(\gamma)<0$. Next, if $\gamma>\frac{\sigma\mu-(1-\alpha)\lambda}{\lambda(\alpha(1+\sigma)-1)}$ then $\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)<0$ and:

$$\sigma(\mu-\alpha\lambda)-\left(-\gamma\lambda\left(1-\alpha(1+\sigma)\right)-\left(\sigma\mu-(1-\alpha)\lambda\right)\right)=$$

$$(1-\gamma)\left(1-\alpha(1+\sigma)\right)+2\left(\gamma\lambda\left(1-\alpha(1+\sigma)\right)+\left(\sigma\mu-(1-\alpha)\lambda\right)\right)<0$$

Hence, $1-\frac{\sigma^2(\mu-\alpha\lambda)^2}{(\gamma\lambda(1-\alpha(1+\sigma))+(\sigma\mu-(1-\alpha)\lambda))^2}>0$ and $\Gamma'(\gamma)<0$.

In the above cases, we have $\Gamma'(\gamma)<0$. Thus, $\Gamma(\gamma)$ decreases in γ. Furthermore, we have

$$(\mu-\alpha\lambda)\left(\alpha^2\sigma^2+(\alpha-1)^2\right)-\alpha(1-\alpha)\Gamma(0)=\mu\left(1-\alpha(1+\sigma)\right)^2\frac{\sigma\mu-(1-\alpha)\lambda-\sigma\alpha\lambda}{\sigma\mu-(1-\alpha)\lambda}>0,$$

and

$$(\mu-\alpha\lambda)\left(\alpha^2\sigma^2+(\alpha-1)^2\right)-\alpha(1-\alpha)\Gamma(1)=(\mu-\alpha\lambda)\left(1-\alpha(1+\sigma)\right)^2>0,$$

so $(\mu-\alpha\lambda)\left(\alpha^2\sigma^2+(\alpha-1)^2\right)-\alpha(1-\alpha)\Gamma(\gamma)>0$. Then, we have $\text{CS}(\gamma)$ increases in γ when $\sigma<1$ and decreases otherwise.

A.3. Proof of Lemma 5 (Page 5)

We consider two cases:

Case $\sigma < 1$:analyzing expression of $P_s^{\sigma<1}$, we observe $(\alpha-1)^2 - \alpha^2\sigma^2 > 0 \Leftrightarrow \alpha < \frac{1}{\sigma+1}$, $\left(\alpha^2\sigma^2 - \sigma(\alpha-1) - (\alpha-1)^2\right) > 0 \Leftrightarrow \alpha > \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$, and $\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)} < \frac{1}{\sigma+1}$. When $\alpha \geq \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$ and $P_s > 0$, P_s is clearly an increasing function of λ. We next consider the case $\alpha < \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$. We derive $\frac{\partial P_s}{\partial\lambda} = \frac{c\lambda\mu}{\sigma(\mu-\alpha\lambda)^3(\sigma\mu-(1+\alpha(\sigma-1))\lambda)^2}\Xi_{P_s}(\lambda)$, where

$$\Xi_{P_s}(\lambda) = \alpha\left((2\sigma-1)\left(1-\sigma^2\right)\alpha^3 + \left(2\sigma^2-6\sigma+3\right)\alpha^2 - 3(\sigma-1)^2\alpha + (1-2\sigma)\right)\lambda^2 + \mu\left(4\alpha^3\sigma^3 - \alpha\left(\alpha^2+2\alpha-3\right)\sigma^2 - 4\alpha(\alpha-1)^2\sigma + (\alpha-1)^3\right)\lambda + 2\sigma\mu^2\left((\alpha-1)^2 - \alpha^2\sigma^2\right).$$

We have $\Xi_{P_s}(0) = 2\sigma\mu^2\left((\alpha-1)^2 - \alpha^2\sigma^2\right) > 0$, because $\alpha < \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)} \Rightarrow (\alpha-1)^2 - \alpha^2\sigma^2 > 0$, and $\Xi_{P_s}(\bar{\mu}) = \frac{\sigma\mu^2(1-\alpha)^3\left(\alpha\sigma^2+1-\alpha\right)}{(\alpha\sigma-\alpha+1)^2} > 0$. If we can prove $\Xi_{P_s}(\lambda)$ decreases in λ for $\lambda \in [0,\bar{\mu}]$, we have $\Xi_{P_s}(\lambda) > 0$ for $\lambda \in [0,\bar{\mu}]$ and further $\frac{\partial P_s}{\partial\lambda} > 0$. We next prove $\Xi_{P_s}(\lambda)$ decreases in λ for $\lambda \in [0,\bar{\mu}]$.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Xi_{P_s}(\lambda) = 2\alpha\left((2\sigma-1)\left(1-\sigma^2\right)\alpha^3 + \left(2\sigma^2-6\sigma+3\right)\alpha^2 - 3(\sigma-1)^2\alpha + (1-2\sigma)\right)\lambda + \mu\left(4\alpha^3\sigma^3 - \alpha\left(\alpha^2+2\alpha-3\right)\sigma^2 - 4\alpha(\alpha-1)^2\sigma + (\alpha-1)^3\right).$$

is a linear function of λ. Thus, if we can prove (i) $\Xi'_{P_s}(0) < 0$ and (ii) $\Xi'_{P_s}(\bar{\mu}) < 0$, we have $\Xi_{P_s}(\lambda)$ decreases in λ for $\lambda \in [0,\bar{\mu}]$.

