Online Appendix to "Market Entry and Competition Under Network Effects"

A. Mathematical Preliminaries for Proofs

A.1. A Stochastic Approximation Process

This subsection reviews some basic definitions and concepts that are useful in the proofs for the theorems in this paper. Consider the stochastic approximation process defined by recurrence relation (2). This process is defined on a n-dimensional unit simplex, denoted by $S=[0,1]^n$. Let $\mathbf{\Phi}(\mathbf{x})$=$\boldsymbol{\phi}(\mathbf{x})$-$\mathbf{x}$ and $\boldsymbol{\zeta}(t$+1)=$\mathbf{e}$(t+1)-$\mathbb{E}[\mathbf{e}(t+1)|\mathbf{x}(t)]$. Then, for t=0,1,2,..., this stochastic approximation process can be rewritten as

$$\mathbf{x}(t+1)-\mathbf{x}(t)=\frac{1}{t+1+n}[\mathbf{\Phi}(\mathbf{x}(t))+\boldsymbol{\zeta}(t+1)], \tag{A.1}$$

where $\mathbb{E}[\boldsymbol{\zeta}(t+1)|\mathbf{x}(t)]=0$.

Let $B=\{\mathbf{x}\in S:\mathbf{\Phi}(\mathbf{x})=\mathbf{0}\}$ be the set of all critical points of $\mathbf{\Phi}(\mathbf{x})$ (or fixed points of $\boldsymbol{\phi}(\mathbf{x})$). A point $\mathbf{x}$ in B must satisfy

$$\frac{q_i^\alpha x_i^\beta}{q_1^\alpha x_1^\beta+\cdots+q_n^\alpha x_n^\beta}=x_i, \tag{A.2}$$

for $i=1,\ldots,n$. If $x_i>0$ for all i and $\beta\neq 1$, from (A.2), we have $\mathbf{x}=(q_1^{\frac{\alpha}{1-\beta}},\ldots,q_n^{\frac{\alpha}{1-\beta}})/\sum_{i=1}^n q_i^{\frac{\alpha}{1-\beta}}$, which is the unique critical point in the interior of S (see Theorem 5.1 in Maldonado et al. 2018, for details). Let B_0 be the intersection of B and the interior of S. We have that $B_0=\{(q_1^{\frac{\alpha}{1-\beta}},\ldots,q_n^{\frac{\alpha}{1-\beta}})/\sum_{i=1}^n q_i^{\frac{\alpha}{1-\beta}}\}$ if $\beta\neq 1$ and is empty otherwise. If $x_i=0$ for some i, then $\mathbf{x}$ satisfying (A.2) must be on the boundary of S. We categorize the boundary critical points into three sets:

Set 1 The set of vertices, denoted by $B_1=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ where $\mathbf{e}_i$ is the unit vector with ith entry being 1 and 0 otherwise.

Set 2 The set of points in the *relative interior* of any facet of the simplex, denoted by $B_2=\{\mathbf{x}: x_i=q_i^{\frac{\alpha}{1-\beta}}/\sum_{i\in I} q_i^{\frac{\alpha}{1-\beta}}$ for $i\in I,\ x_i=0$ for $i\notin I,\ I$ is any subset of $\{1,\ldots,n\},$ with $1<|I|<n\}$. Note that B_2 is nonempty if and only if $\beta\neq 1$.

Set 3 The union of all convex hulls of two or more vertices $\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_k}$ whenever $q_{i_1}=\cdots=q_{i_k}$ and $\beta=1$. We denote this set by B_3. B_3 is a union of intervals or polyhedrons whose vertices are vertices of S. Note that B_3 is nonempty if and only if $\beta=1$ and two or more firms' qualities are equal.

Consequently, we have $B=\bigcup_{i=0}^{3} B_i$. Thus B is a union of finitely many connected components. According to Theorem 7.3 in Nevel'son and Has'minskii (1973), Chapter 2, or Theorem 3.1 in Arthur et al. (1986), the process $\mathbf{x}(t)$ converges with probability 1 either to an isolated point in B or to one of its connected components for any initial condition.

We now consider the stability of a critical point $\mathbf{x}$ in B, depending on whether it has the property: the process $\mathbf{x}(t)$ with any initial point that lies in a sufficiently small neighborhood of $\mathbf{x}$ converges to $\mathbf{x}$ almost surely.

Definition A.1. (Arthur et al. 1986). A point $\mathbf{x} \in B$ is called a *stable point* if for some neighborhood U of $\mathbf{x}$, there is a symmetric positive definite matrix D such that

$$\langle D \cdot \mathbf{\Phi}(\mathbf{v}), \mathbf{v} - \mathbf{x} \rangle < 0, \quad \forall\ \mathbf{v} \neq \mathbf{x} \text{ and } \mathbf{v} \in U \cap S,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product. Similarly, a point $\mathbf{x} \in B$ is called an *unstable point* if for some neighborhood U of $\mathbf{x}$, there is a symmetric positive definite matrix D such that

$$\langle D \cdot \mathbf{\Phi}(\mathbf{v}), \mathbf{v} - \mathbf{x} \rangle > 0, \quad \forall\ \mathbf{v} \neq \mathbf{x} \text{ and } \mathbf{v} \in U \cap S.$$

Definition A.2. (Pemantle 1990). A point $\mathbf{x} \in B$, is *linearly stable* if all the eigenvalues of $\partial \mathbf{\Phi}(\mathbf{x})/\partial \mathbf{x}$ (the *Jacobian* of $\mathbf{\Phi}$ at $\mathbf{x}$) have negative real parts. If some eigenvalue of $\partial \mathbf{\Phi}(\mathbf{x})/\partial \mathbf{x}$ has a positive real part, then $\mathbf{x}$ is called a *linearly unstable* point.

Lemma A.1. *A point $\mathbf{x} \in B$ is linearly stable then it is stable.*

Proof of Lemma A.1. For the proof, see Page 107 in Nevel'son and Has'minskii (1973). □

Theorem A.1. *If $\mathbf{x} \in B$ is a stable point, then $\mathbb{P}\{\mathbf{x}(t) \to \mathbf{x}\} > 0$.*

Proof of Theorem A.1. See Theorem 5.1 in Arthur et al. (1986). □

Theorem A.2. *If $\mathbf{x} \in B$ is linearly unstable, then $\mathbb{P}\{\mathbf{x}(t) \to \mathbf{x}\} = 0$.*

Proof of Theorem A.2. See Theorem 1 in Pemantle (1990). □

A.2. Branching Process

One of the tools to study the distribution of the stochastic sequence (A.1) is the *exponential embedding* method, introduced by Athreya and Karlin (1968). Consider an n-dimensional continuous Markov branching process $\mathbf{Y}(s) = \{Y_1(s), \ldots, Y_n(s)\}$ with initial size $\mathbf{Y}(0) = \{1, \ldots, 1\}$. $\{Y_i(s)\}_{i=1}^n$ are all *pure-birth* processes and mutually independent. We will build the relationship between the branching process $\mathbf{Y}$ and the demand process $\mathbf{d}$ in the base model. For each group i, the kth split time $\gamma_i(k) \triangleq \inf\{s' : Y_i(s+s') = k+1 | Y_i(s) = k\}$ is exponentially distributed with rate $q_i^\alpha k^\beta$. Let $\tau_t,\ t = 1, 2, \ldots$ denote the successive times at which splits occur in the whole collection of n groups. Given τ_t and $\mathbf{Y}(\tau_t)$, we know τ_{t+1} follows an exponential distribution with rate $\sum_{i=1}^n q_i^\alpha Y_i(\tau_t)^\beta$ and $\mathbb{P}\{Y_i(\tau_{t+1}) = Y_i(\tau_t) + 1 | Y_i(\tau_t)\} = q_i^\alpha Y_i(\tau_t)^\beta / \sum_{i=1}^n q_i^\alpha Y_i(\tau_t)^\beta$ for $i = 1, \ldots, n$.

Theorem A.3. *The stochastic processes $\{\mathbf{Y}(\tau_t), t = 0, 1, \ldots\}$ and $\{\mathbf{d}(t), t = 0, 1, \ldots\}$ are equivalent.*

Proof of Theorem A.3. See Theorem 1 in Athreya and Karlin (1968) for the two-dimensional case; see Theorem 2 in Athreya and Ney (1972), Section 9.1, for the case with more than 2 dimensions. □

Let $\Gamma_i(N)=\gamma_i(1)+\cdots+\gamma_i(N)$ be the sum of the first N split times in group i. Thus $Y_i(s)=N+1$ for $\Gamma_i(N)\le s<\Gamma_i(N+1)$. It is known that $\Gamma_i(N)$ follows a *hypoexponential* distribution and its P.D.F. $f_{q_i^\alpha,\beta,N}(s)$ is

$$f_{q_i^\alpha,\beta,N}(s)=\sum_{j=1}^{N} q_i^\alpha j^\beta e^{-(q_i^\alpha j^\beta)s} H_{j,N}, \qquad s\ge 0,$$

where $H_{j,N}\triangleq \prod_{\substack{l=1\\ l\ne j}}^{N}\frac{l^\beta}{l^\beta-j^\beta}$, for $j=1,\ldots,N$. By Lemma 3.2.2 in Zhu (2009), if $\beta>1$, $\Gamma_i(\infty)\triangleq \lim_{N\to\infty}\Gamma_i(N)$ exists almost surely. Let $f_{q_i^\alpha,\beta}(\cdot)$ be its probability density function. Note that, if $\beta>1$, when $q=1$, we define $f_\beta(\cdot)=f_{1,\beta}(\cdot)$ for simplicity. Correspondingly, $F_{q^\alpha,\beta,N}(\cdot)$, $F_{q^\alpha,\beta}(\cdot)$, $F_\beta(\cdot)$ are defined in the same way.

When $\beta>1$, the discrete process $\mathbf{d}(t)$ eventually converges to a *monopoly*, that is, there is a time t such that all subsequent customers select the same product since time t. Define the *attraction time* by $\Gamma_a=\min\{t':\mathbf{e}(t)=\mathbf{e}(t'),\ \forall t>t'\}$.

THEOREM A.4. *If $\beta>1$, then a monopoly occurs almost surely, i.e., $\mathbb{P}\{\Gamma_a<\infty\}=1$.*

Proof of Theorem A.4. See Theorem 3.3.1 in Zhu (2009). □

Next, we consider which product(s) will become a monopoly if $\beta>1$. By Theorems A.3 and A.4, the probability that product i eventually goes to monopoly must be equal to $\mathbb{P}\{\Gamma_i(\infty)<\Gamma_j(\infty),\text{ for all } j\ne i\}$, where "$<$" is equivalent to "$\le$" since all the variables $\Gamma_i(\infty)$ have continuous distribution, which implies that they are distinct with probability one.

THEOREM A.5. *If $\beta>1$, for $i=1,\ldots,n$,*

$$\mathbb{P}\{\Gamma_i(\infty)<\Gamma_j(\infty),\ \textit{for all } j\ne i\}=\int_0^\infty f_{q_i^\alpha,\beta}(s)\prod_{\substack{j=1\\ j\ne i}}^{n}\bar F_{q_j^\alpha,\beta}(s)ds,$$

which can also be rewritten as

$$\mathbb{P}\{\Gamma_i(\infty)<\Gamma_j(\infty),\ \textit{for all } j\ne i\}=\lim_{N\to\infty}\sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N}\frac{q_i^\alpha j_i^\beta}{q_1^\alpha j_1^\beta+\cdots+q_n^\alpha j_n^\beta}\prod_{k=1}^{n}H_{j_k,N},$$

where $H_{j,N}\triangleq\prod_{\substack{l=1\\ l\ne j}}^{N}\frac{l^\beta}{l^\beta-j^\beta}$, for $j=1,\ldots,N$.

Proof of Theorem A.5. For $\beta>1$ and $i=1,\ldots,n$, we have

$$\begin{aligned}
&\mathbb{P}\{\Gamma_i(N)<\Gamma_j(N),\ \text{for all } j\ne i\}\\
&=\int\limits_{s_i<s_j \text{ for all } j\ne i} f_{q_1^\alpha,\beta,N}(s_1)\cdots f_{q_n^\alpha,\beta,N}(s_n)\ ds_1ds_2\cdots ds_n\\
&=\int_0^\infty f_{q_i^\alpha,\beta,N}(s_i)ds_i\int_{s_i}^\infty f_{q_1^\alpha,\beta,N}(s_1)ds_1\cdots\int_{s_i}^\infty f_{q_n^\alpha,\beta,N}(s_n)ds_n
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty f_{q_i^\alpha,\beta,N}(s_i)ds_i \int_{s_i}^\infty f_{q_1^\alpha,\beta,N}(s_1)ds_1 \cdots \int_{s_i}^\infty \sum_{j_n=1}^N q_n^\alpha j_n^\beta e^{-(q_n^\alpha j_n^\beta)s_n} H_{j_n,N} ds_n \\
&= \sum_{j_n=1}^N \left(\int_0^\infty f_{q_i^\alpha,\beta,N}(s_i) e^{-(q_n^\alpha j_n^\beta)s_i} ds_i \int_{s_i}^\infty f_{q_1^\alpha,\beta,N}(s_1)ds_1 \cdots \int_{s_i}^\infty f_{q_{n-1}^\alpha,\beta,N}(s_{n-1})ds_{n-1} \right) H_{j_n,N} \\
&= \cdots \\
&= \sum_{j_1,\dots,j_{i-1},j_{i+1},\dots,j_n=1}^N \left(\int_0^\infty f_{q_i^\alpha,\beta,N}(s_i) e^{-(q_1^\alpha j_1^\beta+\cdots+q_{i-1}^\alpha j_{i-1}^\beta+q_{i+1}^\alpha j_{i+1}^\beta+\cdots+q_n^\alpha j_n^\beta)s_i} ds_i \right) \prod_{\substack{k=1\\k\neq i}}^n H_{j_k,N} \\
&= \sum_{j_1,\dots,j_n=1}^N \frac{q_i^\alpha j_i^\beta}{q_1^\alpha j_1^\beta + \cdots + q_n^\alpha j_n^\beta} \prod_{k=1}^n H_{j_k,N}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\mathbb{P}\{\Gamma_i(\infty) < \Gamma_j(\infty), \text{ for all } j \neq i\} &= \int_0^\infty f_{q_i^\alpha,\beta}(s) \prod_{\substack{j=1\\j\neq i}}^n \bar{F}_{q_j^\alpha,\beta}(s) ds \\
&= \lim_{N\to\infty} \sum_{1\le j_1,\dots,j_n\le N} \frac{q_i^\alpha j_i^\beta}{q_1^\alpha j_1^\beta + \cdots + q_n^\alpha j_n^\beta} \prod_{k=1}^n H_{j_k,N}.
\end{aligned}
$$

In the following, we provide some properties regarding the sum of infinite exponentials.

