Online Appendix to “Flexible-Duration Extended Warranties with Dynamic Reliability Learning”

Appendix A: Proofs

Proof of Lemma 1. Suppose that $R_{t+1}(p_{t+1}) = p_{t+1}c\sum_{i=t+1}^{T}\prod_{j=t+1}^{i}(1-q(j))$. Then,

$$\begin{aligned}
R_t(p_t) &= p_t(1-q(t))(c+R_{t+1}(p_{t+1}^+)) + (1-p_t)(1-q(t))R_{t+1}(p_{t+1}^-)\\
&= p_t(1-q(t))c + (1-q(t))\left(p_tR_{t+1}(p_{t+1}^+) + (1-p_t)R_{t+1}(p_{t+1}^-)\right)\\
&= p_t(1-q(t))c + (1-q(t))\left(p_tp_{t+1}^+ + (1-p_t)p_{t+1}^-\right)c\sum_{i=t+1}^{T}\prod_{j=t+1}^{i}(1-q(j))\\
&= p_t(1-q(t))c + (1-q(t))p_tc\sum_{i=t+1}^{T}\prod_{j=t+1}^{i}(1-q(j)) = p_tc\sum_{i=t}^{T}\prod_{j=t}^{i}(1-q(j)).
\end{aligned}$$

The fourth equality holds because the updating scheme is a martingale and $p_tp_{t+1}^+ + (1-p_t)p_{t+1}^- = p_t$. □

Proof of Proposition 1. For notational convenience, let $M_t(p_t) := R_t(p_t) - W_t(p_t)$. From equations (1)-(3),

$$M_t(p_t) = \max\left\{R_t(p_t) - B_t(p_t),\ 0\right\}, \tag{12}$$

where $R_t(p_t) - B_t(p_t) = (1-q_0(t))p_tc - m + p_t(1-q_0(t))M_{t+1}(p_{t+1}^+) + (1-p_t)(1-q_1(t))M_{t+1}(p_{t+1}^-)$. Suppose that $R_{t+1}(p_{t+1}) - B_{t+1}(p_{t+1})$ is increasing in p_{t+1}, so $M_{t+1}(p_{t+1})$ is increasing in p_{t+1}. For any $p_t' \le p_t$, $p_{t+1}^+ \ge p_{t+1}'^+$ and $p_{t+1}^- \ge p_{t+1}'^-$ from Assumption 2. The situations are different for $q_0(t)=0$ and $q_0(t)>0$, so we will discuss them separately.

Case 1: $q_0(t)=0$. Then,

$$\begin{aligned}
&(R_t(p_t) - B_t(p_t)) - (R_t(p_t') - B_t(p_t')) = (p_t - p_t')c + p_t\left(M_{t+1}(p_{t+1}^+) - M_{t+1}(p_{t+1}'^+)\right) + (p_t - p_t')M_{t+1}(p_{t+1}'^+)\\
&\quad + (1-p_t)(1-q_1(t))\left(M_{t+1}(p_{t+1}^-) - M_{t+1}(p_{t+1}'^-)\right) - (p_t-p_t')(1-q_1(t))M_{t+1}(p_{t+1}'^-)\\
&\ge (p_t-p_t')c + (p_t-p_t')\left(M_{t+1}(p_{t+1}'^+) - (1-q_1(t))M_{t+1}(p_{t+1}'^-)\right) \ge 0.
\end{aligned}$$

The first inequality holds because $M_{t+1}(p_{t+1}^+) \ge M_{t+1}(p_{t+1}'^+)$ and $M_{t+1}(p_{t+1}^-) \ge M_{t+1}(p_{t+1}'^-)$; the second inequality holds because $p_t \ge p_t'$ and $M_{t+1}(p_{t+1}'^+) \ge M_{t+1}(p_{t+1}'^-) \ge (1-q_1(t))M_{t+1}(p_{t+1}'^-)$.

Case 2: $q_0(t)>0$. Suppose $M_{t+1}(p_{t+1}) \le \frac{(1-q_0(t+1))c-m}{q_0(t+1)}$. Then,

$$\begin{aligned}
&(R_t(p_t) - B_t(p_t)) - (R_t(p_t') - B_t(p_t')) = (p_t-p_t')(1-q_0(t))c + p_t(1-q_0(t))\left(M_{t+1}(p_{t+1}^+) - M_{t+1}(p_{t+1}'^+)\right)\\
&\quad + (p_t-p_t')(1-q_0(t))M_{t+1}(p_{t+1}'^+) + (1-p_t)(1-q_1(t))\left(M_{t+1}(p_{t+1}^-) - M_{t+1}(p_{t+1}'^-)\right)\\
&\quad - (p_t-p_t')(1-q_1(t))M_{t+1}(p_{t+1}'^-)\\
&\ge \left((1-q_0(t))c + (1-q_0(t))M_{t+1}(p_{t+1}'^+) - (1-q_1(t))M_{t+1}(p_{t+1}'^-)\right)(p_t-p_t')\\
&\ge \left((1-q_0(t))c - q_0(t)M_{t+1}(p_{t+1}'^-)\right)(p_t-p_t') \ge \left((1-q_0(t))c - q_0(t)\cdot\frac{(1-q_0(t+1))c-m}{q_0(t+1)}\right)(p_t-p_t')\\
&\ge \left((1-q_0(t))c - (1-q_0(t+1))c + m\right)(p_t-p_t') = \left((q_0(t+1)-q_0(t))c + m\right)(p_t-p_t') \ge 0.
\end{aligned}$$