First, we prove $\Xi'_{P_s}(0) = \mu\left(\left(4\sigma^3-\sigma^2-4\sigma+1\right)\alpha^3 + \left(8\sigma-2\sigma^2-3\right)\alpha^2 + \left(3\sigma^2-4\sigma+3\right)\alpha - 1\right) < 0$ when $\alpha < \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$. We first have that $\Xi'_{P_s}(0)$ increases in α, because $\frac{\partial}{\partial\alpha}\Xi'_{P_s}(0) = \mu\Phi(\alpha) = \mu\left(3(4\sigma-1)\left(\sigma^2-1\right)\alpha^2 - 2\left(2\sigma^2-8\sigma+3\right)\alpha + \left(3\sigma^2-4\sigma+3\right)\right) > 0$ which we will prove next.

1. If $\alpha < 1/4$, we have $3(4\sigma-1)\left(\sigma^2-1\right) > 0$, $\Phi(0) = 3\sigma^2-4\sigma+3 > 0$, $\Phi\left(\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}\right) = \frac{\sigma^2}{2(1-\sigma^2)}\left((11-8\sigma)\sqrt{5-4\sigma} + 18\sigma^2 - 30\sigma + 9\right) > 0$, and $\Phi(1) = 4\sigma^2(3\sigma-1) < 0$.
2. If $1/4 < \alpha < 1/3$, we have $3(4\sigma-1)\left(\sigma^2-1\right) < 0$, $\Phi(0) = 3\sigma^2-4\sigma+3 > 0$, $\Phi\left(\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}\right) = \frac{\sigma^2}{2(1-\sigma^2)}\left((11-8\sigma)\sqrt{5-4\sigma} + 18\sigma^2 - 30\sigma + 9\right) > 0$, and $\Phi(1) = 4\sigma^2(3\sigma-1) < 0$.
3. If $\alpha > 1/3$, we have $3(4\sigma-1)\left(\sigma^2-1\right) < 0$, $\Phi(0) = 3\sigma^2-4\sigma+3 > 0$, and $\Phi(1) = 4\sigma^2(3\sigma-1) > 0$. Thus, $\Phi(\alpha) > 0$ for $\alpha \in (0,1)$.

Summarizing, we have $\frac{\partial}{\partial\alpha}\Xi'_{P_s}(0) > 0$ for $\alpha \in \left(0, \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}\right)$. At the upper bound of α, we have $\Xi'_{P_s}(0)\big|_{\alpha=\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}} = -\frac{\mu\sigma^3(1-\sigma)}{2(\sigma^2-1)^2}\left(4-5\sigma+(2-\sigma)\sqrt{5-4\sigma}\right) < 0$. Thus, we have $\Xi'_{P_s}(0) < 0$ when $\alpha < \frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$. We get $\mu\frac{(\alpha-1)^2}{\alpha\sigma-\alpha+1}\left(-3\alpha^2\sigma^3 - \alpha(1-\alpha)\sigma^2 - 3\alpha(1-\alpha)\sigma - (\alpha-1)^2\right) < 0$ when evaluating $\Xi'_{P_s}(\bar{\mu})$. Thus, we proved that $\Xi'_{P_s}(\lambda)$ decreases in λ for $\lambda \in [0,\bar{\mu}]$.

When $\sigma < 1$, $P_s^{\sigma<1}$ can be rewritten as

$$P_s^{\sigma<1} = \frac{c\lambda^2}{\sigma}\frac{\frac{\left(\sigma^2-1\right)\mu-\lambda(\sigma-2)}{\lambda^2} - \frac{1-\sigma^2}{\lambda}\alpha - \frac{\lambda(1-\sigma)\left((\lambda-\mu)^2+\sigma\mu^2\right)\alpha+\mu\left(\sigma\mu(\sigma\mu-\lambda)-(\lambda-\mu)^2\right)}{\lambda^2(\mu-\alpha\lambda)^2}}{\sigma\mu+\alpha(1-\sigma)\lambda-\lambda}$$

which is clearly a decreasing function of α. Using simple algebra, we have $P_s = 0$, if (i) $\alpha \leq \frac{1}{\sigma+1}$ and $\lambda = 0$, or (ii) $\alpha > \frac{1}{\sigma+1}$ and $\lambda = G_1(\alpha)$.

Case $\sigma > 1$: we have $\frac{\partial P_s}{\partial\lambda} = \frac{c\lambda}{\sigma\mu(\sigma\mu-(1+\alpha(\sigma-1))\lambda)^3}\Omega(\lambda)$, where

$$\Omega(\lambda) = \alpha(\alpha(\sigma-1)+1)^2\left(\sigma^2+(\alpha-1)(\sigma-1)\right)\lambda^2 - 3\alpha\sigma\mu\left(\alpha\sigma^3+(\alpha-1)^2\sigma^2-\left(2\alpha^2-3\alpha+1\right)\sigma+(\alpha-1)^2\right)\lambda + 2\sigma^2\mu^2\left(\alpha^2\sigma^2+\alpha(\alpha-1)\sigma-\left(2\alpha^2-3\alpha+1\right)\right).$$