LEMMA A.2. *For $\beta > 1$ and $q > 0$,*

(i) $f_{q^\alpha,\beta}(s)$ is at least first-order continuous and differentiable, and $f_{q^\alpha,\beta}(0) = f'_{q^\alpha,\beta}(0) = \lim_{s\to+\infty} f_{q_i^\alpha,\beta}(s) = \lim_{s\to+\infty} f'_{q_i^\alpha,\beta}(s) = 0$;

(ii) $f_{q^\alpha,\beta}(s)$ is logconcave on $[0,+\infty)$, which means $f_{q^\alpha,\beta}(s)$ is first increasing then decreasing and $f_{q^\alpha,\beta}(s)/\bar{F}_{q^\alpha,\beta}(s)$ is increasing on $[0,+\infty)$;

(iii) $[\ln f_{q^\alpha,\beta}(s)]'$ is decreasing in $q \in (0,+\infty)$ and $\beta \in (1,+\infty)$; $f_{q^\alpha,\beta}(s)/\bar{F}_{q^\alpha,\beta}(s)$ is increasing in $q \in (0,+\infty)$ and $\beta \in (1,+\infty)$;

(iv) $\dfrac{\partial f_{q^\alpha,\beta}(s)}{\partial q} = \dfrac{\alpha}{q}(sf_{q^\alpha,\beta}(s))'$, $\dfrac{\partial^2 f_{q^\alpha,\beta}(s)}{\partial q^2} = \dfrac{\alpha(\alpha-1)}{q^2}(sf_{q^\alpha,\beta}(s))' + \dfrac{\alpha^2}{q^2}(s^2 f'_{q^\alpha,\beta}(s))'$, $\dfrac{\partial \bar{F}_{q^\alpha,\beta}(s)}{\partial q} = -\dfrac{\alpha}{q} s f_{q^\alpha,\beta}(s)$.

Proof of Lemma A.2. We first prove part (i). By Proposition 2 in Smaili et al. (2013), we have $f_{q^\alpha,\beta,N}(0) = f'_{i,N}(0) = 0,\ \forall N \ge 3$. Since $f'_{q^\alpha,\beta,N}(s)$ and $f_{q^\alpha,\beta,N}(s)$ uniformly converge to $f'_{q^\alpha,\beta}(s)$ and $f_{q^\alpha,\beta}(s)$ respectively, as $N \to +\infty$. Hence we have $f_{q^\alpha,\beta}(0) = f'_{q,^\alpha\beta}(0) = 0$. We also have $\lim_{s\to+\infty} f_{q^\alpha,\beta}(s) = \lim_{s\to+\infty} f'_{q^\alpha,\beta}(s) = 0$ because of the integrability of $f_{q^\alpha,\beta}$ and $f'_{q^\alpha,\beta}$.

We now prove part (ii). Since the convolution of logconcave functions is also logconcave, we know $f_{q^\alpha,\beta,N}(s)$ is logconcave for all q^α, β, N. Then by the convergence of $f_{q^\alpha,\beta,N}(s)$ as $N \to \infty$ and the continuity of $f_{q^\alpha,\beta}(s)$, we have $f_{q^\alpha,\beta}(s)$ is logconcave.

We now prove part (iii). It is known that likelihood ratio ordering ($\ge_{lr}$) is preserved under the convolution of log-concave random variables. Hence, for $0 < q_1 < q_2$, we have $f_{q_1^\alpha,\beta}(s)/f_{q_2^\alpha,\beta}(s)$ is increasing in s. It follows that $[\ln f_{q^\alpha,\beta}(s)]'$ is decreasing in $q\in(0,+\infty)$. Similarly, we can deduce that $[\ln f_{q^\alpha,\beta}(s)]'$ is decreasing in β. Since the monotone hazard rate property follows from the monotone likelihood ratios property, we obtain the desired result.

Finally, we prove part (iv). By the definition of $f_{q,\beta,N}(s)$, we can deduce that

$$\begin{aligned}\frac{\partial f_{q^\alpha,\beta,N}(s)}{\partial q} &= \frac{\alpha}{q}(sf_{q^\alpha,\beta,N}(s))',\\ \frac{\partial^2 f_{q^\alpha,\beta,N}(s)}{\partial q^2} &= \frac{\alpha(\alpha-1)}{q^2}(sf_{q^\alpha,\beta,N}(s))' + \frac{\alpha^2}{q^2}(s^2 f'_{q^\alpha,\beta,N}(s))',\\ \frac{\partial \bar{F}_{q^\alpha,\beta,N}(s)}{\partial q} &= -\frac{\alpha}{q}sf_{q^\alpha,\beta,N}(s).\end{aligned}$$

Let $N\to+\infty$, and then this part is proved. □

Theorem A.6. *Suppose Y_1 and Y_2 are two random variables independent to each other with densities $f_{\beta_1}(\cdot)$ and $f_{\beta_2}(\cdot)$. If $1<\beta_1<\beta_2$, then Y_1 is smaller than Y_2 in the star order (denoted by $Y_1\le_* Y_2$), i.e., $F_{\beta_1}^{-1}(y)/F_{\beta_2}^{-1}(y)$ is decreasing in y for $y\in(0,1)$.*

Proof of Theorem A.6. From Page 214 in Shaked and Shanthikumar (2007), we can see that $Y_1\le_* Y_2$ if and only if, for all $b>0$, $F_{\beta_1}(x)-F_{\beta_2}(bx)$ changes sign at most once, and if the change of signs occurs, it is in the order "−, +" as x traverses from 0 to $+\infty$. Note that $F_{\beta_2}(bx)$ is the CDF of Y_2/b.

First, consider the case $b\ge 1$. Recall that Y_1 and Y_2/b are two sums of infinite exponentials with rates $\{1,2^{\beta_1},3^{\beta_1},\dots\}$ and $\{b,b\cdot 2^{\beta_2},b\cdot 3^{\beta_2},\dots\}$. We know that an exponential r.v. with rate λ_1 is larger than another exponential r.v. with rate λ_2 in the likelihood ratio order if they are independent of each other and $0<\lambda_1<\lambda_2$. We also know that likelihood ratio ordering is preserved under the convolution of log-concave random variables. Hence, we have Y_1 is larger than Y_2 in the likelihood ratio order and it follows that $F_{\beta_1}(x)<F_{\beta_2}(bx)$ for $x\in(0,+\infty)$. There is no crossing as required.

Now we consider the case $0<b<1$. To prove this case, we need some preliminary results. Suppose $Z_1,Z_2,\dots,Z_N$ are i.i.d exponential random variable with rate 1. Define $Y=\theta_1Z_1+\dots+\theta_NZ_N$, and its CDF is denoted by $F^\theta(\cdot)$. Then we have the following theorem.

Theorem A.7. *(Alternative version of Proposition 2 and Theorem 4 in Yu 2017) Suppose $0<\theta_1\le\dots\le\theta_N$ and $0<\eta_1\le\dots\le\eta_N$ $(N\ge 2)$ and there exists $1\le k_1\le k_2\le N$ such that $\theta_i\le\eta_i$ for $i\le k_1$, $\theta_i=\eta_i$ for $k_1<i<k_2$ and $\theta_i\ge\eta_i$ for $i\ge k_2$. (a) If $\theta_N>\eta_N$ and $\prod_{i=1}^N\eta_i>\prod_{i=1}^N\theta_i$, then $F^\eta(s)$ crosses $F^\theta(s)$ exactly once, and from below, as s increases from 0 to $+\infty$. (b) If $\prod_{i=1}^N\eta_i\le\prod_{i=1}^N\theta_i$, then $F^\eta(s)\ge F^\theta(s)$ for $s\in(0,+\infty)$.*

Now we continue to prove Theorem A.6 by using Theorem A.7. Let us define $\eta = (1/N^{\beta_1}, 1/(N-1)^{\beta_1}, \ldots, 1/2^{\beta_1}, 1)$ and $\theta = (1/(bN^{\beta_2}), 1/(b(N-1)^{\beta_2}), \ldots, 1/(b2^{\beta_2}), 1/b)$. Then we know $F_{\beta_1,N}(s) = F^{\eta}(s)$ and $F_{\beta_2,N}(bs) = F^{\theta}(s)$ for all $s \in (0,+\infty)$. One can check that for any $b \in (0,1)$ and for any $N \geq 1$, θ and η satisfy either condition (a) or condition (b) in Theorem A.7, and hence $F_{\beta_1,N}(s)$ crosses $F_{\beta_2,N}(bs)$ at most once on $(0,+\infty)$, and always from below.

Let $N \to +\infty$, and then we deduce that for $0 < b < 1$, $F_{\beta_1}(s) - F_{\beta_2}(bs)$ changes sign at most once, and if the change of signs occurs, it is in the order "$-,+$" as s traverses from 0 to $+\infty$, as required. □

B. Proofs

There are a few symbols that occur very often in the following proofs. Specifically, $f_{q^\alpha,\beta}(\cdot)$ is the density of the sum of infinite exponentials with rates $\{q^\alpha, q^\alpha \cdot 2^\beta, q^\alpha \cdot 3^\beta, \ldots\}$; $f_\beta(\cdot) = f_{1,\beta}(\cdot)$. Then $F_{q^\alpha,\beta}(\cdot)$ (resp., $\bar{F}_{q^\alpha,\beta}(\cdot)$), $F_\beta(\cdot)$ (resp., $\bar{F}_\beta(\cdot)$) are corresponding cumulative distribution functions (resp., reliability functions).

Proof of Lemma 1. The proofs for parts (i) and (ii) can be referenced from Theorem 5.1 in Maldonado et al. 2018, Theorem 3 in Hentenryck et al. 2016 and Theorem 2.1 in Chung et al. 2003. Below, we prove them independently.

As mentioned in Appendix A, the stochastic approximation process $\mathbf{x}(t)$ converges with probability 1 either to a point of B or to the boundary of one of its connected components. That is, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*(\mathbf{q}^\alpha, \beta)$ exists almost surely, that may be a single point or a random variable. Next, we will give the explicit form of $\mathbf{x}^*(\mathbf{q}^\alpha, \beta)$ under three cases.

For any β, $0 \leq \beta < 1$, the fixed points set $B = \bigcup_{i=0}^{2} B_i$ (see Section A.1). To prove this part, we resort to a continuous dynamic process:

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{\Phi}(\hat{\mathbf{x}}(t)). \tag{B.1}$$

By Proposition 4.1 in Benaïm (1999), the interpolation of the discrete process $\mathbf{x}(t)$ is almost surely an asymptotic pseudo trajectory of the flow induced by the associated ODE (B.1). Thus, if the continuous flow $\hat{\mathbf{x}}(t)$ defined by (B.1) converges to B_0 for any initial point $\mathbf{x}(0)$ in the interior of S, then the discrete process $\mathbf{x}(t)$ defined by (2) converges to B_0 almost surely, because the discrete process $\mathbf{x}(t)$ with initial point $\mathbf{x}(0) = \{\frac{1}{n}, \ldots \frac{1}{n}\}$ cannot reach the boundary of S in an arbitrarily finite time. The convergence of the flow $\hat{\mathbf{x}}(t)$ has been proved according to the proof of Theorem 5.3 in Maldonado et al. (2018). (They verified the convergence in the case $0 \leq \beta < 1$, but their proof can be extended to $-\infty < \beta < 1$).

For $\beta = 1$, the fixed points set $B = B_1 \cup B_3$, where each connected component of B_1 is an isolated point, and each connected component of B_3 is a convex hull of some vertices of S. Define

$L=\{i: q_i \text{ is the largest entry of } \mathbf{q}\}$ and $B_L=\{\mathbf{x}\in S: \sum_{i\in L} x_i=1, \sum_{i\notin L} x_i=0\}$. Note that $B_L\subset B_3$. We prove $\mathbf{x}(t)$ converges to B_L almost surely in the following.