The first inequality holds because $M_{t+1}(p_{t+1}^+) \ge M_{t+1}(p_{t+1}'^+)$ and $M_{t+1}(p_{t+1}^-) \ge M_{t+1}(p_{t+1}'^-)$; the second inequality holds because $M_{t+1}(p_{t+1}'^+) \ge M_{t+1}(p_{t+1}'^-) \ge 0$; the third inequality holds because $M_{t+1}(p_{t+1}) \le \left((1-q_0(t+1))c-m\right)/q_0(t+1)$; the last inequality holds because $q_0(t+1) \ge q_0(t)$. So $R_t(p_t) - B_t(p_t)$

is increasing in p_t and $M_t(p_t)$ is also increasing in p_t from equation (12). Because $q_0(t) \le q_0(t+1)$ and $M_{t+1}(p_{t+1}) \le \big((1-q_0(t+1))c - m\big)/q_0(t+1)$, then from equation (12)

$$\begin{aligned}
&R_t(p_t) - B_t(p_t) \le (1-q_0(t))cp_t - m + (1-q_1(t) - (q_0(t)-q_1(t))p_t) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)} \\
&= \left((1-q_0(t))c - (q_0(t)-q_1(t)) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)}\right) p_t - m + (1-q_1(t)) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)} \\
&\le (1-q_0(t))c - m + (1-q_0(t)) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)} = \tfrac{(1-q_0(t))c-(1-q_0(t)+q_0(t+1))m}{q_0(t+1)} \\
&\le \tfrac{(1-q_0(t))c-m}{q_0(t+1)} \le \tfrac{(1-q_0(t))c-m}{q_0(t)}.
\end{aligned}$$

The first inequality holds because

$$\begin{aligned}
&(1-q_0(t))c - (q_0(t)-q_1(t)) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)} \ge (1-q_0(t))c - q_0(t) \cdot \tfrac{(1-q_0(t+1))c-m}{q_0(t+1)} \\
&\ge (1-q_0(t))c - ((1-q_0(t+1))c - m) = (q_0(t+1)-q_0(t))c + m \ge 0.
\end{aligned}$$

From equation (12), $M_t(p_t) \le \frac{(1-q_0(t))c-m}{q_0(t)}$. □

Proof of Theorem 1. Part (a) is obvious from equation (3). For part (b), let

$$p_t^* = \inf\{p_t \ge 0 : R_t(p_t) - B_t(p_t) \ge 0\}, \tag{13}$$

where $p_t^* = \infty$ if the set is empty. Because $R_t(p_t) - B_t(p_t)$ is increasing in p_t from Proposition 1, then $R_t(p_t) - B_t(p_t) \ge 0$ for any $p_t \ge p_t^*$ and $R_t(p_t) - B_t(p_t) < 0$ for any $p_t < p_t^*$. Thus, the optimal policy has a threshold structure: it is optimal to keep buying the flexible EW if and only if $p_t \ge p_t^*$. Because $M_{t+1}(p_{t+1}) \ge 0$ for any $0 \le p_{t+1} \le 1$, then for any $p_t \ge \frac{m}{(1-q_0(t))c}$,

$$\begin{aligned}
R_t(p_t) - B_t(p_t) &= (1-q_0(t))p_t c - m + p_t(1-q_0(t))M_{t+1}(p_{t+1}^+) + (1-p_t)(1-q_1(t))M_{t+1}(p_{t+1}^-) \\
&\ge (1-q_0(t))p_t c - m \ge 0.
\end{aligned}$$

Because $R_t(p_t) - B_t(p_t)$ is increasing in p_t, then $p_t^* \le \frac{m}{(1-q_0(t))c}$ from equation (13). □

Proof of Proposition 2. Suppose $R_{t+1}(p_{t+1}) - B_{t+1}(p_{t+1})$ is increasing in p_{t+1}, so $M_{t+1}(p_{t+1})$ is increasing in p_{t+1} from equation (12). For any $p'_t \le p_t$, $p_{t+1}^+ \ge p'^+_{t+1}$ and $p_{t+1}^- \ge p'^-_{t+1}$ from Assumption 2. Then,

$$\begin{aligned}
&(R_t(p_t) - B_t(p_t)) - (R_t(p'_t) - B_t(p'_t)) = (p_t - p'_t)(1-q_0(t))c + p_t(1-q_0(t))\left(M_{t+1}(p_{t+1}^+) - M_{t+1}(p'^+_{t+1})\right) \\
&\quad + (p_t - p'_t)(1-q_0(t))M_{t+1}(p'^+_{t+1}) + (1-p_t)(1-q_1(t))\left(M_{t+1}(p_{t+1}^-) - M_{t+1}(p'^-_{t+1})\right) \\
&\quad - (p_t - p'_t)(1-q_1(t))M_{t+1}(p'^-_{t+1}) \\
&\ge (p_t - p'_t)\big((1-q_0(t))M_{t+1}(p'^+_{t+1}) - (1-q_1(t))M_{t+1}(p'^-_{t+1})\big) \\
&\ge (p_t - p'_t)(1-q_1(t))\big(M_{t+1}(p'^+_{t+1}) - M_{t+1}(p'^-_{t+1})\big) \ge 0.
\end{aligned}$$

The first inequality holds because $M_{t+1}(p_{t+1}^+) \ge M_{t+1}(p'^+_{t+1})$ and $M_{t+1}(p_{t+1}^-) \ge M_{t+1}(p'^-_{t+1})$; the second inequality holds because $q_0(t) \le q_1(t)$; the last inequality holds because $p_t \ge p'_t$ and $M_{t+1}(p'^+_{t+1}) \ge M_{t+1}(p'^-_{t+1})$.
□

Proof of Lemma 2. (a) Let $N_t = a_t - a_1 = a_t - a$ denote the number of failures that occur up to month t. Then, $p_{t+1} = (a + N_{t+1})/(a+b+t)$. So, the Beta updating scheme is Markovian. Since

$$p_{t+1} = \frac{(a+b+t-1)p_t + I_t}{a+b+t} = p_t + \frac{I_t - p_t}{a+b+t},$$

then $p_{t+1}^+ > p_t > p_{t+1}^-$; p_{t+1}^+ and p_{t+1}^- are both increasing in p_t.