Whether $\Omega(\lambda) > 0$ determines the monotonicity of P_s regarding λ. We note that $\sigma^2 + (\alpha-1)(\sigma-1) > 0$ increases in α for $\sigma > 1$, and $\sigma^2 + (\alpha-1)(\sigma-1)|_{\alpha=0} = \sigma^2 - \sigma + 1$ Thus, we have $\sigma^2 + (\alpha-1)(\sigma-1) > 0$. When $\lambda = 0$, we have

$$\Omega(0) = 2\sigma^2\mu^2\left(\alpha^2\sigma^2 + \alpha(\alpha-1)\sigma - \left(2\alpha^2 - 3\alpha + 1\right)\right) = 2\sigma^2\mu^2\left(\left(\sigma^2+\sigma-2\right)\alpha^2 + (3-\sigma)\alpha - 1\right),$$

which is greater than zero for $\alpha > \frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$, because (i) $\sigma^2+\sigma-2 > 0$ for $\sigma > 1$, (ii) $((\sigma^2+\sigma-2)\alpha^2 + (3-\sigma)\alpha - 1)$ evaluated at $\alpha = 0$ equals -1 and evaluated at $\alpha = \frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$ it equals 0. When $\lambda = G_0(\alpha)$, we have

$$\Omega(G_0(\alpha)) = \alpha\sigma^2\mu^2\left(\sigma^2 + (\alpha-1)(\sigma-1)\right)\left(\left(\frac{(\alpha-1)(\alpha\sigma^2-\alpha+1)}{\alpha(\sigma^2+(\alpha-1)(\sigma-1))}+1\right)^2 - \left(\frac{(\alpha-1)(\alpha\sigma^2-\alpha+1)}{\alpha(\sigma^2+(\alpha-1)(\sigma-1))}+1\right)\right).$$

We next prove $\Omega(G_0(\alpha)) < 0$ for $\alpha \in \left(\frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}, 1\right]$. First, $\frac{(\alpha-1)\left(\alpha\sigma^2-\alpha+1\right)}{\alpha(\sigma^2+(\alpha-1)(\sigma-1))}+1$ increases in α, because its derivative wrt α is $\frac{(\sigma-1)\left(\sigma^3+\sigma^2-1\right)\alpha^2+2(\sigma-1)\alpha+\left(\sigma^2-\sigma+1\right)}{\alpha^2(\sigma^2+(\alpha-1)(\sigma-1))^2} > 0$ for $\sigma > 1$. Evaluated at $\alpha = \frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$ it equals 0, and at $\alpha = 1$ it equals 1. Given that $x^2 - x < 0, \forall x \in (0,1)$, we conclude that $\Omega(\lambda)$ decreases from a positive value to a negative value when λ increases in $(0, G_0(\alpha))$. This proves that P_s is a concave function of λ.

Considering derivative of P_s wrt α, $\frac{\partial P_s}{\partial\alpha} = \frac{c\lambda^2}{\sigma\mu(\sigma\mu-(1+\alpha(\sigma-1))\lambda)^3}\Upsilon(\lambda)$ where

$$\Upsilon(\lambda) = \left((\alpha(\sigma-1)+1)^3 + \sigma(\sigma-1)((\sigma-1)\alpha+1)\right)\lambda^2 -$$
$$\sigma\mu\left(3(\sigma-1)^2\alpha^2 + \left(2\sigma^2+\sigma+5\right)(\sigma-1)\alpha + \left(\sigma^2+2\right)\right)\lambda + \sigma^2\mu^2\left(\left(2\sigma^2+2\sigma-4\right)\alpha + (3-\sigma)\right).$$

Whether $\Upsilon(\lambda) > 0$ determines the monotonicity of P_s regarding α. Clearly, $(\alpha(\sigma-1)+1)^3 + \sigma(\sigma-1)((\sigma-1)\alpha+1) > 0$. Also, we have $\Upsilon(0) = \sigma^2\mu^2((2\sigma^2+2\sigma-4)\alpha + (3-\sigma))$ which is positive for $\alpha > \frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$, because (i) $2\sigma^2+2\sigma-4 > 0$ for $\sigma > 1$, and (ii) $(2\sigma^2+2\sigma-4)\alpha + (3-\sigma)$ evaluated at $\alpha = \frac{\sigma-3+\sqrt{5\sigma^2-2\sigma+1}}{2(\sigma^2+\sigma-2)}$ equals $\sqrt{5\sigma^2-2\sigma+1} > 0$. Evaluating $\Upsilon(\lambda)$ at $\mu G_0(\alpha)$, we obtain $\frac{\sigma^2\mu^2(1-\alpha)}{\alpha^2(\alpha(\sigma-1)+1)(\sigma^2+(\alpha-1)(\sigma-1))^2}\Psi(\sigma)$ where

$$\Psi(\sigma) = -(\sigma-1)^4\left(\sigma^2+3\sigma+2\right)\alpha^5 - (\sigma-1)^3\left(-2\sigma^2+5\sigma+8\right)\alpha^4 + (\sigma-1)^2\left(2\sigma^4-\sigma^3+4\sigma^2+2\sigma-13\right)\alpha^3$$
$$+(\sigma-1)\left(2\sigma^4-\sigma^2+9\sigma-11\right)\alpha^2 + (\sigma-1)\left(\sigma^3+2\sigma^2-2\sigma+5\right)\alpha + \left(\sigma^2-\sigma+1\right).$$

Studying derivatives $\Psi_\sigma^{(n)}$ of higher order of $\Psi(\sigma)$, one can observe $\Psi_\sigma^{(6)} = 720\alpha^3(2-\alpha^2) > 0, \forall\alpha \in (0,1)$. Also, when $\sigma = 1$, $\forall\alpha \in (0,1)$, all derivatives till order 5 (including) are positive. Thus, $\Psi(\sigma) > 0$ for $\sigma > 1$. When $\lambda = \bar{\mu}$, we have $\Upsilon(\bar{\mu}) = -\frac{2\sigma^2\mu^2(1-\alpha)(\sigma-1)\left(\alpha\left(\sigma^2-1\right)+1\right)}{\alpha(\sigma-1)+1} < 0$. Thus, we have $\Upsilon(\lambda) > 0$ for $\lambda \in (0, G_0(\alpha))$. This proves that P_s is a increasing function of α in the region of $\lambda \in (0, G_0(\alpha))$.