We first prove that for any point $\mathbf{e}_i\in B_1$ and $i\notin L$, $\mathbb{P}\{\mathbf{x}(t)\to\mathbf{e}_i\}=0$. Without loss of generality, we assume $1\notin L$ and we will prove $\mathbb{P}\{\mathbf{x}(t)\to\mathbf{e}_1\}=0$. Let $T_{\mathbf{x}}$ be the *Jacobian* of $\mathbf{\Phi}$ at $\mathbf{x}$. At the point $\mathbf{x}=\mathbf{e}_1$, we have

$$T_{\mathbf{e}_1}=\begin{bmatrix} 0 & -\frac{q_2^\alpha}{q_1^\alpha} & -\frac{q_3^\alpha}{q_1^\alpha} & \cdots & -\frac{q_n^\alpha}{q_1^\alpha} \\ 0 & \frac{q_2^\alpha}{q_1^\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{q_3^\alpha}{q_1^\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{q_n^\alpha}{q_1^\alpha} \end{bmatrix} - I_n,$$

where I_n is an identity matrix of order n. From the diagonal of $T_{\mathbf{e}_1}$, it can be seen that all eigenvalues of $T_{\mathbf{e}_1}$ are $\{-1,\frac{q_2^\alpha}{q_1^\alpha}-1,\ldots,\frac{q_n^\alpha}{q_1^\alpha}-1\}$ in which at least one element is positive. Thus $\mathbf{e}_1$ is linearly unstable and, by Theorem A.2 which derived from Theorem 1 in Pemantle (1990), $\mathbb{P}\{\mathbf{x}(t)\to\mathbf{e}_1\}=0$. Since B_1 contains finite points, we have $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)\in B_1\backslash B_L\}=0$.

Then we prove that the discrete process $\mathbf{x}(t)$ converges to $B_3\backslash B_L$ with probability 0 if $\beta=1$. Note that, in the light of Theorem A.2, we can prove $\mathbb{P}\{\mathbf{x}(t)\to\mathbf{x}\}=0$, $\forall\mathbf{x}\in B_3\backslash B_L$, but we want to prove that $\mathbf{x}(t)$ with probability 0 converges to $B\backslash B_L$ which contains uncountable number of points. To this end, we arbitrarily choose a connected component of $B_3\backslash B_L$, say the convex hull of $\{\mathbf{e}_1,\mathbf{e}_2\}$ without loss of generality, denoted by $\mathbf{conv}\{\mathbf{e}_1,\mathbf{e}_2\}$, and we will show that $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)\in\mathbf{conv}\{\mathbf{e}_1,\mathbf{e}_2\}\}=0$. To this end, we consider a n-order *auxiliary* matrix

$$A=\begin{bmatrix} 1 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-2}\end{bmatrix}\in\mathbb{R}^{n-1\times n}$$

which maps $(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n$ into $(x_1+x_2,x_3,\ldots,x_n)^T\in\mathbb{R}^{n-1}$ and introduce an *auxiliary* process based on (A.1)

$$A\mathbf{x}(t+1)-A\mathbf{x}(t)=\frac{1}{t+1+n}[A\mathbf{\Phi}(\mathbf{x}(t))+A\boldsymbol{\zeta}(t+1)]. \tag{B.2}$$

Thus, to prove $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)\in\mathbf{conv}\{\mathbf{e}_1,\mathbf{e}_2\}\}=0$, it is sufficient to prove $\mathbb{P}\{\lim_{t\to\infty}A\mathbf{x}(t)=(1,0,\ldots,0)^T\}=0$. Consider a new demand process in which firm 1 has amalgamated with firm 2 by adding up their market shares. Denote by $\tilde{\mathbf{x}}(t)=A\mathbf{x}(t)$ the new vector of market shares of the $n-1$ products. As with the definition of $\mathbf{\Phi}(\mathbf{x}(t))$, $\tilde{\mathbf{\Phi}}(\tilde{\mathbf{x}}(t))$ can be defined in this $n-1$-dimensional problem. Also $\tilde{\mathbf{e}}(t)$ and $\tilde{\boldsymbol{\zeta}}(t)$ can be correspondingly defined. Since each product is selected with probability linear in the market share of this product in the case with $\beta=1$ and $q_1=q_2$, it is not hard to see that the process (B.2) is equivalent to

$$\tilde{\mathbf{x}}(t+1)-\tilde{\mathbf{x}}(t)=\frac{1}{t+1+n}[\tilde{\mathbf{\Phi}}(\tilde{\mathbf{x}}(t))+\tilde{\boldsymbol{\zeta}}(t+1)]. \tag{B.3}$$

That is, to see the convergence of the process $A\mathbf{x}(t)$, we only need to focus on the convergence of the process $\tilde{\mathbf{x}}(t)$, defined by (B.3). As with the previous example in which we proved $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{e}_1\}=0$, we can prove $\tilde{\mathbf{x}}(t)$, defined by (B.3), converges to $(1,0,\ldots,0)^T$ in probability 0 by Theorem A.2. Hence, we have $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)\in\mathbf{conv}\{\mathbf{e}_1,\mathbf{e}_2\}\}=0$.

It is also not hard to show that the cumulated perturbations $\sum_{\tau=0}^{t}\frac{1}{\tau+1+n}\boldsymbol{\zeta}(\tau+1)$, where $\boldsymbol{\zeta}(\cdot)$ is the noise in (A.1), form a martingale and converge. It follows that at a sufficiently large time, there is not enough perturbation for the process to make infinitely many “trips” to-and-from between two or more connected components. Since $B_1\cup B_3\backslash B_L$ has a finite number of connected components, $\mathbf{x(t)}$ converges to $B_1\cup B_3\backslash B_L$ with probability 0. Therefore, $\mathbf{x(t)}$ converges with probability 1 to B_L. Given that the demand will be concentrated on the products in the set L, by Theorem 2.1 in Chung et al. (2003), the limit of $\mathbf{x}(t)$ is uniformly distributed on B_L.

We next prove part (iii). For any $\beta>1$, the fixed points set $B=\bigcup_{i=0}^{2}B_i$. Clearly, B is a union of a finite number of isolated points. Following the same procedure in part (ii), we have that all points in $B_0\cup B_2$ are linearly unstable, and all points in B_1 are linearly stable. By Theorems A.1 and A.2, we have $\mathbb{P}\{\lim_{t\to\infty}\mathbf{x}(t)\in B_1\}=1$. Alternatively, we can prove this result by considering the demand process as a branching process (see Section A.2 for details) and using Theorems A.3 and A.4. Furthermore, Theorem A.5 gives the probability that the process $\mathbf{x}(t)$ converges to each point in B_1. □

Proof of Lemma 2. First, we prove part (i). From Lemma 1(i)(ii), we see that for $0\le\beta\le1$, $\mathbb{E}x_i^*(\mathbf{q}^\alpha,\beta)$ is increasing in q_i and decreasing in q_j for any $j\neq i$. Next, we will consider the case with $\beta>1$. We take firm 1 as an example and then have that

$$\begin{aligned}\frac{\partial}{\partial q_1}\mathbb{E}x_1^*(\mathbf{q}^\alpha,\beta)=&\frac{\partial}{\partial q_1}\mathbb{P}\{\mathbf{x}^*(\mathbf{q}^\alpha,\beta)=\mathbf{e}_1\}\\ =&\frac{\partial}{\partial q_1}\int_0^\infty f_{q_1^\alpha,\beta}(s)\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)ds\\ =&\frac{\alpha}{q_1}\int_0^\infty(sf_{q_1^\alpha,\beta}(s))'\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)ds\\ =&-\frac{\alpha}{q_1}\int_0^\infty sf_{q_1^\alpha,\beta}(s)\frac{\partial}{\partial s}\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)ds\\ >&0,\end{aligned}$$

where the third equality uses Lemma A.2(iii), and the fourth equality uses integration by parts. Hence, the expected market share of firm 1 is increasing in its quality level. Using Lemma A.2(iii) again, we can prove that the expected market share of firm 1 is decreasing in its competitors' quality levels.

Then we prove part (ii). We need only to prove the monotonicity of $\mathbb{E}x_1^*(\mathbf{q}^\alpha,\beta)$, as the proof for the monotonicity of $\mathbb{E}x_n^*(\mathbf{q}^\alpha,\beta)$ is a mirror image. From the expression of $x_1^*(\mathbf{q}^\alpha,\beta)$, we can see that $x_1^*(\mathbf{q}^\alpha,\beta)$ is increasing in β for $\beta<1$, if q_1 is the largest among all q_is. Now, we consider the case $\beta>1$.

We have

$$\begin{aligned}\mathbb{E}x_1^*(\mathbf{q}^\alpha,\beta) &= \int_0^\infty f_{q_1^\alpha,\beta}(s)\prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s)\mathrm{d}s = \int_0^\infty f_\beta(s)\prod_{j=2}^n \bar{F}_{(\frac{q_j}{q_1})^\alpha,\beta}(s)\mathrm{d}s \\ &\overset{s=F_\beta^{-1}(y)}{=\!=\!=} \int_0^1 \prod_{j=2}^n (1-F_{(\frac{q_j}{q_1})^\alpha,\beta}(F_\beta^{-1}(y)))\mathrm{d}y \\ &= \int_0^1 \prod_{j=2}^n (1-F_\beta((\frac{q_j}{q_1})^\alpha F_\beta^{-1}(y)))\mathrm{d}y.\end{aligned}$$

Thus, it suffices to prove that $F_\beta((\frac{q_j}{q_1})^\alpha F_\beta^{-1}(y))$ is increasing in β for $j=2,\dots,n$. For $1<\beta_1<\beta_2$ and $j=2,\dots,n$,

$$\begin{aligned}& F_{\beta_1}((\frac{q_j}{q_1})^\alpha F_{\beta_1}^{-1}(y)) \le F_{\beta_2}((\frac{q_j}{q_1})^\alpha F_{\beta_2}^{-1}(y)),\ \forall y\in(0,1) \\ \overset{x=F_{\beta_1}((\frac{q_j}{q_1})^\alpha F_{\beta_1}^{-1}(y))}{\Longleftrightarrow} & F_{\beta_1}^{-1}(x) = (\frac{q_j}{q_1})^\alpha F_{\beta_1}^{-1}(y),\ F_{\beta_2}^{-1}(x) \le (\frac{q_j}{q_1})^\alpha F_{\beta_2}^{-1}(y),\ \forall y\in(0,1) \\ \Leftarrow & \frac{F_{\beta_1}^{-1}(x)}{F_{\beta_2}^{-1}(x)} \ge \frac{F_{\beta_1}^{-1}(y)}{F_{\beta_2}^{-1}(y)},\ \forall x,y \text{ such that } 0<x\le y<1 \\ \Leftarrow & \frac{F_{\beta_1}^{-1}(y)}{F_{\beta_2}^{-1}(y)} \text{ is decreasing in } y \text{ on } (0,1).\end{aligned}$$

Based on the fact stated in Theorem A.6, we have obtained what we required. We can see that the monotonicity is strict if $q_j<q_1$ for some j. □

Before proving Theorems 1 and 2, we need some preliminary results. Since all firms are identical, we take firm 1 as the focal firm. For $\beta>1,\ n\ge 2$, we introduce some symbols as follows:

$$\begin{aligned} y(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta) &= \int_0^\infty \left[f_{q_1^\alpha,\beta}(s)-\alpha(sf_{q_1^\alpha,\beta}(s))'\right]\prod_{j=2}^n \bar{F}_{q_j,\beta}(s)ds, \\ z(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta) &= -\int_0^\infty \left[(1-\alpha)(sf_{q_1^\alpha,\beta}(s))'-\alpha(s^2 f'_{q_1^\alpha,\beta}(s))'\right]\prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s)ds, \\ y(n,\alpha,\beta) &= \frac{1}{n}-\alpha\int_0^\infty (sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds, \\ z(n,\alpha,\beta) &= -\int_0^\infty \left[(1-\alpha)(sf_\beta(s))'-\alpha(s^2 f'_\beta(s))'\right]\bar{F}_\beta^{n-1}(s)ds.\end{aligned}$$

These symbols will be used in the following lemmas, propositions, and theorems. We give the intuitive meaning of these symbols. First, $y(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)=\mathbb{E}x_1^*(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)-\frac{\partial \mathbb{E}x_1^*(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)}{\partial q_1}q_1$. If q_1 satisfies the FOC of firm 1's problem (i.e., $p\frac{\partial \mathbb{E}x_1^*}{\partial q_1}=c$), given $\mathbf{q}_{-1}$, then $\pi_1=p\mathbb{E}x_1^*-cq_1-K=p\mathbb{E}x_1^*-$

$p\frac{\partial \mathbb{E}x_1^*}{\partial q_1}q_1 - K = py(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) - K$. We can also see that $\frac{\partial^2 \pi_1}{\partial q_1^2} = p\frac{\partial^2 \mathbb{E}x_1^*(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta)}{\partial q_1^2} = \frac{p\alpha}{q_1^2} z(q_1^\alpha, \mathbf{q}_{-\mathbf{1}}^\alpha, \beta)$. Consequently, in a symmetric pure-strategy equilibrium, $y(n,\alpha,\beta)$ and $z(n,\alpha,\beta)$ have the corresponding meaning.

The proofs of Theorem 1 and 2 make use of the following lemmas.