(b) For the Beta updating scheme,

$$E[p_{t+1}|\mathcal{I}_t] = p_t p_{t+1}^+ + (1-p_t)p_{t+1}^- = p_t(p_t + \frac{1-p_t}{a_1+b_1+t}) + (1-p_t)(p_t - \frac{p_t}{a_1+b_1+t}) = p_t,$$

so it is a martingale. □

Proof of Theorem 2. Since the optimal policy has thresholds p_t^* for the estimated failure probabilities and the estimate is updated following $p_t = (a+N_t)/(a+b+t-1)$, the optimal policy also has a threshold structure on the number of observed failures N_t. Then, the optimal x_t^* can be found by solving

$$x_t^* = \min\left\{x_t \in \mathbb{N} : \frac{a+x_t}{a+b+t-1} \geq p_t^*\right\}. \tag{14}$$

The threshold x_t^* can also be expressed as follows: $x_t^* = \lceil (a+b+t-1)p_t^* \rceil - a$, where $\lceil y \rceil$ is the smallest integer number that is greater than or equal to y. □

Proof of Proposition 3. (a) Suppose that $R_{t+1}(p) \geq R_{t+2}(p)$, $B_{t+1}(p) \geq B_{t+2}(p)$ and $W_{t+1}(p) \geq W_{t+2}(p)$ for any $0 \leq p \leq 1$. Because $q_0(t) \leq q_0(t+1)$ and $q_1(t) \leq q_1(t+1)$, from equations (1)-(3),

$$\begin{aligned}
R_t(p) &= p(1-q_0(t))(c+R_{t+1}(p^+)) + (1-p)(1-q_1(t))R_{t+1}(p^-) \\
&\geq p(1-q_0(t+1))(c+R_{t+2}(p^+)) + (1-p)(1-q_1(t+1))R_{t+2}(p^-) = R_{t+1}(p), \\
B_t(p) &= m + p(1-q_0(t))W_{t+1}(p^+) + (1-p)(1-q_1(t))W_{t+1}(p^-) \\
&\geq m + p(1-q_0(t+1))W_{t+2}(p^+) + (1-p)(1-q_1(t+1))W_{t+2}(p^-) = B_{t+1}(p), \\
W_t(p) &= \min\{B_t(p_t),\ R_t(B_t)\} \geq \min\{B_{t+1}(p_{t+1}),\ R_{t+1}(B_{t+1})\} = W_{t+1}(p_{t+1}),
\end{aligned}$$

where p^+ (resp., p^-) represents the failure probability estimate in the next month when the failure probability estimate was p and a failure occurred (resp., did not occur) in the current month. Thus, $R_t(p)$, $B_t(p)$ and $W_t(p)$ are all decreasing in t for $0 \leq p \leq 1$.

(b) It is obvious that $R_T(p) - B_T(p) \geq W_{T+1}(p) - R_{T+1}(p) = 0$. Suppose that $R_{t+1}(p) - B_{t+1}(p) \geq R_{t+2}(p) - B_{t+2}(p)$ for any $0 \leq p \leq 1$. Then, $M_{t+1}(p) \geq M_{t+2}(p)$ for any $0 \leq p \leq 1$ from equation (12).

$$\begin{aligned}
R_t(p) - B_t(p) &= (1-q_0(t))pc - m + p(1-q_0(t))M_{t+1}(p^+) + (1-p)(1-q_1(t))M_{t+1}(p^-) \\
&\geq (1-q_0(t+1))pc - m + p(1-q_0(t+1))M_{t+2}(p^+) + (1-p)(1-q_1(t+1))M_{t+2}(p^-) \\
&= R_{t+1}(p) - B_{t+1}(p).
\end{aligned}$$

The inequality holds because $q_0(t) \leq q_0(t+1)$, $q_1(t) \leq q_1(t+1)$, $R_{t+1}(p^+) - B_{t+1}(p^+) \geq R_{t+2}(p^+) - B_{t+2}(p^+)$ and $R_{t+1}(p^-) - B_{t+1}(p^-) \geq R_{t+2}(p^-) - B_{t+2}(p^-)$. Thus, $R_t(p) - B_t(p)$ is decreasing in t for any $0 \leq p \leq 1$. □

Proof of Theorem 3. Because $R_t(p) - B_t(p)$ is increasing in p from Proposition 1 and it is decreasing in t from Proposition 3, then p_t^* is increasing in t from equation (13). □

Proof of Proposition 4. Denote $q_0(t) = q_0(t+1) = \cdots = q_0(T) = q_0$. Assume that $p_{t+1}^{*+} < p_{t+1}^*$ for some t. Then, $p_{t+1}^{*-} \leq p_{t+1}^{*+} < p_{t+1}^*$, where the first inequality holds because of Assumption 2. From equations (12) and (13), $M_{t+1}(p_{t+1}) = \max\{R_{t+1}(p_{t+1}) - B_{t+1}(p_{t+1}),\ 0\} = 0$ for any $p_{t+1} < p_{t+1}^*$. Then,

$$R_t(p_t^*) - B_t(p_t^*) = (1-q_0)p_t^*c - m + p_t(1-q_0)M_{t+1}(p_{t+1}^{*+}) + (1-p_t)(1-q_1(t))M_{t+1}(p_{t+1}^{*-}) = (1-q_0(t))p_t^*c - m.$$

The second equality holds because $M_{t+1}(p_{t+1}^{*+}) = M_{t+1}(p_{t+1}^{*-}) = 0$. Then, $p_t^* = \frac{m}{(1-q_0)c}$ from equation (13). Because $p_t^* \le p_{t+1}^* \le \cdots \le p_T^* = \frac{m}{(1-q_0)c}$ from Theorem 3, then $p_t^* = p_{t+1}^* = \cdots = p_T^* = \frac{m}{(1-q_0)c}$. Since $p_{t+1}^+ \ge p_t$ for $0 \le p_t \le 1$, then $p_{t+1}^{*+} \ge p_t^* = p_{t+1}^*$, which contradicts the assumption that $p_{t+1}^{*+} < p_{t+1}^*$. □

Proof of Lemma 3. Part (a) is straightforward. For the exponential smoothing mechanism,

$$E[p_{t+1}|\mathcal{I}_t] = p_t p_{t+1}^+ + (1-p_t)p_{t+1}^- = p_t((1-\alpha)p_t + \alpha) + (1-p_t)(1-\alpha)p_t = p_t,$$

so it is a martingale. □

Lemma 5 *(a) (Thomson (1994).) A continuous function $f(\cdot)$ on set C is concave if and only if $f\left(\frac{y_1+y_2}{2}\right) \ge \frac{f(y_1)+f(y_2)}{2}$ for any y_1 and y_2 on C.*

(b) If $f(\cdot)$ is a concave function, then $f(y_1) - f(y_2) \ge f(y_1+\epsilon) - f(y_2+\epsilon)$ for any $y_1 \ge y_2$ and $\epsilon \ge 0$.

