Electronic Companion
Dynamic Pricing of Perishable Assets under Competition

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A. Demand Structures

We consider several of the most frequently-used classes of demand functions and verify that they indeed satisfy pseudo-convexity of the demand rate function and pseudo-concavity of the revenue rate function.

General Time-Varying Attraction Models

In the attraction models, customers choose each firm with probability proportional to its attraction value. Specifically, we have the following demand rate functions: for all i,

$$d_i(t,\vec{p}) = \lambda(t)\frac{a_i(t,p_i)}{\sum_{j=0}^m a_j(t,p_j)},$$

where $\lambda(t) > 0$, $a_i(t,p_i) \geqslant 0$ is the attraction value for firm i at time t, and

$$a_0(t) \equiv a_0(t,p_0) > 0$$

is interpreted as the value of the no-purchase option at time t. We emphasize that in order to have pseudo-convexity of the demand rate function holds with respect to one's own price (Proposition 1(i)), we need the no-purchase value to be positive. Since $\lambda(t)$ is always positive, it does not have impact on the signs of derivatives we will consider, hence we drop it in the following discussion.

Lemma 1 (Sufficient Condition of Pseudo-Convexity). *If a twice continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ satisfies that $f'(x)=0 \Longrightarrow f''(x)>0$, then f is pseudo-convex, i.e., for any x_1 and x_2, $(x_1-x_2)f'(x_2)\geqslant 0 \Longrightarrow f(x_1)\geqslant f(x_2)$.*

Proof of Lemma 1. For each x_0 with $f'(x_0)=0$, we have $f''(x_0)>0$. This means that whenever the function f' reaches the value 0, it is strictly increasing. Therefore it can reach the value 0 at most once. If f' does not reach the value 0 at all, then f is either strictly decreasing or strictly increasing, and therefore pseudo-convex: if f is strictly decreasing, then $(x_1-x_2)f'(x_2)\geqslant 0 \Longrightarrow x_1 \leqslant x_2 \Longrightarrow f(x_1)\geqslant f(x_2)$; if f is strictly increasing, then $(x_1-x_2)f'(x_2)\geqslant 0 \Longrightarrow x_1\geqslant x_2 \Longrightarrow f(x_1)\geqslant f(x_2)$. Otherwise f' must reach the value 0 exactly once, say at x_0. Since $f''(x_0)>0$, it follows that

$f'(x) < 0$ for $x < x_0$, and $f'(x) > 0$ for $x > x_0$. Again in this case, f is pseudo-convex: if $x_2 = x_0$, we always have $f(x_1) \geqslant f(x_2) = f(x_0)$ for any x_1; if $x_2 < x_0$, then $(x_1 - x_2)f'(x_2) \geqslant 0 \Longrightarrow x_1 \leqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$; and if $x_2 > x_0$, then $(x_1 - x_2)f'(x_2) \geqslant 0 \Longrightarrow x_1 \geqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$. □

We can extend Lemma 1 to the functional space.

LEMMA 2. *Let V be a Hilbert space with associated scalar product $\langle \cdot, \cdot \rangle$ and $J[f]$ be a functional that is Gateaux-differentiable on V. If for any functional $f \in V$,*

$$\delta J[f; h] = 0 \text{ for all directions } h \neq 0 \text{ a.e.} \Longrightarrow \delta^2 J[f; h, h] > 0 \text{ for all such directions}, \tag{A1}$$

then $J[f]$ is pseudo-convex in the functional f, i.e., for any $f, f' \in V$,

$$\langle f' - f, \nabla_f J[f] \rangle \geqslant 0 \Longrightarrow J[f'] \geqslant J[f]. \tag{A2}$$

Proof of Lemma 2. Consider any functional $f, f' \in V$. By the definition of Gateaux differential (see, e.g., Luenberger 1997, Section 7.2), for any direction h,

$$\delta J[f; h] = \lim_{\alpha \to 0} \frac{J[f + \alpha h] - J[f]}{\alpha} = \frac{d}{d\alpha} J[f + \alpha h] \bigg|_{\alpha = 0}, \tag{A3}$$

$$\delta^2 J[f; h, h] = \lim_{\alpha \to 0} \frac{\delta J[f + \alpha h; h] - \delta J[f; h]}{\alpha} = \frac{d^2}{d\alpha^2} J[f + \alpha h] \bigg|_{\alpha = 0}. \tag{A4}$$