Lemma B.1. *For $\beta > 1$,*

(i) $\pi^*(n,\alpha,\beta,K) = p\cdot y(n,\alpha,\beta) - K$ *if* $y(n,\alpha,\beta) \geq 0$;

(ii) $y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta)|_{q_1=\cdots=q_n>0} = y(n,\alpha,\beta)$, $z(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta)|_{q_1=\cdots=q_n>0} = z(n,\alpha,\beta)$;

(iii) $y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) \geq 0$ *implies* $z(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) < 0$, *for any* $\mathbf{q}$ *in the interior of* E*; especially,* $y(n,\alpha,\beta) \geq 0$ *implies* $z(n,\alpha,\beta) < 0$;

(iv) given $\mathbf{q}_{-1}$*, if* $y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) \geq 0$ *at some* q_1*, then* $y((q_1')^\alpha, \mathbf{q}_{-1}^\alpha, \beta) > 0$ *for any* $q_1' > q_1$*; given* $\mathbf{q}_{-1}$*, there exists a threshold such that* π_1 *is concave in* q_1 *if* q_1 *is larger than the threshold; given* $\mathbf{q}_{-1}$*, if* q_1^* *is the optimal decision for firm 1 and* $q_1^* > 0$*, then* q_1^* *must lie in the concave part of* π_1*;*

(v) $y(n,\alpha,\beta) \geq 0$ *implies* $y(n-1,\alpha,\beta) > y(n,\alpha,\beta) > y(n+1,\alpha,\beta)$, $\forall n \geq 2$;

(vi) there exists an integer m *such that* $py(n,\alpha,\beta) \geq K$ *if* $n \leq m$*, and* $py(n,\alpha,\beta) < K$ *otherwise;* $y(n,\alpha,\beta)$ *is decreasing in* n *for* $n \leq m$*; Here,* m *has the same meaning with* $\tilde{n}(\alpha,\beta,K)$ *defined in Theorem 2;*

(vii) $y(n,\alpha,\beta)$ *is increasing in* β, $\forall n \geq 2$.

Proof of Lemma B.1. Parts (i) and (ii) are obvious. We now prove part (iii). From definition of $y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta)$, we have

$$y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) = \int_0^\infty \left[(1-\alpha) - \alpha s \frac{f'_{q_1^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)}\right] f_{q_1^\alpha,\beta}(s) \prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s) ds.$$

By Lemma A.2(ii), we know $\dfrac{\partial}{\partial s}\ln\left(\prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s)\right)$ is negative and decreasing for $s \in (0,+\infty)$, and that $f_{q_1^\alpha,\beta}(s)$ is first increasing then decreasing on $[0,+\infty)$. We also know $(1-\alpha) - \alpha s f'_{q_1^\alpha,\beta}(s)/f_{q_1^\alpha,\beta}(s)$ is first negative then positive as s varies from 0 to $+\infty$. Suppose $(1-\alpha) - \alpha s f'_{q_1^\alpha,\beta}(s)/f_{q_1^\alpha,\beta}(s)$ is negative on $(0,\epsilon)$ and positive on $(\epsilon,+\infty)$. Since $y(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) \geq 0$ and

$$\begin{aligned} z(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) &= \int_0^\infty \left[(1-\alpha) s f_{q_1^\alpha,\beta}(s) - \alpha s^2 f'_{q_1^\alpha,\beta}(s)\right] \prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s) \frac{\partial}{\partial s}\ln\left(\prod_{j=2}^n \bar{F}_{q_j,\beta}(s)\right) ds \\ &= \int_0^\infty \left[(1-\alpha) - \alpha s \frac{f'_{q_1^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)}\right] s f_{q_1^\alpha,\beta}(s) \prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s) \frac{\partial}{\partial s}\ln\left(\prod_{j=2}^n \bar{F}_{q_j,\beta}(s)\right) ds, \end{aligned} \tag{B.4}$$

by folowing Mean Value Theorem for Integrals, we have that there exist $s_1 < s_2$ such that

$$z(q_1^\alpha, \mathbf{q}_{-1}^\alpha, \beta) = \int_0^\epsilon \left[(1-\alpha) s f_{q_1^\alpha,\beta}(s) - \alpha s^2 f'_{q_1^\alpha,\beta}(s)\right] \prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s) \frac{\partial}{\partial s}\ln\left(\prod_{j=2}^n \bar{F}_{q_j,\beta}(s)\right) ds$$

$$+\int_{\epsilon}^{\infty}\left[(1-\alpha)sf_{q_1^\alpha,\beta}(s)-\alpha s^2 f'_{q_1^\alpha,\beta}(s)\right]\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)\frac{\partial}{\partial s}\ln\left(\prod_{j=2}^{n}\bar{F}_{q_j,\beta}(s)\right)ds$$

$$=s_1\frac{\partial}{\partial s}\ln\left(\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)\right)\Big|_{s=s_1}\int_0^{\epsilon}\left[(1-\alpha)-\alpha s\frac{f'_{q_1^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)}\right]f_{q_1^\alpha,\beta}(s)\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)ds$$

$$+s_2\frac{\partial}{\partial s}\ln\left(\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)\right)\Big|_{s=s_2}\int_{\epsilon}^{\infty}\left[(1-\alpha)-\alpha s\frac{f'_{q_1^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)}\right]f_{q_1^\alpha,\beta}(s)\prod_{j=2}^{n}\bar{F}_{q_j^\alpha,\beta}(s)ds$$

$$<0.$$

Analogously, we can prove that $y(n,\alpha,\beta)\geq 0$ implies $z(n,\alpha,\beta)<0$.

We now prove part (iv). By part (iii), we have that if $y(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)\geq 0$, then $\mathbb{E}x_1^*(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)$ is concave in q_1. We also know that if $\mathbb{E}x_1^*(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)$ is concave in q_1, then $\frac{\partial y(q_1^\alpha,\mathbf{q}_{-1}^\alpha,\beta)}{\partial q_1}=-\frac{\partial^2\mathbb{E}x_1^*}{\partial q_1^2}q_1>0$. According to these, we can conclude that the first and second statements in Part (iv) hold. Since the equilibrium quality q_1^* satisfies $y((q_1^*)^\alpha,\mathbf{q}_{-1}^\alpha,\beta)\geq 0$, it follows that the third statement is true.

We now prove part (v). Since $\int_0^\infty f_\beta(s)\bar{F}_\beta^{n-1}(s)ds=1/n$, we have

$$y(n,\alpha,\beta)=\int_0^\infty\left[(1-\alpha)-\alpha s\frac{f'_\beta(s)}{f_\beta(s)}\right]f_\beta(s)\bar{F}_\beta^{n-1}(s)ds.$$

Retracing the proof of part (iv), we can deduce that there exist $s_1<s_2$ such that

$$y(n,\alpha,\beta)-y(n+1,\alpha,\beta)=F_\beta(s_1)\int_0^{\epsilon}\left[(1-\alpha)-\alpha s\frac{f'_\beta(s)}{f_\beta(s)}\right]f_\beta(s)\bar{F}_\beta^{n-1}(s)ds$$
$$+F_\beta(s_2)\int_{\epsilon}^{\infty}\left[(1-\alpha)-\alpha s\frac{f'_\beta(s)}{f_\beta(s)}\right]f_\beta(s)\bar{F}_\beta^{n-1}(s)ds>0;$$
$$y(n,\beta)-y(n-1,\beta)=(1-\frac{1}{\bar{F}_\beta(s_1)})\int_0^{\epsilon}\left[(1-\alpha)-\alpha s\frac{f'_\beta(s)}{f_\beta(s)}\right]f_\beta(s)\bar{F}_\beta^{n-1}(s)ds$$
$$+(1-\frac{1}{\bar{F}_\beta(s_2)})\int_0^{\epsilon}\left[(1-\alpha)-\alpha s\frac{f'_\beta(s)}{f_\beta(s)}\right]f_\beta(s)\bar{F}_\beta^{n-1}(s)ds<0.$$

Part (vi) follows immediately from part (v).

Now prove part (vii). Since $\int_0^\infty f_\beta(s)\bar{F}_\beta^{n-1}(s)ds=1/n$, we have

$$y(n,\alpha,\beta)=\frac{1}{n}-\alpha(n-1)\int_0^\infty sf_\beta^2(s)\bar{F}_\beta^{n-2}(s)ds.$$

Then it suffices to prove $\int_0^\infty sf_\beta^2(s)\bar{F}_\beta^{n-2}(s)ds$ is decreasing in β. We also have

$$\int_0^\infty sf_\beta^2(s)\bar{F}_\beta^{n-2}(s)ds\overset{s=F_\beta^{-1}(y)}{=\!=\!=\!=}\int_0^1 F_\beta^{-1}(y)f_\beta(F_\beta^{-1}(y))(1-y)^{n-2}ds.$$

Now, it suffices to prove that for $y\in(0,1)$, $F_\beta^{-1}(y)f_\beta(F_\beta^{-1}(y))$ is decreasing in $\beta\in(1,+\infty)$. For $1<\beta_1<\beta_2$,

$$F_{\beta_1}^{-1}(y)f_{\beta_1}(F_{\beta_1}^{-1}(y))>F_{\beta_2}^{-1}(y)f_{\beta_2}(F_{\beta_2}^{-1}(y)),\ \forall y\in(0,1)$$

$$\Leftarrow (\ln F_{\beta_1}^{-1}(y))' < (\ln F_{\beta_2}^{-1}(y))', \ \forall y \in (0,1)$$
$$\Leftarrow \left(\ln \frac{F_{\beta_1}^{-1}(y)}{F_{\beta_2}^{-1}(y)}\right)' < 0, \ \forall y \in (0,1)$$
$$\Leftarrow \frac{F_{\beta_1}^{-1}(y)}{F_{\beta_2}^{-1}(y)} \text{ is decreasing in } y \in (0,1),$$

which together with the fact in Theorem A.6, proves this part.

LEMMA B.2. *For $i = 1, \ldots, n$, let $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$ be a solution to the first-order condition (FOC) of firm i's problem and satisfy the second-order condition (SOC), given $\mathbf{q}_{-i}$. For $n \geq 2$, $\beta \in (-\infty, +\infty)$, and $\mathbf{q}_{-i} > \mathbf{0}$, if $q_j \geq q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$ for some $j \neq i$, then $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$ is decreasing at q_j.*

Proof of Lemma B.2. If $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) = 0$, which means $\pi_i \leq 0$ for any positive q_i, then decreasing q_j can increase π and hence can increase the incentive for firm i to offer a positive quality. The lemma holds for this case. Therefore, in the following, we focus on the case $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$.

First, consider the case $\beta < 1$. For the simplicity of notation, we write $x_i^*(\mathbf{q}^\alpha, \beta)$ as x_i^*, and $\pi_i(q_i, \mathbf{q}_{-i}, \alpha, \beta, K)$ as π_i . Since $x_i^* = q_i^{\frac{\alpha}{1-\beta}} / \sum_{i=1}^n q_i^{\frac{\alpha}{1-\beta}}$ and $\pi_i = p x_i^* - c q_i$, we have

$$\frac{\partial \pi_i}{\partial q_i} = \frac{p\alpha}{q_i(1-\beta)} x_i^*(1 - x_i^*) - c,$$
$$\frac{\partial^2 \pi_i}{\partial q_i \partial q_j} = -\frac{p\alpha^2}{q_i q_j (1-\beta)^2} x_i^* x_j^* (1 - 2x_i^*),$$
$$\frac{\partial^2 \pi_i}{\partial q_i^2} = \frac{p\alpha}{q_i^2(1-\beta)} x_i^*(1 - x_i^*)[\frac{\alpha}{1-\beta}(1 - 2x_i^*) - 1].$$

We can see that the second-order derivative is negative if and only if $q_i^{\frac{\alpha}{1-\beta}} > \frac{\alpha+\beta-1}{1+\alpha-\beta} \sum_{j \neq i} q_j^{\frac{\alpha}{1-\beta}}$. Hence π_i is first convex then concave for $q_i \in (0, +\infty)$. Thus $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$ and is the local maximum of the function π_i. If $\pi \geq 0$ at $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$, then $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$ is global maximum. Because $q_j \geq q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$, $x_i^* < 1/2$ which means $\frac{\partial^2 \pi_i}{\partial q_i \partial q_j} < 0$, and therefore $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$ decreases as q_j increases.

Note that we do not need to consider the case $\beta = 1$. When $\beta = 1$, only the product with the highest quality can have a nonzero market share. Therefore, it is impossible for $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)0$ to be a local maximum if $q_j \geq q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$.

Then consider the case $\beta > 1$. Without loss of generality, we take $i = 1$ and $j = 2$. For $\beta > 1$, we have

$$\frac{\partial \pi_1}{\partial q_1} = \frac{p\alpha}{q_1} \int_0^\infty (s f_{q_1^\alpha, \beta}(s))' \prod_{j=2}^n \bar{F}_{q_j^\alpha, \beta}(s) ds - c,$$
$$\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} = -\frac{p\alpha^2}{q_1 q_2} \int_0^\infty (s f_{q_1^\alpha, \beta}(s))' (s f_{q_2^\alpha, \beta}(s)) \prod_{j=3}^n \bar{F}_{q_j^\alpha, \beta}(s) \mathrm{d}s,$$
$$\frac{\partial^2 \pi_1}{\partial q_1^2} = \frac{p\alpha}{q_1^2} z(q_1^\alpha, \mathbf{q}_{-\mathbf{1}}^\alpha, \beta),$$

where $z(q_1^\alpha, \mathbf{q}_{-\mathbf{1}}^\alpha, \beta)$ is defined above Lemma B.1. By Lemma B.1 parts (iii) and (iv), we have that $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$ and $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K)$ must be the local maximum of the function π_i. If $\pi \geq 0$ at $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$, then $q_i^*(\mathbf{q}_{-i}, \alpha, \beta, K) > 0$ is global maximum. To complete the proof, it suffices to prove $\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} < 0$ if $q_2 \geq q_1^*(\mathbf{q}_{-1}, \alpha, \beta, K) > 0$. Since $\frac{f_{q_2^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)}$ is decreasing in s if $q_2 \geq q_1^*(\mathbf{q}_{-1}, \alpha, \beta, K) > 0$, it follows that

$$\begin{aligned}
\frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} &= -\frac{p\alpha^2}{2q_1 q_2} \int_0^\infty (s^2 f^2_{q_1^\alpha,\beta}(s))' \frac{f_{q_2^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)} \prod_{j=3}^n \bar{F}_{q_j^\alpha,\beta}(s) \mathrm{d}s, \\
&= \frac{p\alpha^2}{2q_1 q_2} \int_0^\infty s^2 f^2_{q_1^\alpha,\beta}(s) \frac{\partial}{\partial s} \left(\frac{f_{q_2^\alpha,\beta}(s)}{f_{q_1^\alpha,\beta}(s)} \prod_{j=3}^n \bar{F}_{q_j^\alpha,\beta}(s) \right) \mathrm{d}s < 0,
\end{aligned}$$

as required.