(c) If $f(\cdot)$ is a concave function, then $\theta f(y_1) + (1-\theta)f(y_2) \ge \theta f(y_1') + (1-\theta)f(y_2')$ for any $y_1' \le y_1 \le y_2 \le y_2'$ such that $\theta y_1 + (1-\theta)y_2 = \theta y_1' + (1-\theta)y_2'$, where $0 \le \theta \le 1$.

Proof of Lemma 5. (a) See page 121 on Thomson (1994).

(b) Because $f(\cdot)$ is concave, $y_1 = \frac{y_1-y_2}{y_1-y_2+\epsilon}\cdot(y_1+\epsilon) + \frac{\epsilon}{y_1-y_2+\epsilon}\cdot y_2$ and $y_2+\epsilon = \frac{\epsilon}{y_1-y_2+\epsilon}\cdot(y_1+\epsilon) + \frac{y_1-y_2}{y_1-y_2+\epsilon}\cdot y_2$, then

$$f(y_1) \ge \frac{y_1-y_2}{y_1-y_2+\epsilon}\cdot f(y_1+\epsilon) + \frac{\epsilon}{y_1-y_2+\epsilon}\cdot f(y_2) \text{ and } f(y_2+\epsilon) \ge \frac{\epsilon}{y_1-y_2+\epsilon}\cdot f(y_1+\epsilon) + \frac{y_1-y_2}{y_1-y_2+\epsilon}\cdot f(y_2).$$

Thus, $f(y_1) + f(y_2+\epsilon) \ge f(y_1+\epsilon) + f(y_2)$.

(c) Let $z = \theta y_1 + (1-\theta)y_2 = \theta y_1' + (1-\theta)y_2'$. So $y_2 = (z-\theta y_1)/(1-\theta)$ and $y_2' = (z-\theta y_1')/(1-\theta)$. Then,

$$\begin{aligned}&\theta f(y_1) + (1-\theta)f(y_2) - \theta f(y_1') - (1-\theta)f(y_2') = \theta(f(y_1) - f(y_1')) + (1-\theta)\left(f\left(\tfrac{z-\theta y_1}{1-\theta}\right) - f\left(\tfrac{z-\theta y_1'}{1-\theta}\right)\right)\\ &\ge \theta\left(f\left(y_1 + \tfrac{z-y_1}{1-\theta}\right) - f\left(y_1' + \tfrac{z-y_1}{1-\theta}\right)\right) + (1-\theta)\left(f\left(\tfrac{z-\theta y_1}{1-\theta}\right) - f\left(\tfrac{z-\theta y_1'}{1-\theta}\right)\right)\\ &= f\left(\tfrac{z-\theta y_1}{1-\theta}\right) - \left(\theta\cdot f\left(y_1' + \tfrac{z-y_1}{1-\theta}\right) + (1-\theta)f\left(\tfrac{z-\theta y_1'}{1-\theta}\right)\right) \ge 0.\end{aligned}$$

The first inequality holds because of part (b) and $z \ge y_1$; the second equality holds because $y_1 + \frac{z-y_1}{1-\theta} = \frac{z-\theta y_1}{1-\theta}$; the last inequality holds because $f(\cdot)$ is concave and $\theta\big(y_1' + \frac{z-y_1}{1-\theta}\big) + (1-\theta)\big(\frac{z-\theta y_1'}{1-\theta}\big) = \frac{z-\theta y_1}{1-\theta}$. □

Proof of Proposition 5. (a) Denote $q_0(t) = q_1(t) = q(t)$ for each $t = 1, 2, \ldots, T$. Because $R_t(p_t) = p_t c \sum_{i=t}^{T}\prod_{j=t}^{i}(1-q(j))$ and it is independent of α, it is sufficient to prove that $B_t(p_t)$ is decreasing concave in α, which is equivalent to showing that $p_t W_{t+1}((1-\alpha)p_t+\alpha) + (1-p_t)W_{t+1}((1-\alpha)p_t)$ is decreasing concave in α. For any $\alpha' \le \alpha$, $p_t((1-\alpha)p_t+\alpha) + (1-p_t)(1-\alpha)p_t = p_t((1-\alpha')p_t+\alpha') + (1-p_t)(1-\alpha')p_t = p_t$ and $(1-\alpha)p_t \le (1-\alpha')p_t \le p_t \le (1-\alpha')p_t + \alpha' \le (1-\alpha)p_t + \alpha$. Because $W_{t+1}(p_{t+1})$ is concave in p_{t+1}, from Lemma 5,

$$p_t W_{t+1}((1-\alpha)p_t+\alpha) + (1-p_t)W_{t+1}((1-\alpha)p_t) \le p_t W_{t+1}((1-\alpha')p_t+\alpha') + (1-p_t)W_{t+1}((1-\alpha')p_t).$$

Thus, both $B_t(p_t)$ and $W_t(p_t)$ are decreasing in α. For the concavity of $W_t(p_t)$ with respect to α, consider

$$p_t W_{t+1}\left((1-\tfrac{\alpha+\alpha'}{2})p_t+\tfrac{\alpha+\alpha'}{2}\right)+(1-p_t)W_{t+1}\left((1-\tfrac{\alpha+\alpha'}{2})p_t\right)$$
$$\geq \tfrac{p_t}{2}\Big(W_{t+1}((1-\alpha)p_t+\alpha)+W_{t+1}((1-\alpha')p_t+\alpha')\Big)+\tfrac{1-p_t}{2}\Big(W_{t+1}((1-\alpha)p_t)+W_{t+1}((1-\alpha')p_t)\Big)$$
$$=\tfrac{1}{2}\Big(p_tW_{t+1}((1-\alpha)p_t+\alpha)+(1-p_t)W_{t+1}((1-\alpha)p_t)\Big)+\tfrac{1}{2}\Big(p_tW_{t+1}((1-\alpha')p_t+\alpha')+(1-p_t)W_{t+1}((1-\alpha')p_t)\Big).$$

The inequality holds because $W_{t+1}(p_{t+1})$ is concave in p_{t+1}. Therefore, both $B_t(p_t)$ and $W_t(p_t)$ are concave in α.