A.4. Proof of Lemma 6 (Page 6)

$\Theta(\lambda)$ is positive. Thus, we have $\Delta S > 0$ if $\sigma > 1$, and $\Delta S < 0$ if $\sigma < 1$. Next, we show that $\Theta(\lambda)$ decreases in λ. We have:

$$\frac{\mathrm{d}\Theta(\lambda)}{\mathrm{d}\lambda} = \begin{cases} \frac{1}{\sigma-1}\left(\alpha(\sigma-1)+1-\frac{\sigma\mu^2\left(\alpha\left(\sigma^2-1\right)+1\right)}{(\alpha\lambda(\sigma-1)+\mu)^2}\right) & \text{if } \sigma > 1 \\ \frac{1}{1-\sigma}\frac{1}{\sigma-1}\left(\frac{\sigma\mu^2(1-\alpha(1-\sigma^2))}{(\mu-\lambda\alpha(1-\sigma))^2} - (1-\alpha(1-\sigma))\right) & \text{if } \sigma < 1 \end{cases},$$

which increases in λ. In addition, we have $\Theta'(\bar{\mu}) = -(1-\alpha)\frac{\alpha(\sigma-1)+1}{\alpha(\sigma^2-1)+1} < 0$. Thus, $\Theta'(\lambda) < 0$, i.e., $\Theta(\lambda)$ decreases in λ. Then, ΔS increases in λ if $\sigma > 1$, and decreases otherwise. In the $\sigma > 1$ case, when $\lambda = G_0(\alpha)$, we calculate

$$\Delta S = \frac{c(\sigma-1)(\alpha\sigma^2-\alpha+1)(\sigma^2+(\alpha-1)\sigma+(1-\alpha))(\alpha^2\sigma^2+\alpha(\alpha-1)\sigma-(2\alpha^2-3\alpha+1))^2}{(1-\alpha)(\alpha\sigma^3+\sigma^2-\alpha+1)\left(\alpha\sigma^3+(\alpha-1)^2\sigma^2-(2\alpha^2-3\alpha+1)\sigma+(\alpha-1)^2\right)^2}$$

which is positive because $\sigma^2+(\alpha-1)\sigma+(1-\alpha) = (\sigma-1+\alpha)\sigma+(1-\alpha) > 0$. In the $\sigma < 1$ case, we have $\lim_{\lambda\to\bar{\mu}}\Theta(\lambda) = 0 \Rightarrow \lim_{\lambda\to\bar{\mu}}\Delta S = -\infty$. Also, when $\lambda = G_1(\alpha) = \frac{\mu\left(\alpha^2\sigma^2-(\alpha-1)^2\right)}{\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)}$ and $\alpha > \frac{1}{\sigma+1}$, we have $\Delta S = \frac{c\alpha(\sigma-1)\left(\alpha^2\sigma^2-\alpha^2+2\alpha-1\right)^2}{\sigma(\alpha-1)^2(-(1-\sigma^2)\alpha^2+(2-\sigma)\alpha-(1-\sigma))}$, in which $-(1-\sigma^2)\alpha^2+(2-\sigma)\alpha-(1-\sigma) > 0$ when $\alpha > \frac{1}{\sigma+1}$ because

$$\begin{aligned} -\left(1-\sigma^2\right)\alpha^2+(2-\sigma)\alpha-(1-\sigma)\big|_{\alpha=\frac{1}{\sigma+1}} &= \frac{\sigma^2}{\sigma+1} > 0 \\ \frac{\partial}{\partial\alpha}\left(-\left(1-\sigma^2\right)\alpha^2+(2-\sigma)\alpha-(1-\sigma)\right) &= 2-\sigma-2\alpha\left(1-\sigma^2\right) > 0. \end{aligned}$$

Thus, we have $\Delta S < 0$ when $\lambda = G_1(\alpha)$. Clearly, if $\lambda = 0$ and $\alpha \le \frac{1}{\sigma+1}$, we have $\Delta S = 0$.

A.5. Proof of Lemma 9 (Page 12)