Proof of Theorem 1. For the simplicity of notation, we write $x_i^*(\mathbf{q}^\alpha, \beta)$ as x_i^*, and $\pi_i(q_i, \mathbf{q}_{-i}, \alpha, \beta, K)$ as π_i . We only prove parts (ii) and (iii) since part (i) is trivial.

For $0 \leq \beta < 1$ and $n \geq 2$, $x_i^* = q_i^{\frac{\alpha}{1-\beta}} / \sum_{i=1}^n q_i^{\frac{\alpha}{1-\beta}}$, $\pi_i = px_i^* - cq_i$. For $i = 1, \ldots, n$, the first-order derivative (FOD) and the second-order derivative (SOD) of firm i's problem are given by

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i} &= \frac{p\alpha}{q_i(1-\beta)} x_i^*(1-x_i^*) - c, \\
\frac{\partial^2 \pi_i}{\partial q_i^2} &= \frac{p\alpha}{q_i^2(1-\beta)} x_i^*(1-x_i^*)[\frac{\alpha}{1-\beta}(1-2x_i^*) - 1],
\end{aligned}$$

which is negative if and only if $q_i^{\frac{\alpha}{1-\beta}} > \frac{\alpha+\beta-1}{1+\alpha-\beta} \sum_{j \neq i} q_j^{\frac{\alpha}{1-\beta}}$. Hence π_i is first convex then concave for $q_i \in (0, +\infty)$.

For $0 \leq \beta < 1$ and $n \geq 2$, it can be seen from the FOCs for all firms' problems that a symmetric equilibrium of $\mathcal{G}$, if it exists, must be $\mathbf{q} = \dfrac{(n-1)p\alpha}{n^2 c(1-\beta)} \cdot (1, \ldots, 1)$ where $n \geq 2$. Thus $\mathbf{q}$ is a symmetric equilibrium of $\mathcal{G}$ if and only if it satisfies the following two conditions: (a) Each firm's profit at $\mathbf{q}$ is nonnegative; (b) The SOCs hold for each firm's problem. More precisely, these two conditions can be written as

$$\begin{aligned}
\pi_i &= p\frac{1}{n} - c\frac{(n-1)p\alpha}{n^2 c(1-\beta)} - K \geq 0, \ i = 1, \ldots, n, \\
\frac{\partial^2 \pi_i}{\partial q_i^2} &= \frac{c}{q_i}[\frac{\alpha}{1-\beta}(1-\frac{2}{n}) - 1] < 0, \ i = 1, \ldots, n.
\end{aligned}$$

We can find that $\pi_i \geq -K$ implies $\frac{\partial^2 \pi_i}{\partial q_i^2} < 0$, and that $\pi_i \geq 0$ if and only if $n \leq \frac{2p\alpha}{\sqrt{(1-\alpha-\beta)^2 p^2 + 4p\alpha K(1-\beta)} - p(1-\alpha-\beta)}$. Define $\tilde{n}(\alpha, \beta, K) = \lfloor \frac{2p\alpha}{\sqrt{(1-\alpha-\beta)^2 p^2 + 4p\alpha K(1-\beta)} - p(1-\alpha-\beta)} \rfloor$ if $K > 0$ or $\alpha + \beta > 1$ and $\tilde{n}(\alpha, \beta, K) = +\infty$ otherwise. Thus we have that if $2 \leq n \leq \tilde{n}(\alpha, \beta, K)$, $\mathbf{q}$ must be an equilibrium of $\mathcal{G}$, and $\pi^* \geq 0$ if and only if $n \leq \tilde{n}(\alpha, \beta, K)$; if $n > \tilde{n}(\alpha, \beta, K)$, then there is no symmetric pure-strategy equilibrium since at least one of conditions (a) and (b) does not hold at the point $\mathbf{q}$.

For $\beta = 1$ and $n \geq 2$, only the firms whose qualities are largest among the n firms can have positive profits. Thus, any firm has an incentive to increase its quality in order to win the whole market (as in Bertrand competition) unless its current quality is $(p-K)/c$, at which the production cost is sufficiently high such that the firm always gets a nonpositive profit even if he wins the whole market. When all firms offer the quality $(p-K)/c$, we can see that all wish to deviate and choose the quality of 0. Define $\tilde{n}(\alpha,1,K) = 1$ for all $0 \leq K < p$ and $\tilde{n}(\alpha,1,K) = 0$ otherwise. Thus if $n > \tilde{n}(\alpha,1,K)$, no pure-strategy equilibrium exists.

We next prove in any asymmetric pure-strategy equilibrium, at most $\tilde{n}(\alpha,\beta,K)$ firms offer nonzero-quality products. Suppose there exists an asymmetric pure-strategy equilibrium in which there are $n(> \tilde{n}(\alpha,\beta,K))$ firms offering nonzero quality levels and obtaining nonnegative expected profit. For $i = 1,\dots,n$, let $q_i^*(\mathbf{q}_{-i},\alpha,\beta,K)$ be firm i's best response to $\mathbf{q}_{-i}$, and q_i^* and π_i^* be firm i's quality and expected profit in the asymmetric equilibrium respectively. Assuming $q_1^* \geq q_2^* \geq \cdots \geq q_n^* > 0$ without loss of generality, it follows from the asymmetry that $q_1^* > q_n^* > 0$. Since the positive equilibrium quality vector must satisfy FOCs, by Lemma B.2, we have that q_n^* increases as q_1 decreases. By Lemma 2, we also have π_n^* in creases as q_1 decreases. Consequently, there must be a quality level q^* such that $q_i = q^*$ for $i = 1,\dots,n$ is a symmetric equilibrium, which contradicts to the fact that there is no pure-strategy equilibrium if $n > \tilde{n}(\alpha,\beta,K)$.

For $\beta = 1$, we know that $\tilde{n}(\alpha,1,K) \leq 1$. If $n > \tilde{n}(\alpha,1,K)$, then there is no (symmetric or asymmetric) pure-strategy equilibrium. Therefore, the theorem still holds. □

Proof of Theorem 2. In this proof, shortened forms of notation have the same meaning in the proof of Theorem 1. We need only to prove parts (ii) and (iii) since part (i) is trivial.

For $\beta > 1$, $\pi_i = p \cdot \mathbb{P}\{\mathbf{x}^* = \mathbf{e}_i\} - cq_i$. In view of Lemma A.2(iv), in the interior of E, the FOD and SOD of firm 1's problem are given by

$$\frac{\partial \pi_1}{\partial q_1} = \frac{p\alpha}{q_1} \int_0^\infty (s f_{q_1^\alpha,\beta}(s))' \prod_{j=2}^n \bar{F}_{q_j^\alpha,\beta}(s) ds - c,$$
$$\frac{\partial^2 \pi_1}{\partial q_1^2} = \frac{p\alpha}{q_1^2} z(q_1^\alpha, \mathbf{q}_{-\mathbf{1}}^\alpha, \beta),$$

where $z(q_1^\alpha, \mathbf{q}_{-\mathbf{1}}^\alpha, \beta)$ is defined above Lemma B.1. From the FOCs we see that if there exists a symmetric equilibrium then it must be

$$\begin{aligned}\mathbf{q} &= (1,\dots,1) \cdot \frac{p\alpha}{c} \int_0^\infty (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds \\ &= (1,\dots,1) \cdot \frac{p}{c} \left(\frac{1}{n} - y(n,\alpha,\beta) \right),\end{aligned}$$

where $y(n,\alpha,\beta)$ is defined above Lemma B.1, or briefly

$$q^*(n,\alpha,\beta) = \frac{p}{c} \left(\frac{1}{n} - y(n,\alpha,\beta) \right), \tag{B.5}$$

where we use the fact $\int_0^\infty f_{q_1,\beta}(s)\prod_{j=2}^{n}\bar{F}_{q_j,\beta}(s)ds|_{q_1=\cdots=q_n} = \int_0^\infty f_\beta(s)\bar{F}_\beta^{n-1}(s)ds = \frac{1}{n}$. Thus $\mathbf{q}$ is a symmetric equilibrium of $\mathcal{G}$ if and only if it satisfies $\pi_i|_{\mathbf{q}} \geq 0$ and $\frac{\partial^2\pi_i}{\partial q_i^2}|_{\mathbf{q}} < 0$, which are equivalent to $y(n,\alpha,\beta) \geq K/p$ and $z(n,\alpha,\beta) < 0$ (see Lemma B.1(i)). By Lemma B.1(iii), we have that $y(n,\alpha,\beta) \geq 0$ implies $z(n,\alpha,\beta) < 0$. Define $\tilde{n}(\alpha,\beta,K) = \max\{n : \pi^*(n,\alpha,\beta,K) \geq 0,\ z(n,\alpha,\beta) < 0\}$=$\max\{n : y(n,\alpha,\beta) \geq K/p\}$. By Lemma B.1(vi), we have that, if $2 \leq n \leq \tilde{n}(\alpha,\beta,K)$, $\mathbf{q}$ must be an equilibrium of $\mathcal{G}$. For any $K \geq 0$ and $n \geq 2$, $\pi^*(n,\alpha,\beta,K) \geq 0$ if and only if $n \leq \tilde{n}(\alpha,\beta,K)$. If $n > \tilde{n}(\alpha,\beta,K)$, there is no symmetric pure-strategy equilibrium.

We can prove Theorem 2(iii) by using the same ideas we used to prove Theorem 1(iii). □

Proof of Proposition 1. Part (i) follows from Lemma B.1 parts (i) and (v). Part (ii) is obvious from Theorem 1(ii) and Theorem 2(ii). Part (iii) follows from Theorem 1(ii). By Equation (B.5) and Lemma B.1(vii), we can also obtain part (iv). □

Proof of Proposition 2. Parts (i) and (ii) follow immediately from the definition of $\tilde{n}(\alpha,\beta,K)$.

By Lemma B.1(vii), we know that $y(n,\alpha,\beta)$ is increasing in β for $\beta > 1$. By Lemma B.1(v), we also have that $y(n,\alpha,\beta)$ is decreasing in n if $\beta > 1$.The two facts imply that $\tilde{n}(\alpha,\beta,K) = \max\{n : y(n,\alpha,\beta) \geq K/p\}$ is increasing in β for $\beta > 1$. □

The proof of Proposition 3 uses the following lemma.

Lemma B.3. *There exists $\tilde{\beta} \geq 1$, such that $\int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds \geq \int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n}(s)ds$ for $\beta > \tilde{\beta}$ and $n \geq 2$.*

Proof of Lemma B.3. Using the techniques in the proof of Lemma B.1(iv), we deduce that $\int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds \geq \int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n}(s)ds$ implies $\int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n}(s)ds \geq \int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n+1}(s)ds$. Hence it suffices to prove

$$\int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta(s)ds \geq \int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^2(s)ds$$

or equivalently,

$$\int_0^{+\infty} sf_\beta^2(s)ds > 2\int_0^{+\infty} sf_\beta^2(s)\bar{F}_\beta(s)ds. \tag{B.6}$$

Since $\lim_{\beta\to+\infty} f_\beta(s) = \lim_{\beta\to+\infty}\bar{F}_\beta(s) = \exp(-s)$, there must be a $\tilde{\beta} \geq 1$ such that (B.6) holds for $\beta > \tilde{\beta}$, which completes the proof. □

Proof of Proposition 3. If there are $\tilde{n}(\alpha,\beta,K)$ firms entering the market, then in the symmetric equilibrium, $q^*(\alpha,\beta) = q^*(\tilde{n}(\alpha,\beta,K),\alpha,\beta)$. We first prove part (i), i.e., the case when $\beta < 1$. Let $U = \{\beta : \tilde{n}(\alpha,\beta,K) \geq 2\}$. If $\beta \in (-\infty,1)\cap U$, we can see from Theorem 1 that $q^*(\alpha,\beta) = \frac{\tilde{n}(\alpha,\beta,K)-1}{\tilde{n}(\alpha,\beta,K)^2}\frac{p\alpha}{c(1-\beta)}$. By using the fact that $\frac{\tilde{n}(\alpha,\beta,K)-1}{\tilde{n}(\alpha,\beta,K)^2}$ is decreasing in $\tilde{n}(\alpha,\beta,K)$ if $\tilde{n}(\alpha,\beta,K) \geq 2$ and the fact that $\tilde{n}(\alpha,\beta,K)$ is decreasing in α and β for $\beta < 1$ (see Proposition 2(ii)), we have that $q^*(\alpha,\beta)$ is increasing in α and $\beta \in (-\infty,1)\cap U$.