(b) Because $R_t(p_t)$ is independent of α and $B_t(p_t)$ is decreasing in α, then $R_t(p_t)-B_t(p_t)$ is increasing in α. From Proposition 1 and equation (13), p_t^* is decreasing in α for all t. □

Proof of Proposition 6. We first show that $\pi_t(p_t,m,\mathcal{Q}_t)$ is increasing in p_t by induction. Suppose that $\pi_{t+1}(p_{t+1},m,\mathcal{Q}_{t+1})$ is increasing in p_{t+1}. For any two points $p_t>p_t'$, if $p_t'<p_t^*$, then $\pi_t(p_t',m,\mathcal{Q}_t)=0\leq\pi_t(p_t,m,\mathcal{Q}_t)$; if $p_t'>p_t^*$, we have

$$\pi_t(p_t,m,\mathcal{Q}_t)=m+\lambda(1-q_0(t))\left(\pi_{t+1}(p_{t+1}^+,m,\mathcal{Q}_{t+1})-\beta c\right)+(1-\lambda)(1-q_1(t))\pi_{t+1}(p_{t+1}^-,m,\mathcal{Q}_{t+1})$$
$$\geq m+\lambda(1-q_0(t))\left(\pi_{t+1}(p_{t+1}'^+,m,\mathcal{Q}_{t+1})-\beta c\right)+(1-\lambda)(1-q_1(t))\pi_{t+1}(p_{t+1}'^-,m,\mathcal{Q}_{t+1})=\pi_t(p_t',m,\mathcal{Q}_t).$$

The inequality holds because $p_{t+1}^+\geq p_{t+1}'^+$, $p_{t+1}^-\geq p_{t+1}'^-$ and $\pi_{t+1}(p_{t+1}^+,m,\mathcal{Q}_{t+1})\geq\pi_{t+1}(p_{t+1}'^+,m,\mathcal{Q}_{t+1})$, $\pi_{t+1}(p_{t+1}^-,m,\mathcal{Q}_{t+1})\geq\pi_{t+1}(p_{t+1}'^-,m,\mathcal{Q}_{t+1})$. Thus, $\max_{m\geq0}\pi_t(p_t,m,\mathcal{Q}_t)\geq\max_{m\geq0}\pi_t(p_t',m,\mathcal{Q}_t)$.

For the monotonicity of the profit for the traditional EW, we only need to show that $R_1(p_1)$ is increasing in p_1 because the support cost $S_1(\mathcal{Q}_1)$ is independent of p_1. The value $R_1(p_1)-B_1(p_1)$ is increasing in p_1 for any m by Propositions 1 and 2. For the special case with monthly premium $m=0$, $B_1(p_1)=0$ for any p_1, so $R_1(p_1)$ is increasing in p_1. □

Proof of Theorem 4. We will show that the flexible EW is strictly profitable if and only if $p_1<p_1^o$, where p_1^o is expressed as follows

$$p_1^o=\inf\left\{p_1\geq0:\ \pi_1(p_1,m^*(p_1),\mathcal{Q}_1)\leq(p_1-\lambda\beta)c\textstyle\sum_{i=1}^T\prod_{j=1}^i(1-q(j))\right\}.$$

The customer's maximum willingness-to-pay for the traditional EW is $R_1(p_1)$. Since $\pi_1(p_1,m^*(p_1),\mathcal{Q}_1)\geq0$, the flexible EW is more profitable if and only if $\pi_1(p_1,m^*(p_1),\mathcal{Q}_1)\geq R_1(p_1)-S_1(\mathcal{Q}_1)$. For the existence of the threshold, we will show that $\pi_1(p_1,m,\mathcal{Q}_1)-(R_1(p_1)-S_1(\mathcal{Q}_1))$ is decreasing in p_1, which is equivalent to showing that $\pi_1(p_1,m,\mathcal{Q}_1)-R_1(p_1)$ is decreasing in p_1 because $S_1(\mathcal{Q}_1)$ is independent of p_1.

Suppose that $\pi_{t+1}(p_{t+1},m,\mathcal{Q}_{t+1})-R_{t+1}(p_{t+1})$ is decreasing in p_{t+1}. For $p_t<p_t^*$, $\pi_t(p_t,m,\mathcal{Q}_t)-R_t(p_t)=-p_tc\sum_{i=t}^T\prod_{j=t}^i(1-q(j))$, which is decreasing in p_t. For $p_t\geq p_t^*$,

$$\pi_t(p_t,m,\mathcal{Q}_t)-R_t(p_t)=m+\lambda(1-q(t))\left(\pi_{t+1}(p_{t+1}^+,m,\mathcal{Q}_{t+1})-\beta c\right)+(1-\lambda)(1-q(t))\pi_{t+1}(p_{t+1}^-,m,\mathcal{Q}_{t+1})$$
$$-p_t(1-q(t))(c+R_{t+1}(p_{t+1}^+))-(1-p_t)(1-q(t))R_{t+1}(p_{t+1}^-)$$
$$=m-(1-q(t))(\lambda\beta c+p_tc)+\lambda(1-q(t))\left(\pi_{t+1}(p_{t+1}^+,m,\mathcal{Q}_{t+1})-R_{t+1}(p_{t+1}^+)\right)$$
$$+(1-\lambda)(1-q(t))\left(\pi_{t+1}(p_{t+1}^-,m,\mathcal{Q}_{t+1})-R_{t+1}(p_{t+1}^-)\right)+(1-q(t))(\lambda-p_t)\left(R_{t+1}(p_{t+1}^+)-R_{t+1}(p_{t+1}^-)\right)$$
$$=m-(1-q(t))(\lambda\beta c+p_tc)+\lambda(1-q(t))\left(\pi_{t+1}(p_{t+1}^+,m,\mathcal{Q}_{t+1})-R_{t+1}(p_{t+1}^+)\right)$$
$$+(1-\lambda)(1-q(t))\left(\pi_{t+1}(p_{t+1}^-,m,\mathcal{Q}_{t+1})-R_{t+1}(p_{t+1}^-)\right)+(1-q(t))(\lambda-p_t)(p_{t+1}^+-p_{t+1}^-)c\textstyle\sum_{i=t+1}^T\prod_{j=t+1}^i(1-q(j)).$$