From (2), we have

$$U_w(0,\lambda) = R - c\left(\frac{\alpha(\alpha-1)(\sigma-1)^2}{\sigma\mu(1+\alpha(\sigma-1))} + \frac{1+\alpha(\sigma^2-1)}{1+\alpha(\sigma-1)}\frac{1}{\sigma\mu-\lambda(1+\alpha(\sigma-1))}\right)$$

in which, $1+\alpha(\sigma^2-1) > 0$ and $1+\alpha(\sigma-1) > 0$ for $\alpha\in[0,1]$ and $\sigma > 0$. Therefore, $\frac{\alpha(\alpha-1)(\sigma-1)^2}{\sigma\mu(1+\alpha(\sigma-1))} + \frac{\alpha\left(\sigma^2-1\right)+1}{1+\alpha(\sigma-1)}\frac{1}{\sigma\mu-\lambda(1+\alpha(\sigma-1))}$ increases in λ. Thus, $U_w(0,\lambda)$ decreases in λ. We also have $\lim_{\lambda\to\bar{\mu}}U_w(0,\lambda) = -\infty$ and $U_w(0,0) = \frac{c}{\mu}\left(\nu-\frac{1+\alpha(\sigma-1)}{\sigma}\right) > 0$. This means that $U_w(0,\lambda) = 0$ has a unique solution $\bar{\lambda}_w = \frac{\sigma\mu(\sigma\nu-1-\alpha(\sigma-1))}{\sigma\nu(\alpha(\sigma-1)+1)-\alpha(\alpha-1)(\sigma-1)^2} \in (0,\bar{\mu})$, where $\nu = R\mu/c$.

From (2), we have

$$U_d(1,\lambda) = R - c\frac{1}{\mu-\alpha\lambda}\left(\alpha\frac{\sigma(2\alpha-1)-2\alpha+2}{1+\alpha(\sigma-1)} + \frac{\mu(1-\alpha)(1+\alpha(\sigma^2-1))}{1+\alpha(\sigma-1)}\frac{1}{\sigma\mu-\lambda(1+\alpha(\sigma-1))}\right)$$

decreasing in λ. A solution to $U_d(1,\lambda) = 0$ should thus solve:

$$\nu\alpha(\alpha\sigma-\alpha+1)\lambda^2 + \mu(\alpha(2\alpha\sigma-2\alpha-\sigma+2)-\nu(2\alpha\sigma-\alpha+1))\lambda + \mu^2(\nu\sigma-\alpha\sigma+\alpha-1) = 0,$$

where $(\alpha\sigma-\alpha+1) > 0$. If evaluated at $\lambda = 0$, this expression yields $\mu^2(\nu\sigma-\alpha\sigma+\alpha-1) > 0$ because $R \ge c/\bar{\mu}$; if evaluated at $\lambda = \bar{\mu}$, it yields $-\frac{\mu^2(1-\alpha)\left(\alpha\sigma^2-\alpha+1\right)}{(\alpha\sigma-\alpha+1)} < 0$. Thus, there exists unique solution $\bar{\lambda}_d \in (0,\bar{\mu})$. Moreover, $\bar{\lambda}_d$ increases in R since $U_d(1,\lambda)$ increases in R.

Lemma 10. *$\bar{\lambda}_w < \bar{\lambda}_d$ if $\sigma < 1$ and $\bar{\lambda}_w > \bar{\lambda}_d$ otherwise.*

Proof The difference between $U_w(0,\lambda)$ and $U_d(1,\lambda)$ is

$$U_w(0,\lambda) - U_d(1,\lambda) = \frac{c\lambda\alpha(1-\alpha)}{\sigma\mu(\sigma\mu-\lambda(1+\alpha(\sigma-1)))}\frac{\mu+\alpha(\sigma-1)\lambda}{\mu-\alpha\lambda}(\sigma-1).$$

One can verify that $\mu+\alpha(\sigma-1)\lambda > 0$ and $\mu-\alpha\lambda > 0$, by using $\lambda < \bar{\mu}$. Hence, we have $\begin{cases} U_w(0,\lambda) - U_d(1,\lambda) < 0 \text{ if } \sigma < 1 \\ U_w(0,\lambda) - U_d(1,\lambda) > 0 \text{ if } \sigma > 1 \end{cases}$, which leads to $\begin{cases} \bar{\lambda}_w < \bar{\lambda}_d \text{ if } \sigma < 1 \\ \bar{\lambda}_w > \bar{\lambda}_d \text{ if } \sigma > 1 \end{cases}$. □

Next, we investigate how $G_0(\alpha)$ and $G_1(\alpha)$ intersect with $\bar{\lambda}_w$ and $\bar{\lambda}_d$. Recall from the proof of Proposition 1 that $G_0(\alpha)-G_1(\alpha)=0 \Leftrightarrow \alpha=\frac{1}{\sigma^{3/2}+1}$.

Intersection of $G_1(\alpha)$ and $\bar{\lambda}_w$: when $\sigma<1$, intersection point can be derived by solving

$$\frac{\mu\left(\alpha^2\sigma^2-(\alpha-1)^2\right)}{\alpha\left(\alpha^2\sigma^2-(\alpha-1)\sigma-(\alpha-1)^2\right)}=\frac{\sigma\mu\left(\sigma\nu-1-\alpha\left(\sigma-1\right)\right)}{\sigma\nu\left(\alpha\left(\sigma-1\right)+1\right)-\alpha\left(\alpha-1\right)\left(\sigma-1\right)^2}$$

Which is equivalent to solving:

$$K_1\left(\alpha\right)\equiv\left(1-\sigma\right)\alpha^3+\left(2\sigma-\sigma^2+\sigma\nu-2\right)\alpha^2+\left(1-2\sigma\nu-\sigma\right)\alpha+\sigma\nu=0$$

$\lim_{\alpha\to-\infty}K_1\left(\alpha\right)<0,\ K_1\left(0\right)=\sigma\nu>0,\ K_1\left(1\right)=-\sigma^2<0,$ and $\lim_{\alpha\to\infty}K_1\left(\alpha\right)>0$. Hence, $G_1(\alpha)$ and $\bar{\lambda}_w$ have a unique intersection point $\alpha_{s1}\in(0,1)$. Moreover, we have $G_1(\alpha)=\mu>\bar{\lambda}_w=\frac{\nu-1}{\nu}\mu$ at $\alpha=1$, which leads to $G_1(\alpha)\le\bar{\lambda}_w\Leftrightarrow\alpha\le\alpha_{s1}$ and $G_1(\alpha)\ge\bar{\lambda}_w\Leftrightarrow\alpha\ge\alpha_{s1}$.