Next, we prove part (ii). We have known that

$$q^*(n,\alpha,\beta) = \frac{p\alpha}{c}\int_0^{+\infty}(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds.$$

According to Lemma B.3, we have that there exists a $\tilde{\beta}\geq 1$ such that $q^*(n,\alpha,\beta)$ is decreasing in n if $\beta>\tilde{\beta}$ and $n\geq 2$. When $\beta>1$, we have the fact that $\tilde{n}(\alpha,\beta,K)$ is decreasing in α and increasing in β and the fact that $q^*(n,\alpha,\beta)$ is increasing in α and decreasing in β. Therefore, $q^*(\alpha,\beta)$ is increasing in α and decreasing in $\beta\in(\tilde{\beta},+\infty)\cap U$. □

Proof of Proposition 4. First, we prove part (i). Let firm 1 be the focal firm.

For $\beta<1$, we have

$$\begin{aligned} x_1^*(\mathbf{q}^\alpha\exp(-\mathbf{p}),\beta) &= (q_1^\alpha\exp(-p_1))^{\frac{1}{1-\beta}} / \sum_{i=1}^n (q_i^\alpha\exp(-p_i))^{\frac{1}{1-\beta}}, \\ \pi_1(\mathbf{q}^\alpha\exp(-\mathbf{p}),\beta,K) &= p_1x_1^*(\mathbf{q}^\alpha\exp(-\mathbf{p}),\beta)-cq_1-K. \end{aligned}$$

The FOCs are

$$\begin{aligned} \frac{\partial\pi_1}{\partial p_1} &= x_1^*-\frac{p_1}{1-\beta}x_1^*(1-x_1^*)=0, \\ \frac{\partial\pi_1}{\partial q_1} &= \frac{p_1\alpha}{q_1(1-\beta)}x_1^*(1-x_1^*)-c=0. \end{aligned}$$

In a symmetric equilibrium, we must have $x_1^*=1/n$, and all prices and qualities are the same. Then the FOCs become

$$\frac{1}{n}-p^*\frac{1}{1-\beta}\frac{n-1}{n^2}=0,$$

$$\frac{p^*\alpha}{q^*(1-\beta)}\frac{n-1}{n^2}-c=0.$$

Accordingly, we can obtain the equilibrium results:

$$p^*(n,\alpha,\beta)=\frac{n(1-\beta)}{n-1},\ q^*(n,\alpha,\beta)=\frac{\alpha}{nc}.$$

We also need to make sure that each firm earns a nonnegative profit in the equilibrium, that is,

$$\frac{1}{n}p^*(n,\alpha,\beta)-cq^*(n,\alpha,\beta)-K\geq 0,$$

which is equivalent to

$$n\leq\tilde{n}(\alpha,\beta,K)\triangleq\begin{cases}\lfloor\frac{2\alpha}{\sqrt{(K+1-\alpha-\beta)^2+4\alpha K}-(K+1-\alpha-\beta)}\rfloor, & \text{if } K>0 \text{ or } \alpha+\beta>1+K,\\ +\infty, & \text{otherwise.}\end{cases}$$

Therefore, $p^*(n,\alpha,\beta)$ and $q^*(n,\alpha,\beta)$ are equilibrium results if $2\leq n\leq\tilde{n}(\alpha,\beta,K)$.

For $\beta > 1$, we have

$$\mathbb{E}x_1^*(\mathbf{q}^\alpha \exp(-\mathbf{p}), \beta) = \int_0^{+\infty} f_{q_1^\alpha \exp(-p_1),\beta}(s) \prod_{j=2}^n \bar{F}_{q_j^\alpha \exp(-p_j),\beta}(s) ds,$$
$$\pi_1(\mathbf{q}^\alpha \exp(-\mathbf{p}), \beta, K) = p_1 \mathbb{E}x_1^*(\mathbf{q}^\alpha \exp(-\mathbf{p}), \beta) - cq_1 - K.$$

The FOCs are

$$\frac{\partial \pi_1}{\partial p_1} = \mathbb{E}x_1^* - p_1 \int_0^{+\infty} \left(s f_{q_1^\alpha \exp(-p_1),\beta}(s)\right)' \prod_{j=2}^n \bar{F}_{q_j^\alpha \exp(-p_j),\beta}(s) ds = 0,$$
$$\frac{\partial \pi_1}{\partial q_1} = \frac{p_1 \alpha}{q_1} \int_0^{+\infty} \left(s f_{q_1^\alpha \exp(p_1),\beta}(s)\right)' \prod_{j=2}^n \bar{F}_{q_j^\alpha \exp(-p_j),\beta}(s) ds - c = 0.$$

In a symmetric equilibrium, we must have $\mathbb{E}x_1^* = 1/n$, and all prices and qualities are the same. Then FOCs become

$$\frac{1}{n} - p^* \int_0^{+\infty} (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds = 0,$$
$$\frac{p^* \alpha}{q^*} \int_0^{+\infty} (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds - c = 0.$$

Accordingly, we can obtain the equilibrium results:

$$p^*(n, \alpha, \beta) = \frac{1}{n \int_0^{+\infty} (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds}, \; q^*(n, \alpha, \beta) = \frac{\alpha}{nc}.$$

We still need to make sure that each firm earns a nonnegative profit in the equilibrium, that is,

$$\frac{1}{n} p^*(n, \alpha, \beta) - cq^*(n, \alpha, \beta) - K \geq 0,$$

which is equivalent to

$$n \leq \tilde{n}(\alpha, \beta, K) \triangleq \max \left\{ n \in \mathbb{N} : \int_0^{+\infty} (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds \leq \frac{1}{n(nK + \alpha)} \right\}.$$

Therefore, $p^*(n, \alpha, \beta)$ and $q^*(n, \alpha, \beta)$ are equilibrium results if $2 \leq n \leq \tilde{n}(\alpha, \beta, K)$.

Parts (ii) and (iii) are obvious because $\int_0^{+\infty} (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) ds$ is decreasing in β for $\beta > 1$, which we have shown in the proof of Proposition 1. □

Proof of Proposition 5. All parts follow directly from the definition of $\tilde{n}(\alpha, \beta, K)$. □

Proof of Proposition 6. All parts follow directly from Propositions 4 and 5. □

Proof of Theorem 3. First, we prove parts (i) and (ii). Let firm 1 be the focal firm. Consider $\beta < 1$. We have

$$x_1^*(\mathbf{m}, \mathbf{q}^\alpha, \beta) = m_1 q_1^{\frac{\alpha}{1-\beta}} / \sum_{i=1}^n m_i q_i^{\frac{\alpha}{1-\beta}},$$
$$\pi_1(\mathbf{m}, \mathbf{q}^\alpha, \beta, K) = p x_1^*(\mathbf{m}, \mathbf{q}^\alpha, \beta) - m_1(cq_1 + K).$$

The FODs of firm 1's problem are

$$\frac{\partial \pi_1}{\partial q_1} = \frac{p\alpha}{q_1(1-\beta)} x_1^*(1-x_1^*) - cm_1,$$
$$\frac{\partial \pi_1}{\partial m_1} = \frac{p}{m_1} x_1^*(1-x_1^*) - cq_1 - K.$$

By the first FOC and the symmetry, we have $m^* q^* = \frac{p\alpha(n-1)}{cn^2(1-\beta)}$. Substituting it into the second FOD, since $m_i \geq 1$, $\forall i$, we have

$$m^* = \max\left\{\frac{p(n-1)(1-\alpha-\beta)}{Kn^2(1-\beta)}, 1\right\}.$$

To make sure $(\mathbf{m}^*, \mathbf{q}^*)$ is an equilibrium, we must have that each firm earns a nonnegative profit at this point. If $\frac{n-1}{n^2} \geq \frac{K(1-\beta)}{p(1-\alpha-\beta)}$, that is, $m^* = \frac{p(n-1)(1-\alpha-\beta)}{Kn^2(1-\beta)}$, firm 1's profit at $(\mathbf{m}^*, \mathbf{q}^*)$ is

$$\pi_1^* = \frac{p}{n} - m^*(cq^* + K) = \frac{p}{n^2},$$

which is always positive. If $m^* = 1$, which implies $q^* = \frac{p\alpha(n-1)}{cn^2(1-\beta)}$, firm 1's profit at $(\mathbf{m}^*, \mathbf{q}^*)$ is

$$\pi_1^* = \frac{p}{n} - \frac{p\alpha(n-1)}{n^2(1-\beta)} - K,$$

which is the same as in the base model. Therefore, $\tilde{n}(\alpha, \beta, K)$ can be defined as the same as in Theorem 1, and part (i) holds immediately.

When $\beta = 1$, we clearly have $\tilde{n}(\alpha, \beta, K) = m^*(n, \alpha, \beta, K) = 1$ and hence parts (i) still holds.

From the result in part (i), we see that the main idea of the proof for part (ii) is the same as that for Theorem 1(iii). □

Proof of Theorem 4. First, we prove parts (i) and (ii). Let firm 1 be the focal firm. For $\beta > 1$,

$$\mathbb{E}x_1^*(\mathbf{m}, \mathbf{q}^\alpha, \beta) = \int_0^{+\infty} m_1 f_{q_1^\alpha, \beta}(s) \bar{F}_{q_1^\alpha, \beta}^{m_1 - 1}(s) \prod_{j=2}^n \bar{F}_{q_j^\alpha, \beta}^{m_j}(s) ds,$$
$$\pi_1(\mathbf{m}, \mathbf{q}^\alpha, \beta, K) = p\mathbb{E}x_1^*(\mathbf{m}, \mathbf{q}^\alpha, \beta) - m_1(cq_1 + K).$$

The FODs of firm 1's problem are

$$\frac{\partial \pi_1}{\partial q_1} = \frac{p\alpha}{q_1} \int_0^{+\infty} m_1 \big(s f_{q_1^\alpha, \beta}(s) \bar{F}_{q_1^\alpha, \beta}^{m_1 - 1}(s)\big)' \prod_{j=2}^n \bar{F}_{q_j^\alpha, \beta}^{m_j}(s) ds - cm_1,$$
$$\frac{\partial \pi_1}{\partial m_1} = -p \int_0^{+\infty} \big(\bar{F}_{q_1^\alpha, \beta}^{m_1}(s) \ln \bar{F}_{q_1^\alpha, \beta}^{m_1}(s)\big)' \prod_{j=2}^n \bar{F}_{q_j^\alpha, \beta}^{m_j}(s) ds - cq_1 - K.$$

By the first FOC and the symmetry, we have

$$q^* = \frac{p\alpha}{c} \int_0^{+\infty} (s f_\beta(s) \bar{F}_\beta^{m^*-1}(s))' \bar{F}_\beta^{(n-1)m^*}(s) ds. \tag{B.7}$$

Substituting it into the second FOD, by symmetry, we have that if $m^* > 1$, m^* must satisfy

$$-p \int_0^{+\infty} \big(\bar{F}_\beta^{m^*}(s) \ln \bar{F}_\beta^{m^*}(s)\big)' \bar{F}_\beta^{(n-1)m^*}(s) ds - p\alpha \int_0^{+\infty} (s f_\beta(s) \bar{F}_\beta^{m^*-1}(s))' \bar{F}_\beta^{(n-1)m^*}(s) ds - K = 0.$$

Since

$$-\int_0^{+\infty}\left(\bar{F}_\beta^{m^*}(s)\ln\bar{F}_\beta^{m^*}(s)\right)'\bar{F}_\beta^{(n-1)m^*}(s)ds\overset{s=F_\beta^{-1}(y)}{=\!=\!=}\frac{n-1}{n^2m^*},$$

the second FOC becomes

$$\frac{p(n-1)}{n^2}-pm^*\alpha\int_0^{+\infty}(sf_\beta(s)\bar{F}_\beta^{m^*-1}(s))'\bar{F}_\beta^{(n-1)m^*}(s)ds-m^*K=0, \tag{B.8}$$

or equivalently,

$$\frac{p(n-1)}{n^2}=m^*(cq^*+K). \tag{B.9}$$

Obviously, Equation (B.8) has at least one solution. Let M_0 be the set of real roots of Equation (B.8). For each $m_0\in M_0$, we obtain an equilibrium, and in the equilibrium,

$$m^*=\max\{m_0,1\},$$

and q^* can be solved by Equation (B.7). See from Equation (B.9) that, if there are multiple equilibria, at each equilibrium (m^*,q^*), the expected profit for each firm is p/n^2, independent of an equilibrium itself.

To make sure $(\mathbf{m}^*,\mathbf{q}^*)$ is an equilibrium, we must have that each firm earns a nonnegative profit at this point. Based on the above analysis, we need to consider the case $m^*=1$. When $m^*=1$, the equilibrium profit for each firm is the same as in the base model. Therefore, $\tilde{n}(\alpha,\beta,K)$ can be defined as the same as in Theorem 2, and part (i) holds immediately.

Using the result in part (i), we know that the main idea of the proof for part (ii) is the same as that for Theorem 2(iii). □

Proof of Proposition 7. For $\beta\le 1$, parts (i), (ii), and (iii) hold according to the definitions of $m^*(n,\alpha,\beta,K)$ and $q^*(n,\alpha,\beta,K)$. Next, we consider the case $\beta>1$.