The last equality holds by Lemma 1. Since $p_{t+1}^+ - p_{t+1}^-$ is independent of p_t, then $(1-q(t))(\lambda - p_t)(p_{t+1}^+ - p_{t+1}^-)c\sum_{i=t+1}^T \prod_{j=t+1}^i (1-q(j))$ is decreasing in p_t. Because both $\pi_{t+1}(p_{t+1}^+, m, \mathcal{Q}_{t+1}) - R_{t+1}(p_{t+1}^+)$ and $\pi_{t+1}(p_{t+1}^-, m, \mathcal{Q}_{t+1}) - R_{t+1}(p_{t+1}^-)$ are decreasing in p_t, then $\pi_t(p_t, m, \mathcal{Q}_t) - R_t(p_t)$ is decreasing in p_t. Therefore, $\pi_1(p_1, m, \mathcal{Q}_1) - R_1(p_1)$ is decreasing in p_1. Next, we will show that $\pi_1(p_1, m^*(p_1), \mathcal{Q}_1) - R_1(p_1)$ is decreasing in p_1. For any $p_1' \le p_1$, then,

$$
\begin{aligned}
&(\pi_1(p_1, m^*(p_1), \mathcal{Q}_1) - R_1(p_1)) - (\pi_1(p_1', m^*(p_1'), \mathcal{Q}_1) - R_1(p_1')) \\
&\le (\pi_1(p_1, m^*(p_1), \mathcal{Q}_1) - R_1(p_1)) - (\pi_1(p_1', m^*(p_1), \mathcal{Q}_1) - R_1(p_1')) \le 0
\end{aligned}
$$

The first inequality holds because $\pi_1(p_1', m^*(p_1'), \mathcal{Q}_1) \ge \pi_1(p_1', m^*(p_1), \mathcal{Q}_1)$; the second inequality holds because $\pi_1(p_1, m, \mathcal{Q}_1) - R_1(p_1)$ is decreasing in p_1. Therefore, $\pi_1(p_1, m^*(P_1), \mathcal{Q}_1) - R_1(p_1)$ is decreasing in p_1. Thus, $\pi_1(p_1, m^*(p_1), \mathcal{Q}_1) > R_1(p_1) - S_1(\mathcal{Q}_1)$ for $p_1 < p_1^o$. $\square$

Proof of Theorem 5. We will show that the thresholds $\hat{p}_1^H$ and $\check{p}_1^H$ can be expressed as follows

$$
\hat{p}_1^H = \sup\left\{p_1^H \ge 0:\ \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \le \tfrac{1}{\gamma^H}(p_1^L - \lambda\beta)c\sum_{i=1}^T \prod_{j=1}^i (1-q(j)) - \tfrac{\gamma^L}{\gamma^H}\cdot \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L)\right\},
$$

$$
\check{p}_1^H = \inf\left\{p_1^H \ge 0:\ \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \le (p_1^L - \lambda\beta)c\sum_{i=1}^T \prod_{j=1}^i (1-q(j)) - \tfrac{\gamma^L}{\gamma^H}\cdot \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L)\right\}.
$$

Under Assumption 4, $S_1^L(\mathcal{Q}_1^L) = S_1^H(\mathcal{Q}_1^H) = \lambda\beta c\sum_{i=1}^T \prod_{j=1}^i (1-q(j))$ by Lemma 4. From Proposition 7, it is optimal for the traditional EW only to capture market segment H if and only if $p_1^H \ge (p_1^L - \lambda\beta\gamma^L)/\gamma^H$. We will consider the two cases separately.

If $p_1^H \ge (p_1^L - \lambda\beta\gamma^L)/\gamma^H$, the traditional only captures segment H, and that the flexible EW is more profitable is equivalent to

$$
\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \ge \gamma^H\left(R_1^H(p_1^H) - S_1^H(\mathcal{Q}_1^H)\right).
$$

By the same argument as in the proof of Theorem 4, we can show that $\pi_1(p_1^H, m, \mathcal{Q}_1^H) - R_1^H(p_1^H)$ is decreasing in p_1^H for any m. We will next prove that $\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1^H)\right)$ is decreasing in p_1^H. For any $p_1'^H \le p_1^H$,

$$
\begin{aligned}
&\Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1^H)\right)\Big) - \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) \\
&\quad + \gamma^H\left(\pi_1(p_1'^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H) - R_1^H(p_1'^H)\right)\Big) \\
&\le \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1^H)\right)\Big) - \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) \\
&\quad + \gamma^H\left(\pi_1(p_1'^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1'^H)\right)\Big) \\
&= \gamma^H\Big(\left(\pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1^H)\right) - \left(\pi_1(p_1'^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1'^H)\right)\Big) \le 0.
\end{aligned}
$$

The first inequality holds because $\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1'^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H\right) \le \gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1'^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H\right)$; the second inequality holds because $\pi_1(p_1^H, m, \mathcal{Q}_1^H) - R_1^H(p_1^H)$ is decreasing in p_1^H for any m. Therefore, $\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H\left(\pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) - R_1^H(p_1^H)\right)$ is decreasing in p_1^H and there exists a threshold $\check{p}_1^H$ such that the flexible EW is more profitable if $(p_1^L - \lambda\beta\gamma^L)/\gamma^H \le p_1^H < \check{p}_1^H$, where $\check{p}_1^H$ is

$$
\check{p}_1^H = \inf\left\{p_1^H \ge 0:\ \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \le (p_1^L - \lambda\beta)c\sum_{i=1}^T \prod_{j=1}^i (1-q(j)) - \tfrac{\gamma^L}{\gamma^H}\cdot \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L)\right\}.
$$