To show that $J[f]$ is pseudo-convex in the functional f, let $h' = f' - f$ and consider the class of functionals $\{f(\epsilon) = f + \epsilon h', \epsilon \in \mathbb{R}\}$. Note that $f(\epsilon = 1) = f'$ and $f(\epsilon = 0) = f$. By Equations (A3) and (A4), the stipulation (A1) is equivalent to

$$\frac{d}{d\alpha} J[f + \alpha h] \bigg|_{\alpha = 0} = 0 \Longrightarrow \frac{d^2}{d\alpha^2} J[f + \alpha h] \bigg|_{\alpha = 0} > 0. \tag{A5}$$

Note that the stipulation (A5) applies for all functionals f and directions h. In particular, applying (A5) to the class of functionals $f(\epsilon) = f + \epsilon h'$ and the direction $h' = f' - f$, we have

$$\frac{d}{d\alpha} J[f(\epsilon) + \alpha h'] \bigg|_{\alpha = 0} = 0 \Longrightarrow \frac{d^2}{d\alpha^2} J[f(\epsilon) + \alpha h'] \bigg|_{\alpha = 0} > 0. \tag{A6}$$

By change of variables, we have

$$\frac{d}{d\alpha} J[f(\epsilon) + \alpha h'] \bigg|_{\alpha = 0} = \frac{d}{d\alpha} J[f + \alpha h'] \bigg|_{\alpha = \epsilon} = 0 \Longrightarrow \frac{d^2}{d\alpha^2} J[f(\epsilon) + \alpha h'] \bigg|_{\alpha = 0} = \frac{d^2}{d\alpha^2} J[f + \alpha h'] \bigg|_{\alpha = \epsilon} > 0. \tag{A7}$$

Define $\mathbf{f}(\epsilon) \equiv J[f + \epsilon h']$. Note that $\mathbf{f}(\epsilon)$ is a *one-dimensional function* of the real variable ϵ. We can equivalently rewrite (A7) as: for any ϵ,

$$\mathbf{f}'(\epsilon) = 0 \Longrightarrow \mathbf{f}''(\epsilon) > 0.$$

By Lemma 1, $\mathbf{f}(\epsilon)$, as a one-dimensional function, is pseudo-convex in ϵ, and in particular, applying the pseudo-convexity property to $\epsilon_1 = 1$ and $\epsilon_2 = 0$, we have

$$0 \leqslant (1-0)\cdot \mathbf{f}'(\epsilon=0) = \mathbf{f}'(\epsilon=0) = \frac{d}{d\epsilon}J[f+\epsilon h']\Big|_{\epsilon=0} \Longrightarrow J[f'] = \mathbf{f}(\epsilon=1) \geqslant \mathbf{f}(\epsilon=0) = J[f]. \quad \text{(A8)}$$

It remains to verify that $\langle f'-f, \nabla_f J[f]\rangle \geqslant 0$ is equivalent to $\mathbf{f}'(\epsilon=0) = \frac{d}{d\epsilon}J[f+\epsilon h']\big|_{\epsilon=0} \geqslant 0$. To see this,

$$\langle f'-f, \nabla_f J[f]\rangle = \delta J[f; f'-f] = \frac{d}{d\epsilon}J[f+\epsilon h']\Big|_{\epsilon=0},$$

where the first equation is due to the definition of the scalar product $\langle \cdot,\cdot\rangle$ (see, e.g., Friesz 2010, Definition 4.35) and the second equation is due to Equation (A3). Hence (A8) is equivalent to

$$\langle f'-f, \nabla_f J[f]\rangle \geqslant 0 \Longrightarrow J[f'] \geqslant J[f].$$

This completes the proof. □

LEMMA 3. *If $g(t,f)$ satisfies that for almost all t, $\partial g(t,f)/\partial f = 0 \Longrightarrow \partial^2 g(t,f)/\partial f^2 > 0$, then $J[f] \equiv \int_0^T g(t,f(t))\,dt$ is pseudo-convex in the functional $f = \{f(t), 0\leqslant t\leqslant T\}$.*

Proof of Lemma 3. We consider any stationary point $\hat f$ such that for *all* directions $h \neq 0$ a.e.,

$$\delta J[\hat f; h] = \int_0^T \frac{\partial g(t,f)}{\partial f}\Big|_{f=\hat f} h(t)\,dt = 0,$$

where the first equation is due to Luenberger (1997, Example 2 in section 7.2).