Then, we derive the range of ν such that $\bar{\lambda}_w$ does not cross region II, where customers' equilibrium behavior is $\gamma^*\in(0,1)$. Recall that $\bar{\lambda}_w$ increases in R. Substituting α with $\frac{1}{\sigma^{3/2}+1}$ (intersection point of $G_0(\alpha)$ and $G_1(\alpha)$) into $K_1\left(\alpha\right)$ and equating it to zero gives the lower bound of ν such that $\bar{\lambda}_w$ does not cross region II: $\frac{(\sigma-1)\sqrt{\sigma}+1}{\sigma^2(\sigma-\sqrt{\sigma}+1)}$. Thus, $\bar{\lambda}_w$ does not cross region II iff $R\ge\frac{c((\sigma-1)\sqrt{\sigma}+1)}{\mu\sigma^2(\sigma-\sqrt{\sigma}+1)}$.

Intersection of $G_1(\alpha)$ and $\bar{\lambda}_d$: one has to solve

$$0=U_d\left(1\right)=R-c\frac{1}{\mu-\alpha\bar{\lambda}_d}\left(\alpha\frac{\sigma\left(2\alpha-1\right)-2\alpha+2}{1+\alpha\left(\sigma-1\right)}+\frac{\mu\left(1-\alpha\right)\left(1+\alpha\left(\sigma^2-1\right)\right)}{1+\alpha\left(\sigma-1\right)}\frac{1}{\sigma\mu-\bar{\lambda}_d\left(1+\alpha\left(\sigma-1\right)\right)}\right).$$

To solve for the intersection point, we substitute $\bar{\lambda}_d$ with $G_1(\alpha)$ in $U_d\left(1\right)$ to obtain $R-\frac{c}{\sigma\mu}Q_1(\alpha)$ where:

$$\begin{aligned}Q_1(\alpha)\equiv&\frac{\alpha\left(\left(1-\sigma\right)\alpha-1\right)\left(\left(1-\sigma^2\right)\alpha^2-\left(2-\sigma\right)\alpha+\left(1-\sigma\right)\right)}{\left(\alpha-1\right)^2}=\\&\frac{\left(1-\sigma\right)\left(1-\sigma^2\right)}{(\alpha-1)^2}\alpha\left(\alpha-\frac{1}{1-\sigma}\right)\left(\alpha-\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2\left(\sigma^2-1\right)}\right)\left(\alpha-\frac{\sigma-2-\sigma\sqrt{5-4\sigma}}{2\left(\sigma^2-1\right)}\right)\end{aligned}$$

When $\sigma<1$, rewrite $Q_1\left(\alpha\right)=\left(1-\sigma\right)\left(1-\sigma^2\right)\frac{(\alpha-\alpha_1)(\alpha-\alpha_2)(\alpha-\alpha_4)(\alpha-\alpha_5)}{(\alpha-\alpha_3)^2}$, where $\alpha_3=1$ is the vertical asymptote, and $\alpha_1=0$, $\alpha_2=\frac{\sigma-2-\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$, $\alpha_4=\frac{\sigma-2+\sigma\sqrt{5-4\sigma}}{2(\sigma^2-1)}$, and $\alpha_5=\frac{1}{1-\sigma}$ are zero points. In addition, we have $\alpha_1<\alpha_2<\alpha_3<\alpha_4<\alpha_5$. Simple algebraic manipulation reveals that the first derivative $Q_1'(\alpha)>0,\forall\alpha\in(\alpha_2,\alpha_3)$. Hence $Q_1\left(\alpha\right)$ increases from zero to ∞ when α increases from α_2 to α_3. When $\sigma>1$, rewrite $Q_1\left(\alpha\right)=\left(1-\sigma\right)\left(\left(1-\sigma^2\right)\alpha^2-\left(2-\sigma\right)\alpha+\left(1-\sigma\right)\right)\left(\alpha-\frac{1}{1-\sigma}\right)\frac{\alpha}{(\alpha-1)^2}$, where $\frac{\alpha}{(\alpha-1)^2}>0$ is an increasing function. Derivative $Q_1'(\alpha)>0$. Thus, $Q_1\left(\alpha\right)$ increases. Note that $Q_1\left(0\right)=0$ and $\lim_{\alpha\to1}Q_1\left(\alpha\right)=\infty$. Hence, $Q_1\left(\alpha\right)$ increases from zero to ∞ when α increases from 0 to 1. Combining the two case for $\sigma<1$ and $\sigma>1$, we have that $R-\frac{c}{\sigma\mu}Q_1\left(\alpha\right)$ has a unique root $\alpha_{s2}\in(0,1)$, which is also the intersection point of $G_1(\alpha)$ and $\bar{\lambda}_d$.

Since $\bar{\lambda}_w<\bar{\lambda}_d$ when $\sigma<1$, we should have $\alpha_{s1}<\alpha_{s2}$; otherwise, if $\alpha_{s2}\le\alpha_{s1}$, from the result that $G_1(\alpha)\le\bar{\lambda}_w\Leftrightarrow\alpha\le\alpha_{s1}$, we have $G_1(\alpha_{s2})=\bar{\lambda}_d|_{\alpha=\alpha_{s2}}\le\bar{\lambda}_w$, which conflicts with the result $\bar{\lambda}_w<\bar{\lambda}_d$.