If $p_1^H < (p_1^L - \lambda\beta\gamma^L)/\gamma^H$, the traditional EW captures both segments L and H, and that the flexible EW is more profitable is equivalent to

$$\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \geq R_1^L(p_1^L) - S_1(\mathcal{Q}_1) = (p_1^L - \lambda\beta)c\sum_{i=1}^T \prod_{j=1}^i (1-q(j)).$$

The profit of the traditional EW is independent of the prior p_1^H as long as $p_1^H > p_1^L$ and we only need to show that $\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H)$ is increasing in p_1^H. Similarly, for any $p_1'^H \leq p_1^H$, we have

$$\begin{aligned}
&\Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H)\Big) - \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1'^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H)\Big) \\
&\geq \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H)\Big) - \Big(\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1'^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H)\Big) \\
&= \gamma^H \Big(\pi_1(p_1^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H) - \pi_1(p_1'^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H)\Big) \geq 0.
\end{aligned}$$

The first inequality holds because $m^*(p_1^L, p_1^H)$ is the optimal monthly premium in the heterogeneous market of two segments with prior estimates of failure probabilities (p_1^L, p_1^H) and $\gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \geq \gamma^L \pi_1(p_1^L, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^L) + \gamma^H \pi_1(p_1^H, m^*(p_1^L, p_1'^H), \mathcal{Q}_1^H)$; the second inequality holds because $\pi_1(p_1^H, m), \mathcal{Q}_1^H)$ is increasing in p_1^H for any m by Proposition 6. Therefore, there exists a threshold $\hat{p}_1^H$ such that the flexible EW is strictly more profitable if $\hat{p}_1^H < p_1^H < (p_1^L - \lambda\beta\gamma^L)/\gamma^H$, where

$$\hat{p}_1^H = \sup\left\{p_1^H \geq 0: \ \pi_1(p_1^H, m^*(p_1^L, p_1^H), \mathcal{Q}_1^H) \leq \tfrac{1}{\gamma^H}(p_1^L - \lambda\beta)c\sum_{i=1}^T \prod_{j=1}^i (1-q(j)) - \tfrac{\gamma^L}{\gamma^H} \cdot \pi_1(p_1^L, m^*(p_1^L, p_1^H), \mathcal{Q}_1^L)\right\}.$$

□

Proof of Proposition 7. The maximum willingness-to-pay of type-L and type-H customers are $R_1^L(p_1^L)$ and $R_1^H(p_1^H)$, respectively. Under martingale updating schemes and Assumption 4, $R_1^n(p_1^n) = p_1^n c\sum_{i=1}^T \prod_{j=1}^i (1-q(j))$, $n \in \{L, H\}$. Moreover, $R_1^H(p_1^H) \geq R_1^L(p_1^L)$ because $p_1^H \geq p_1^L$. From Lemma 4, $S_1^L(\mathcal{Q}_1^L)) = \lambda^L \beta c\sum_{i=1}^T \prod_{j=1}^i (1-q(j))$ and $S_1^H(\mathcal{Q}_1^H)) = \lambda^H \beta c\sum_{i=1}^T \prod_{j=1}^i (1-q(j))$.

When the traditional EW captures two market segments, the optimal price is $r = R_1^L(p_1^L)$ and the profit is $\pi_{HL}^t = \gamma^L(R_1^L(p_1^L) - S_1^L(\mathcal{Q}_1^L)) + \gamma^H(R_1^L(p_1^L) - S_1^H(\mathcal{Q}_1^H))$; when it only captures segment H, the optimal price is $r = R_1^H(p_1^H)$ and the profit is $\pi_H^t = \gamma^H(R_1^H(p_1^H) - S_1^H(\mathcal{Q}_1^H))$. That $\pi_{HL}^t > \pi_H^t$ is equivalent to $\gamma^H R_1^H(p_1^H) < R_1^L(p_1^L) - \gamma^L S_1^L(\mathcal{Q}_1^L)$. If the termination probabilities are the same for type-L and type-H customers, i.e., $\mathcal{Q}_1^L = \mathcal{Q}_1^H$, then $\pi_{HL}^t > \pi_H^t$ can be further simplified to $p_1^H < \frac{p_1^L - \lambda_L\beta\gamma^L}{\gamma^H}$. □

Proof of Theorem 6. Note that the thresholds of the optimal policies for the two segments are also the same but the purchase durations are different due to different true failure probabilities. Denote the expected cost to both types of customers as $R_1(p_1)$ if buying pay-as-you-go services. For the traditional EW provider, denote the expected support cost to a type-L (resp., type-H) customer as $S_1^L(\mathcal{Q}_1)$ (resp., $S_1^H(\mathcal{Q}_1)$). The total expected profit is equal to

$$\gamma^L\big(R_1(p_1) - S_1^L(\mathcal{Q}_1)\big) + \gamma^H\big(R_1(p_1) - S_1^H(\mathcal{Q}_1)\big).$$

Let $\pi_t^L(p_t, m, \mathcal{Q}_t)$ (resp., $\pi_t^H(p_t, m, \mathcal{Q}_t)$) be the flexible EW provider's expected profit per type-L (resp., type-H) customer. Similar to the case in a homogeneous market, $\pi_t^L(p_t, m, \mathcal{Q}_t)$ and $\pi_t^H(p_t, m, \mathcal{Q}_t)$ can be found by solving the following dynamic programs.