We first prove part (i). By the definition of $m^*(n,\alpha,\beta,K)$ and the proof of Theorem 4, we have

$$m^*(n,\alpha,\beta,K)=\max\{m\ge 1:\ \frac{p(n-1)}{n^2}\ge m(cq^*(n,\alpha,\beta,m)+K)\}, \tag{B.10}$$

where $q^*(n,\alpha,\beta,m)$ is defined by Equation (B.7). It is easy to see from Equation (B.10) that $m^*(n,\alpha,\beta,K)$ is decreasing in K. Using the techniques in the proof of Proposition 1, we can obtain that, given m, $q^*(n,\alpha,\beta,m)$ is increasing α. Then we can see from Equation (B.10) that $m^*(n,\alpha,\beta,K)$ is decreasing in α.

We now prove part (ii). See from Equation (B.10) that it suffices to prove $q^*(n,\alpha,\beta,m)$ is decreasing in m. By Equation (B.7), we have

$$q^*(n,\alpha,\beta,m)=\frac{p\alpha}{c}\int_0^{+\infty}(sf_\beta(s)\bar{F}_\beta^{m^*-1}(s))'\bar{F}_\beta^{(n-1)m^*}(s)ds$$

$$= \frac{p\alpha(n-1)}{cn} \int_0^{+\infty} s \frac{f_\beta(s)}{\bar{F}_\beta(s)} [nm f_\beta(s) \bar{F}_\beta^{nm-1}(s)] ds.$$

Consider $nm(1-y)^{nm-1}$ as a density of a random variable. This random variable is stochastically decreasing in m. Since $s\frac{f_\beta(s)}{\bar{F}_\beta(s)}$ is increasing in s, it follows that $q^*(n,\alpha,\beta,m)$ is decreasing in m.

We now prove part (iv). Using the techniques in the proof of Proposition 1, we can obtain that, given m, $q^*(n,\alpha,\beta,m)$ is decreasing β. Then we can see from Equation (B.10) that $m^*(n,\alpha,\beta,K)$ is increasing in β. According to the proof of part (iii), $q^*(n,\alpha,\beta,m)$ is decreasing in m. Therefore, by Equation (B.7), we have $q^*(n,\alpha,\beta,K)$ is decreasing in β. □

Proof of Corollary 1. The Corollary holds according to the definition of $m^*(n,\alpha,\beta,K)$ in Theorems 3 and 4. □

The proof of Proposition 8 makes use of the following lemmas.

LEMMA B.4. *For any $C>0$, there exists $\tilde{\beta}(C)$ such that $|H_k| > C|H_{k+1}|$ (where H_k is specified in Appendix A.2), for $k=1,2,\ldots,$ if $\beta > \tilde{\beta}(C)$.*

Proof of Lemma B.4. We can see from the expression of H_k that, for sufficiently large β, $\{|H_k|\}_{k=1}^{\infty}$ is bounded. For $k=1,2,\ldots,$ $\frac{H_{k+1}}{H_k} = -\prod_{\substack{l\neq k\\ l\neq k+1}} \frac{l^\beta - k^\beta}{l^\beta-(k+1)^\beta}\frac{k^\beta}{(k+1)^\beta}$. Since $\forall l \neq k,\ l\neq k+1$, $\frac{l^\beta-k^\beta}{l^\beta-(k+1)^\beta}$ is decreasing in β and $\frac{k^\beta}{(k+1)^\beta}$ is decreasing in β, we have that $\frac{|H_{k+1}|}{|H_k|}$ converges to 0 as $\beta\to\infty$, for any given k. Since $\prod_{\substack{l\neq k\\ l\neq k+1}} \frac{l^\beta-k^\beta}{l^\beta-(k+1)^\beta}$ is decreasing in k, $\frac{|H_{k+1}|}{|H_k|}$ uniformly converges to 0 as $\beta\to\infty$. Therefore, for any $C>0$, there exists $\tilde{\beta}(C)$ such that $|H_k| > C|H_{k+1}|$, for $k=1,2,\ldots,$ if $\beta > \tilde{\beta}(C)$. □

LEMMA B.5. *For any given nonnegative n, there exists a $\hat{\beta}$ such that $\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s))\mathrm{e}^{-ns}\mathrm{d}s < 0$ if $\beta > \hat{\beta}$.*

Proof of Lemma B.5. Using the technique we used to prove Theorem A.5, we have

$$\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s))\mathrm{e}^{-ns}\mathrm{d}s = \frac{\partial}{\partial\beta}\sum_{j_1=1}^{\infty}\frac{j_1^\beta}{(j_1^\beta+n)^2}H_{j_1}$$
$$= -\sum_{j_1=1}^{\infty}\frac{j_1^\beta}{(j_1^\beta+n)^2}\left[\frac{j_1^\beta-n}{j_1^\beta+n}\ln j_1 + \sum_{l\neq j_1}\frac{\ln l - \ln j_1}{l^\beta - j_1^\beta} j_1^\beta\right] H_{j_1}$$

Define

$$\psi(j_1,\beta,n) = -\frac{j_1^\beta}{(j_1^\beta+n)^2}\left[\frac{j_1^\beta-n}{j_1^\beta+n}\ln j_1 + \sum_{l\neq j_1}\frac{\ln l - \ln j_1}{l^\beta - j_1^\beta} j_1^\beta\right].$$

Since $\sum_{l\neq j_1}\frac{\ln l - \ln j_1}{l^\beta - j_1^\beta} j_1^\beta > \ln j_1$, $\forall j_1 \geq 1$, we have that $\psi(j_1,\beta,n) < 0$ for all $j_1\geq 1$, $n\geq 0$ and $\beta > 1$. From the expression of $\psi(j_1,\beta,n)$, we can have that for any given $n(\geq 0)$, there exists $\tilde{\beta}$ such that $|\frac{\psi(j_1+1,\beta,n)}{\psi(j_1,\beta,n)}| < C$ for all $j_1 \geq 1$ if $\beta > \tilde{\beta}$.

Notice that $\{H_{j_1}\}$ is a infinite sequence with alternating signs. We have that

$$\sum_{j_1=1}^{\infty}\psi(j_1,\beta,n)H_{j_1}=\sum_{k=1}^{\infty}[\psi(2k-1,\beta,n)H_{2k-1}-\psi(2k,\beta,n)H_{2k}].$$

By Lemma B.4, there exists a $\hat{\beta}(C)$ such that if $\beta>\hat{\beta}(C)$ then

$$\psi(2k-1,\beta,n)H_{2k-1}-\psi(2k,\beta,n)H_{2k}<0,\ \forall k\geq 1.$$

Therefore, $\int_0^{\infty}s(\frac{\partial}{\partial\beta}f_{\beta}(s))\mathrm{e}^{-ns}\mathrm{d}s<0$ if $\beta>\max\{\tilde{\beta},\hat{\beta}(C)\}$. □

Proof of Proposition 8. For the simplicity of notation, we write $x_i^*(\mathbf{q}^{\alpha},\beta)$ as x_i^*, and $\pi_i(q_i^{\alpha},\mathbf{q}_{-i}^{\alpha},\beta)$ as π_i.

We first prove part (i). If a symmetric pure-strategy equilibrium exists, then the equilibrium quality can be solved by the first-order and second-order conditions:

$$\frac{\partial\pi_i}{\partial q_i}=\frac{p\alpha}{q_i(1-\beta)}x_i^*(1-x_i^*)-c=0,$$
$$\frac{\partial^2\pi_i}{\partial q_i^2}=\frac{p\alpha}{q_i^2(1-\beta)}x_i^*(1-x_i^*)[\frac{\alpha}{1-\beta}(1-2x_i^*)-1]\leq 0.$$

Thus, the quality q in a symmetric pure-strategy equilibrium must satisfy

$$\frac{p\alpha}{q(1-\beta)}\frac{q^{\frac{\alpha}{1-\beta}}((n-1)q^{\frac{\alpha}{1-\beta}}+q_0^{\frac{\alpha}{1-\beta}})}{(nq^{\frac{\alpha}{1-\beta}}+q_0^{\frac{\alpha}{1-\beta}})^2}=c. \tag{B.11}$$

If $\frac{\partial^2\pi_i}{\partial q_i^2}\leq 0$ at q_i, then the left side of the (B.11) is decreasing at $q=q_i$. Since π_i is concave in q_i for all $q_i\geq q$ as long as π_i is concave at $q_i=q$, we can have that the symmetric pure-strategy equilibrium must be unique if it exists. Using the same approach we used to prove Theorem 1, we have that a symmetric solution to FOCs is the symmetric equilibrium if it is positive and ensures that each firm earns a nonnegative profit. Define $\tilde{n}(\alpha,\beta,K,q_0)\triangleq\max\{n\in\mathbb{N}:\ (B.11)$ has a solution q such that $p\mathbb{E}[x_1^*((q_0/q)^{\alpha},\mathbf{1},\beta)]-cq-K\geq 0$ and $q>0\}$. Then we can have that there exists a unique symmetric pure-strategy equilibrium if and only if $n\leq\tilde{n}(\alpha,\beta,K,q_0)$, and there is no symmetric pure-strategy equilibrium if $n>\tilde{n}(\alpha,\beta,K,q_0)$.

Now we show the monotonicity of $q^*(n,\alpha,\beta,q_0)$ in β. By the first-order condition, we have

$$\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}=\frac{\frac{p\alpha x_i^*(1-x_i^*)}{q_i(1-\beta)^3}\left[(1-\beta)+\alpha x_0^*[\ln q_i-\ln q_0]\frac{1-2x_i^*}{1-x_i^*}\right]\Big|_{q_i=q^*(n,\alpha,\beta,q_0)}}{-\frac{\partial^2\pi_i}{\partial q_i^2}\Big|_{q_i=q^*(n,\alpha,\beta,q_0)}}.$$

It is obvious to see that if $q^*(n,\alpha,\beta,q_0)\geq q_0$, then $\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}\geq 0$, that is, the equilibrium quality is increasing in β. In the following, we focus on the case $q^*(n,\alpha,\beta,q_0)<q_0$. In this case, for any β such that $\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}\leq 0$, that is, the equilibrium quality is decreasing in β, both x_0^* and $\frac{1-2x_i^*}{1-x_i^*}$

are increasing in β. This implies that $(1-\beta)+\alpha x_0^*[\ln q^*(n,\alpha,\beta,q_0)-\ln q_0]\frac{1-2x_i^*}{1-x_i^*}$ is decreasing in β. Consequently, if $\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}\le 0$ for some value of β, then it is also true for any value greater than β. Since $\lim_{\beta\to-\infty}=0$, it follows that $q^*(n,\alpha,\beta,q_0)$ is increasing in β if β is close to $-\infty$. Thus, $q^*(n,\alpha,\beta,q_0)$ is either an increasing function or first-increasing-then-decreasing function of β. From the first-order condition, we can see also that $q^*(n,\alpha,\beta,q_0)$ is decreasing in q_0. Hence we can deduce that there exists a threshold $\tilde{q}_0$ such that $q^*(n,\alpha,\beta,q_0)$ is increasing in β on $\{\beta:\ \beta<1, n\le\tilde{n}(\alpha,\beta,K,q_0)\}$ if $q_0\le\tilde{q}_0$, and $q^*(n,\alpha,\beta,q_0)$ is first-increasing-then-decreasing in β on $\{\beta:\ \beta<1, n\le\tilde{n}(\alpha,\beta,K,q_0)\}$ if $q_0>\tilde{q}_0$.

Now we prove part (ii). If a symmetric pure-strategy equilibrium exists, then the equilibrium quality can be solved by the first-order and second-order conditions:

$$\frac{\partial\pi_i}{\partial q_i}=\frac{p\alpha}{q_i}\int_0^\infty(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)\bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s)\mathrm{d}s-c=0,$$
$$\frac{\partial^2\pi_i}{\partial q_i^2}=-\frac{p\alpha}{q_i^2}\int_0^\infty\left[(1-\alpha)(sf_\beta(s))'-\alpha(s^2f'_\beta(s))'\right]\bar{F}_\beta^{n-1}(s)\bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s)\mathrm{d}s\le 0.$$

Thus, the quality q in a symmetric pure-strategy equilibrium must satisfy

$$\frac{p\alpha}{q}\int_0^\infty(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)\bar{F}_{(\frac{q_0}{q})^\alpha,\beta}(s)ds=c.$$

If $\frac{\partial^2\pi_i}{\partial q_i^2}\le 0$ at q_i, then the left side of the (B.11) is decreasing at $q=q_i$. Then by Lemma B.1 (iii) and (iv), we have that π_i is concave in q_i for all $q_i\ge q$ as long as π_i is concave at $q_i=q$. Hence, the symmetric pure-strategy equilibrium must be unique if it exists. Using the same approach we used to prove Theorem 2 and the ideas we used to prove the case with $\beta<1$, we can have that there exists a unique symmetric pure-strategy equilibrium if and only if $n\le\tilde{n}(\alpha,\beta,K,q_0)$, and there is no symmetric pure-strategy equilibrium if $n>\tilde{n}(\alpha,\beta,K,q_0)$.