Thus, when $\sigma<1$ and $R\mu/c\ge\frac{(\sigma-1)\sqrt{\sigma}+1}{\sigma^2(\sigma-\sqrt{\sigma}+1)}$, customers' equilibrium join-up-to level is $\bar{\lambda}_w$ for $\alpha\in[0,\alpha_{s1}]$ (see, e.g., region I in Figure 10(a)), $G_1(\alpha)$ for $\alpha\in(\alpha_{s1},\alpha_{s2}]$ (see, e.g., region III in Figure 10(a)), and $\bar{\lambda}_d$ for $\alpha\in[\alpha_{s2},1]$ (see, e.g., region IV in Figure 10(a)).

We next consider the case $\sigma < 1$ and $R\mu/c < \frac{(\sigma-1)\sqrt{\sigma}+1}{\sigma^2(\sigma-\sqrt{\sigma}+1)}$, where $\bar{\lambda}_w$ may cross region II. In this region II, customers' equilibrium information disclosure strategy $\gamma^* \in (0,1)$ given in Proposition 1 is a function of α and λ, and it fulfills $U_w(\gamma^*) = U_d(\gamma^*)$. To find the join-up-to level in the region II, we need to fix α and solve for $\bar{\mu}_m$ from $\gamma U_d(\gamma^*) + (1-\gamma) U_w(\gamma^*) = 0$, equivalent to $U_w(\gamma^*) = 0$, which can be rewritten as $R - L(\lambda) = 0$ with

$$L(\lambda) \equiv -\frac{c\alpha(1-\alpha)(1-\alpha(1-\sigma)(\sigma^2+1))(1-\alpha(\alpha(\sigma-1)-\sigma)(\sigma-1))}{(1-\alpha(1-\sigma))(1-\alpha\sigma)(1-\alpha(1-\sigma))(\alpha^2(\sigma-1)+\alpha(\sigma^2-\sigma+1))}\frac{(\lambda-\lambda_4)(\lambda-\lambda_5)}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}.$$

Here $-\frac{\mu\alpha(1-\alpha)\left(1-\alpha(1-\sigma)\left(\sigma^2+1\right)\right)(1-\alpha(\alpha(\sigma-1)-\sigma)(\sigma-1))}{(1-\alpha(1-\sigma))(1-\alpha\sigma)(1-\alpha(1-\sigma))(\alpha^2(\sigma-1)+\alpha(\sigma^2-\sigma+1))} < 0$, $\bar{\lambda}_d = \frac{\alpha(1-\sigma)}{1-\alpha\sigma}\frac{\sigma\mu}{1+\alpha(\sigma-1)}$, $\lambda_2 = \frac{\sigma\mu}{1+\alpha(\sigma-1)}$, and $\lambda_3 = \frac{\mu(\alpha-1)(\sigma-1)}{\alpha(\sigma^2+(\alpha-1)(\sigma-1))}$ are three positive vertical asymptotes of $L(\lambda)$. Further, $\lambda_4 = \frac{\sigma\mu(1-\sigma)}{\alpha(\sigma-1)(\sigma^2+1)+1}$ and $\lambda_5 = \frac{\mu(1-\sigma)(1-\alpha+\alpha\sigma)}{1-\alpha(\alpha(\sigma-1)-\sigma)(\sigma-1)}$ are two positive zero points of $L(\lambda)$. One can show that $\bar{\lambda}_d < \lambda_i$ for $i = 2,\ldots,5$, when $\alpha < \frac{1}{\sigma^{3/2}+1}$, where $\frac{1}{\sigma^{3/2}+1}$ (intersection point of $G_1(\alpha)$ and $G_0(\alpha)$). We also have $L(0) = \frac{(1+\alpha(\sigma-1))}{\sigma} \le \frac{R\mu}{c}$, $L(\lambda) > 0$ for $\lambda < \bar{\lambda}_d$, and $\lim_{\lambda\nearrow\lambda_1} L(\lambda) = \infty$. Investigating derivative $\frac{\partial}{\partial\lambda}\left(\frac{(\lambda-\lambda_4)(\lambda-\lambda_5)}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}\right)$ for $\lambda < \lambda_1$, we have $L'(\lambda) > 0$ for $\lambda < \lambda_1$.

Combining the above results we have that when λ increases from zero to λ_1, customers equilibrium utility under the equilibrium information disclosure strategy γ^* increases from $\frac{(1+\alpha(\sigma-1))}{\sigma}$ to ∞. Hence, there is a unique solution $\bar{\lambda}_m$ (see, e.g., region II in Figure 10(b)).