If $R_t(p_t) - B_t(p_t) \geq 0$,

$$\pi_t^L(p_t, m, \mathcal{Q}_t) = m + \lambda^L(1-q_0(t))\left(\pi_{t+1}(p_{t+1}^+, m, \mathcal{Q}_{t+1}) - \beta c\right) + (1-\lambda^L)(1-q_1(t))\pi_{t+1}(p_{t+1}^-, m, \mathcal{Q}_{t+1}),$$

$$\pi_t^H(p_t, m, \mathcal{Q}_t) = m + \lambda^H(1-q_0(t))\left(\pi_{t+1}(p_{t+1}^+, m, \mathcal{Q}_{t+1}) - \beta c\right) + (1-\lambda^H)(1-q_1(t))\pi_{t+1}(p_{t+1}^-, m, \mathcal{Q}_{t+1}).$$

If $R_t(p_t) - B_t(p_t) < 0$, $\pi_t^L(p_t, m, \mathcal{Q}_t) = \pi_t^H(p_t, m, \mathcal{Q}_t) = 0$.

Similar to the proof of Theorem 4, we can easily show that $\gamma^L(\pi_1^L(p_1, m, \mathcal{Q}_1) - R_1(p_1)) + \gamma^H(\pi_1^H(p_1, m, \mathcal{Q}_1) - R_1(p_1))$ is decreasing in p_1. Therefore, there exists a threshold $p_1^\sharp$ for the profitability of the flexible EW compared to the traditional EW in a heterogeneous market with two segments only differing in the true failure probability. $\square$

Appendix B: Single Flexible EW vs. Multiple Traditional EWs

As an alternative to offering a flexible EW to differentiate customers and achieve market segmentation, a provider may instead offer multiple traditional EWs with different coverage durations at different prices. In this section we consider this alternative and compare the profitability of a single flexible EW versus multiple traditional EWs in a heterogeneous market.

Suppose that the provider can offer two traditional EWs with durations T and τ respectively, assuming $\tau < T$. Using the same notations, the willingness-to-pay for the traditional EW with duration τ can be expressed by $R_1^n(p_1^n) - E[R_{\tau+1}^n(p_{\tau+1}^n)]$ and the support cost to the warranty provider is $S_1^n(\mathcal{Q}_1^n) - S_{\tau+1}^n(\mathcal{Q}_{\tau+1}^n)$ for a type-n customer, where $E[\cdot]$ denotes the expectation with respect to $p_{\tau+1}^n$, $n \in \{L, H\}$. Again, each customer will select the alternative with the lowest total support cost under the assumption of individual rationality.

The provider's problem is to determine prices for the two traditional EWs, denoted by r and r_τ respectively, to maximize the total expected profit:

$$\max_{r\geq 0, r_\tau \geq 0} \sum_{n\in\{L,H\}} \gamma^n \Big\{ (r - S_1^n(\mathcal{Q}_1^n)) \cdot \mathbf{1}\big(r \leq R_1^n(p_1^n), r \leq r_\tau + E[R_{\tau+1}^n(p_{\tau+1}^n)]\big) + \big(r_\tau - S_1^n(\mathcal{Q}_1^n) + S_{\tau+1}^n(\mathcal{Q}_{\tau+1}^n)\big) \cdot \mathbf{1}\big(r_\tau + E[R_{\tau+1}^n(p_{\tau+1}^n)] \leq R_1^n(p_1^n), r_\tau + E[R_{\tau+1}^n(p_{\tau+1}^n)] < r\big) \Big\}, \tag{15}$$

where the indicator functions $\mathbf{1}(\cdot)$ state the incentive compatibility constraints under which customers will select the alternative with the lowest support cost.

We assume that a customer will buy the traditional EW with longer duration if she is indifferent in the total expected support cost between the EWs with durations T and τ. We will show the profitability comparison by the following numerical example.

Example 7. We continue with Example 4 in a heterogeneous market of two segments with proportions $\gamma^L = 65\%$ and $\gamma^H = 35\%$. The termination probabilities are: $q_0^L(t) = q_1^L(t) = 0$ for $t = 1, 2, \ldots, 6$; $q_0^L(t) = 25\%, q_1^L(t) = 5\%$ for $t = 7, 8, \ldots, 12$; $q_0^H(t) = 15\%, q_1^H(t) = 5\%$ for $t = 1, 2, \ldots, 12$. We will fix the prior monthly failure probability of type-H customers at $p_1^H = 3\%$ and study the profitability variation with respect to p_1^L, the prior failure probability of type-L customers.

Figure 6 **Single Flexible EW vs. Two Traditional EWs in a Heterogeneous Market**

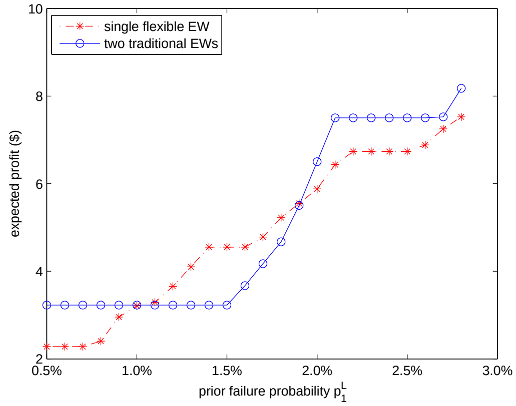

Figure 6 shows the profitability comparison between a single flexible EW and two traditional EWs with different durations. For $0 \leq p_1^L \leq 1.5\%$, the optimal traditional EWs are priced such that only type-H customers buy the warranty with duration T and the total profit are constantly equal to \$3.05; for $1.6\% \leq p_1^L < 1.9\%$, type-L customers purchase the traditional EW with duration $\tau = 6$ and type-H customers buy the one with duration $T = 12$; for $1.9\% \leq p_1^L \leq 2.7\%$, both the segments buy the traditional EW with duration $T = 12$; for $p_1^L \geq 2.7\%$, the optimal combination of two traditional EWs only captures type-L customers.

As shown by Figure 6, there are clearly instances where the flexible EW is more profitable than a menu of traditional EWs with varying lengths, i.e., $1.1\% \leq p_1^L \leq 1.9\%$. For example, when $p_1^L = 1.5\%$, the profit of a single flexible EW is \$4.55, which is 41% improvement than the menu of two traditional EWs. This underscores the finding that the flexible EWs are advantageous when the market contains customers who initially underestimate the failure probability. □