Now we show the monotonicity of $q^*(n,\alpha,\beta,q_0)$ in β when q_0 is sufficiently small. From the first-order condition, we can also see that

$$\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}=\frac{\frac{p\alpha}{q_i}\frac{\partial}{\partial\beta}\int_0^\infty(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)\bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s)ds\Big|_{q_i=q^*(n,\alpha,\beta,q_0)}}{-\frac{\partial^2\pi_i}{\partial q_i^2}\Big|_{q_i=q^*(n,\alpha,\beta,q_0)}}.$$

We also have $q^*(n,\alpha,\beta,q_0)$ is decreasing in q_0. Then, it is easy to have $\frac{\partial q^*(n,\alpha,\beta,q_0)}{\partial\beta}$ uniformly converges to $\frac{p\alpha}{c}\int_0^\infty(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds$ for all $\beta\ge 1+\delta$, where δ is a sufficiently small positive number, as q_0 goes to zero. Since we have proved in Proposition 1 that $\frac{p\alpha}{c}\int_0^\infty(sf_\beta(s))'\bar{F}_\beta^{n-1}(s)ds$ is decreasing in β, it follows that there exists a threshold $\hat{q}_0$ such that $q^*(n,\alpha,\beta,q_0)$ is decreasing in β if $q_0\le\hat{q}_0$.

Now we show the monotonicity of $q^*(n,\alpha,\beta,q_0)$ in β when β is sufficiently large. Before proving this, we first presents some properties of $f_{q^\alpha,\beta}(\cdot)$ and $\bar{F}_{q^\alpha,\beta}(\cdot)$. Due to the logconcavity of $f_{q^\alpha,\beta}(s)$

we have that for $\beta > 1$, $\frac{\partial}{\partial\beta} f_{q^\alpha,\beta}(s)$ is first positive then negative for $s \in [0,+\infty)$. Recall that $\bar{F}_{q^\alpha,\beta}(s)$ is the tail distribution of the convolution of exponentials with parameters $1, 2^\beta, 3^\beta, \ldots$. As $\beta \to \infty$, $f_{q^\alpha,\beta}(s) \to q^\alpha e^{-q^\alpha s}$, $\bar{F}_{q^\alpha,\beta}(s) \to e^{-q^\alpha s}$. Since $f_{q^\alpha,\beta_1}(s)$ dominates $f_{q^\alpha,\beta_2}(s)$ in the likelihood ratio ordering for $1 < \beta_1 < \beta_2$ and $q > 0$, it follows that $\bar{F}_{q^\alpha,\beta}(s)$ is decreasing in β, and both $\frac{\bar{F}_{q^\alpha,\beta}(s)}{e^{-q^\alpha s}}$ and $\frac{f_{q^\alpha,\beta}(s)}{q^\alpha e^{-q^\alpha s}}$ are increasing in s, $\forall s \geq 0$.

We have

$$\begin{aligned}
&\frac{\partial}{\partial\beta}\int_0^\infty (s f_\beta(s))' \bar{F}_\beta^{n-1}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) ds \\
&= (n-1)\frac{\partial}{\partial\beta}\int_0^\infty s f_\beta^2(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) ds + \frac{\partial}{\partial\beta}\int_0^\infty s f_\beta(s) f_{(\frac{q_0}{q_i})^\alpha,\beta}(s) \bar{F}_\beta^{n-1}(s) ds.
\end{aligned}$$

Since $\frac{\partial}{\partial\beta}\bar{F}_\beta(s) < 0$ and $\frac{\partial}{\partial\beta}\bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) < 0$, for sufficiently large β, to prove $\frac{\partial}{\partial\beta} q^*(n,\alpha,\beta,q_0) < 0$, it suffices to prove

$$\begin{aligned}
&\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) f_\beta(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) ds < 0, \\
&\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) f_{(\frac{q_0}{q_i})^\alpha,\beta}(s) \bar{F}_\beta^{n-1}(s) ds < 0, \\
&\int_0^\infty s(\frac{\partial}{\partial\beta} f_{(\frac{q_0}{q_i})^\alpha,\beta}(s)) f_\beta(s) \bar{F}_\beta^{n-1}(s) ds < 0.
\end{aligned}$$

In the following, we only prove the first condition since the proofs for the the other two are alike.

We also have

$$\begin{aligned}
&\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) f_\beta(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) ds \\
&= \int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) e^{-(n-1+(\frac{q_0}{q_i})^\alpha)s} \left(\frac{f_\beta(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s)}{e^{-(n-1+(\frac{q_0}{q_i})^\alpha)s}} \right) ds.
\end{aligned}$$

By Lemma B.5, we have that there exist $\hat{\beta}$ such that $= \int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) e^{-(n-1+(\frac{q_0}{q_i})^\alpha)s} ds < 0$ if $\beta > \hat{\beta}$. Since $\frac{\partial}{\partial\beta} f_\beta(s)$ is first positive then negative for $s \in [0,+\infty)$ and $\frac{f_\beta(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s)}{e^{-(n-1+(\frac{q_0}{q_i})^\alpha)s}}$ is increasing in s, we can deduce that $\int_0^\infty s(\frac{\partial}{\partial\beta} f_\beta(s)) f_\beta(s) \bar{F}_\beta^{n-2}(s) \bar{F}_{(\frac{q_0}{q_i})^\alpha,\beta}(s) ds < 0$ if $\beta > \hat{\beta}$. This completes the proof. □

Proof of Lemma 3. We first prove $q_i^* \geq q_j^*$ for any i and j such that $1 \leq i \leq j \leq n$. We use proof by contradiction to prove it. If $i \leq j$, then

$$\frac{\partial \pi_i}{\partial q_i}\Big|_{q_i = q_i^*} = c_i \leq c_j = \frac{\partial \pi_j}{\partial q_j}\Big|_{q_i = q_j^*}.$$

Suppose $q_i^* < q_j^*$ for some i and j such that $i \leq j$. By Lemma B.2, there exists q between q_i^* and q_j^* such that

$$\frac{\partial \pi_i}{\partial q_i}\Big|_{q_i = q_j = q} = \frac{\partial \pi_j}{\partial q_j}\Big|_{q_i = q_j = q} = c_i.$$

From Lemma B.1 (iv), we see that π_j is concave in q_j on $(q, +\infty)$. Thus, $\frac{\partial \pi_j}{\partial q_j}|_{q_i=q, q_j=q_j^*} < c_i$, which is a contradiction.

From the proofs of Theorem 1 and Lemma B.1(iii)(iv), we see that for $n \geq 2$, $\mathbf{q}$ is an equilibrium if and only if it satisfies FOCs and each firm earns a nonnegative profit at $\mathbf{q}$. Since $\pi_i \geq \pi_j$ in equilibrium if $i < j$, it follows that if firm n's expected profit at $\mathbf{q} = \mathbf{q}^*$ is nonnegative, then $\mathbf{q}^*$ is a pure-strategy equilibrium. □

Proof of Lemma 4. For $i = 1, \ldots, n$, define $\tilde{\pi}_i$ as the concavification of π, where the concavification is defined as the pointwise smallest concave function which is no smaller than the function being concavified. Let $\tilde{\mathcal{G}}$ be the n-firm game with $\tilde{\pi}_i$s being payoff functions. By Lemmas 1 and B.1(iv), we also have that the maximum point of π, given $\mathbf{q}_{-i}$, is either 0 or the local maximum that satisfies the FOC. Consequently, we can deduce that if $\tilde{\mathcal{G}}$ has an equilibrium in which each q_i is located on the concave part of π_i, then the equilibrium must be an equilibrium of $\mathcal{G}$. Otherwise, if $\tilde{\mathcal{G}}$ has no such equilibrium, then in any pure-strategy equilibrium of $\mathcal{G}$ (if it exists), at least one product has a quality of 0. In addition, we have also shown that if q_i satisfying the FOC is positive and $\pi_i \geq 0$ at q_i, then q_i must be located on the concave part of π_i. By Theorem 1 in Rosen (1965), we have that the game $\tilde{\mathcal{G}}$ has at least one pure-strategy equilibrium. Therefore, for each n, we can determine whether $\mathcal{G}$ has a pure-strategy equilibrium with all entries being positive by examining whether $\tilde{\mathcal{G}}$ has an equilibrium in which $\mathbf{q} > \mathbf{0}$ and $\pi_i \geq 0$ at $\mathbf{q}$ for each i.

Define $\tilde{n}(\alpha, \beta, K)$ as the largest integer n such that the n-firm game $\tilde{\mathcal{G}}$ has an equilibrium in which $\mathbf{q} > \mathbf{0}$ and $\pi_i \geq 0$ at $\mathbf{q}$ for each i. Then the lemma holds naturally. □

Proof of Proposition 9. Given $n = 2$, assume (q_1^*, q_2^*) satisfies FOCs of the two firms' problems:

$$
\begin{aligned}
\frac{\partial \pi_1}{\partial q_1} &= \frac{p\alpha}{q_1(1-\beta)} x_1^* x_2^* - c_1 = 0, && \text{if } \beta < 1, \\
\frac{\partial \pi_2}{\partial q_2} &= \frac{p\alpha}{q_2(1-\beta)} x_1^* x_2^* - c_2 = 0, && \text{if } \beta < 1, \\
\frac{\partial \pi_1}{\partial q_1} &= \frac{p\alpha}{q_1} \int_0^\infty s f_{q_1^\alpha, \beta}(s) f_{q_2^\alpha, \beta}(s) \mathrm{d}s - c_1 = 0, && \text{if } \beta > 1, \\
\frac{\partial \pi_2}{\partial q_2} &= \frac{p\alpha}{q_2} \int_0^\infty s f_{q_1^\alpha, \beta}(s) f_{q_2^\alpha, \beta}(s) \mathrm{d}s - c_2 = 0, && \text{if } \beta > 1.
\end{aligned}
$$

We can see that $c_1 q_1^* = c_2 q_2^*$. Using the ideas we used to prove Theorem 1, we have that (q_1^*, q_2^*) is an equilibrium if it is positive and ensures that each firm earns a nonnegative profit. Since

$$
\pi_2|_{q_1=q_1^*, q_2=q_2^*} = \begin{cases} p \frac{(q_2^*)^{\frac{\alpha}{1-\beta}}}{(q_1^*)^{\frac{\alpha}{1-\beta}} + (q_2^*)^{\frac{\alpha}{1-\beta}}} \left(1 - \frac{\alpha}{1-\beta} \frac{(q_1^*)^{\frac{\alpha}{1-\beta}}}{(q_1^*)^{\frac{\alpha}{1-\beta}} + (q_2^*)^{\frac{\alpha}{1-\beta}}}\right) - K, & \text{if } \beta < 1, \\ p \int_0^{+\infty} f_{(\frac{q_2^*}{q_1^*})^\alpha, \beta}(s) (\bar{F}_\beta(s) - \alpha s f_\beta(s)) \mathrm{d}s - K, & \text{if } \beta > 1, \end{cases}
$$

and $\frac{q_2^*}{q_1^*} = \frac{c_1}{c_2}$, it follows that $\pi_2 \geq 0$ at (q_1^*, q_2^*) if the following condition holds:

$$\begin{cases} p\frac{(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}{1+(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}(1-\frac{\alpha}{1-\beta}\frac{1}{1+(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}) \geq K, & \text{if } \beta < 1, \\ p\int_0^{+\infty} f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)(\bar{F}_\beta(s) - \alpha s f_\beta(s))\mathrm{d}s \geq K, & \text{if } \beta > 1. \end{cases} \tag{B.12}$$

When Condition (B.12) holds, there must be a unique pure-strategy equilibrium, namely

$$q_1^* = \begin{cases} \frac{p\alpha}{c_1(1-\beta)}\frac{(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}{1+(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}\frac{1}{1+(\frac{c_1}{c_2})^{\frac{\alpha}{1-\beta}}}, & \text{if } \beta < 1, \\ \frac{p}{c_1}\int_0^{+\infty} f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)f_\beta(s)\mathrm{d}s, & \text{if } \beta > 1, \end{cases}$$
$$q_2^* = \frac{c_1}{c_2}q_1^*.$$

From the expression of (B.12), we see that π_2^* is decreasing in β for $\beta < 1$. Hence, there is $\hat{\beta}_1 < 1$ such that $\pi_2^* \geq 0$ if $\beta \geq \hat{\beta}_1$. From Lemma 2(ii), we also see that $\int_0^{+\infty} f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)\bar{F}_\beta(s)\mathrm{d}s$ is increasing in β for $\beta > 1$. Consequently, to complete the proof, it remains to prove that there exists $\hat{\beta}_2$ such that $\int_0^{+\infty} s f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)f_\beta(s)\mathrm{d}s$ is decreasing in β for $\beta > \hat{\beta}_2$, i.e.,

$$\int_0^{\infty} s(\frac{\partial}{\partial\beta}f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s))f_\beta(s)\mathrm{d}s < 0 \text{ and } \int_0^{\infty} s(\frac{\partial}{\partial\beta}f_\beta(s))f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)\mathrm{d}s < 0.$$

Using the techniques we used to prove Proposition 9, we can have that

$$\int_0^{\infty} s(\frac{\partial}{\partial\beta}f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s))f_\beta(s)\mathrm{d}s = \int_0^{\infty} s(\frac{\partial}{\partial\beta}f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)\mathrm{e}^{-s})\frac{f_\beta(s)}{\mathrm{e}^{-s}}\mathrm{d}s < 0,$$
$$\int_0^{\infty} s(\frac{\partial}{\partial\beta}f_\beta(s))f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)\mathrm{d}s = \int_0^{\infty} s(\frac{\partial}{\partial\beta}f_\beta(s)\mathrm{e}^{-(\frac{c_1}{c_2})^{\alpha}s})\frac{f_{(\frac{c_1}{c_2})^{\alpha},\beta}(s)}{\mathrm{e}^{-(\frac{c_1}{c_2})^{\alpha}s}}\mathrm{d}s < 0,$$

as required. Similarly, we can obtain that firm 1's equilibrium quality is increasing in β for $\beta < \hat{\beta}_1$ and decreasing in β for $\beta > \hat{\beta}_2$. □

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