Intersection of $G_0(\alpha)$ and $\bar{\lambda}_w$: when $\sigma > 1$, the intersection point can be derived by solving:

$$K_2(\alpha) \equiv (1-\sigma)^2\alpha^3 + (1-\sigma)(2\sigma+\sigma\nu-1)\alpha^2 - \sigma(2\nu-\sigma\nu+1)\alpha + \sigma\nu = 0$$

where $\lim_{\alpha\to-\infty} K_2(\alpha) < 0$, $K_2(0) = \sigma\nu > 0$, $K_2(1) = -\sigma^2 < 0$, and $\lim_{\alpha\to\infty} K_2(\alpha) > 0$. Hence, $G_0(\alpha)$ and $\bar{\lambda}_w$ have a unique intersection point $\alpha_{l2} \in (0,1)$. Moreover, we have $G_0(\alpha) = \mu > \bar{\lambda}_w = \frac{\nu-1}{\nu}\mu$ at $\alpha = 1$, which leads to $G_0(\alpha) \le \bar{\lambda}_w \Leftrightarrow \alpha \le \alpha_{l2}$ and $G_0(\alpha) \ge \bar{\lambda}_w \Leftrightarrow \alpha \ge \alpha_{l2}$.

Intersection of $G_0(\alpha)$ and $\bar{\lambda}_d$: intersection point satisfies $U_d(1) = 0$ or

$$R - c\frac{1}{\mu-\alpha\bar{\lambda}_d}\left(\alpha\frac{\sigma(2\alpha-1)-2\alpha+2}{1+\alpha(\sigma-1)} + \frac{\mu(1-\alpha)(1+\alpha(\sigma^2-1))}{1+\alpha(\sigma-1)}\frac{1}{\sigma\mu-\bar{\lambda}_d(1+\alpha(\sigma-1))}\right) = 0.$$

To solve for the intersection point, we substitute $\bar{\lambda}_d$ with $G_0(\alpha)$ in $U_d(1)$ to get $R - \frac{c}{\mu\sigma}Q_2(\alpha)$, where

$$Q_2(\alpha) = \frac{\alpha((2\sigma^2-\sigma-1)\alpha+\sigma+1)((\sigma-1)\alpha+(\sigma^2-\sigma+1))}{(1-\alpha)((\sigma^3-1)\alpha+(\sigma^2+1))}.$$

When $\sigma < 1$, we can rewrite

$$Q_2(\alpha) = \frac{(2\sigma^2-\sigma-1)(1-\sigma)}{(\sigma^3-1)}\cdot\frac{(\alpha-\alpha_1)(\alpha-\alpha_4)(\alpha-\alpha_5)}{(\alpha-\alpha_2)(\alpha-\alpha_3)},$$

where $\alpha_2 = 1$ and $\alpha_3 = \frac{\sigma^2+1}{1-\sigma^3} > 1$ are the vertical asymptote, and $\alpha_1 = 0$, $\alpha_4 = \frac{\sigma+1}{1+\sigma-2\sigma^2}$, and $\alpha_5 = \frac{\sigma^2-\sigma+1}{1-\sigma}$ are all positive zero points. In addition, we have $\alpha_1 = 0 < \alpha_2 = 1 < \alpha_3 < \alpha_4, \alpha_5$. One can verify that the first derivative $Q_2'(\alpha) > 0, \forall\alpha \in (\alpha_1, \alpha_2)$. Hence, $Q_2(\alpha)$ increases from zero to ∞ when α increases from 0 towards 1.

When $\sigma > 1$, we can rewrite

$$Q_2(\alpha) = \frac{\alpha}{1-\alpha}\frac{((2\sigma^2-\sigma-1)\alpha+\sigma+1)((\sigma-1)\alpha+(\sigma^2-\sigma+1))}{((\sigma^3-1)\alpha+(\sigma^2+1))},$$

in which $\frac{\alpha}{1-\alpha}$ increases in α, and one can easily verify that $\frac{\left(\left(2\sigma^2-\sigma-1\right)\alpha+\sigma+1\right)\left((\sigma-1)\alpha+\left(\sigma^2-\sigma+1\right)\right)}{(\sigma^3-1)\alpha+(\sigma^2+1)}$ also increases in α because its first derivative is positive when $\sigma > 1$. Note that $Q_2(0) = 0$ and $\lim_{\alpha\to 1} Q_1(\alpha) = \infty$. Hence, $Q_2(\alpha)$ increases from zero to ∞ when α increases from 0 towards 1. Combining the above two case, we have that $R - \frac{c}{\sigma\mu} Q_2(\alpha)$ has a unique root $\alpha_{s1} \in (0,1)$, which is also the intersection point of $G_0(\alpha)$ and $\bar{\lambda}_d$.

Since $\bar{\lambda}_d < \bar{\lambda}_w$ when $\sigma > 1$, we should have $\alpha_{l1} < \alpha_{l2}$; otherwise, if $\alpha_{l1} \geq \alpha_{l2}$, from the result that $G_0(\alpha) \geq \bar{\lambda}_w \Leftrightarrow \alpha \geq \alpha_{l2}$, we have $G_0(\alpha_{l1}) = \bar{\lambda}_d|_{\alpha=\alpha_{l1}} \geq \bar{\lambda}_w$, which conflicts with the result $\bar{\lambda}_d < \bar{\lambda}_w$.

Thus, when $\sigma > 1$, the join-up-to level is $\bar{\lambda}_w$ for $\alpha \in [0, \alpha_{l1}]$ (see, e.g., region V in Figure 10(c)), $\begin{cases} \bar{\lambda}_d & \text{if } \Lambda \leq G_0(\alpha) \\ \bar{\lambda}_w & \text{if } \Lambda > G_0(\alpha) \end{cases}$ for $\alpha \in (\alpha_{s1}, \alpha_{s2}]$ (see, e.g., region VI in Figure 10(c)), and $\bar{\lambda}_d$ for $\alpha \in [\alpha_{l2}, 1]$ (see, e.g., region VII in Figure 10(c)).