We must have $\frac{\partial g(t,f)}{\partial f} = 0$ for almost all t. Then by stipulation that $\partial g(t,f)/\partial f = 0 \Longrightarrow \partial^2 g(t,f)/\partial f^2 > 0$, we have $\partial^2 g(t,f(t))/\partial f^2 > 0$ for almost all t. Hence, for all directions $h\neq 0$ a.e.,

$$\delta^2 J[\hat f; h,h] = \int_0^T \frac{\partial^2 g(t,f)}{\partial f^2}\Big|_{f=\hat f} h^2(t)\,dt > 0,$$

where the first equation is due to repeatedly applying the result of Luenberger (1997, Example 2 in section 7.2) to the first variation $\delta J[\hat f; h]$. Hence we just showed that for any functional f,

$$\begin{aligned}&\delta J[f;h] = 0 \text{ for all directions } \; h\neq 0 \text{ a.e.}\\ &\qquad\qquad \Longrightarrow \delta^2 J[f;h,h] > 0 \text{ for all such directions.}\end{aligned}$$

By Lemma 2, $J[f]$ is pseudo-convex in the functional f. □

We have the following structural results on the general attraction demand functions. We assume $a_i(t)$ is twice continuously differentiable. For notation simplicity, we drop arguments and let $a_0 \equiv a_0(t) > 0$, $a_i \equiv a_i(t,p_i), a_i' \equiv \partial a_i(t,p_i)/\partial p_i$ and $a_i'' \equiv \partial^2 a_i(t,p_i)/\partial {p_i}^2$.

PROPOSITION 1 (PSEUDO PROPERTIES OF ATTRACTION MODELS). *The following pseudo-properties of general attraction models hold:*

(i) if $a_i'' > (resp. <) 0$ for all i, $d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i for all i;

(ii) if $a_i > 0$, $a_i'' < (resp. >) 0$ for all i, $d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_j for all $j \neq i$;

(iii) if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$ for all i, $r_i(t,\vec{p}) - \mu d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i for all i and $\mu \in \mathbb{R}$.

Proof of Proposition 1. (i) Taking the first order derivative of $d_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial d_i}{\partial p_i} = \frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $d_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial^2 d_i}{\partial {p_i}^2} = \frac{a_i'' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \frac{2(a_i')^2 \sum_{j\neq i} a_j}{(\sum_j a_j)^3} = \frac{a_i'' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \left(\frac{\partial d_i}{\partial p_i}\right) \frac{2a_i'}{\sum_j a_j}.$$

Since $a_0 > 0$, then $\sum_{j\neq i} a_j > 0$. Hence whenever $\partial d_i/\partial p_i = 0$, $\partial^2 d_i/\partial {p_i}^2 > (resp. <) 0$ if $a_i'' > (resp. <) 0$. By Lemma 1, $d_i(t,\vec{p})$ is pseudo-convex (pseudo-concave) in p_i if $a_i'' > (resp. <) 0$.

(ii) Taking the first order derivative of $d_i(t,\vec{p})$ with respect to p_j,

$$\frac{\partial d_i}{\partial p_j} = -\frac{a_i a_j'}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $d_i(t,\vec{p})$ with respect to p_j,

$$\frac{\partial^2 d_i}{\partial {p_j}^2} = -\frac{a_i a_j''}{(\sum_j a_j)^2} + \frac{2a_i (a_j')^2}{(\sum_j a_j)^3} = -\frac{a_i a_j''}{(\sum_j a_j)^2} - \left(\frac{\partial d_i}{\partial p_j}\right) \frac{2a_j'}{\sum_j a_j}.$$

Whenever $\partial d_i/\partial p_j = 0$, $\partial^2 d_i/\partial {p_j}^2 > (resp. <) 0$ if $a_i > 0$, $a_j'' < (resp. >) 0$. By Lemma 1, $d_i(t,\vec{p})$ is pseudo-convex (pseudo-concave) in p_j for all $j \neq i$ if $a_i > 0$, $a_i'' < (resp. >) 0$ for all i.

(iii) Taking the first order derivative of $r_i(t,\vec{p}) - \mu d_i(t,\vec{p}) = (p_i - \mu) d_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial r_i}{\partial p_i} = d_i + (p_i - \mu)\frac{\partial d_i}{\partial p_i} = \frac{a_i}{\sum_j a_j} + (p_i - \mu)\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2}.$$

Taking the second order derivative of $r_i(t,\vec{p})$ with respect to p_i,

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{\partial d_i}{\partial p_i} + (p_i - \mu)\frac{\partial^2 d_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} + (p_i - \mu)\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} \left(\frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j}\right).$$

Whenever $\partial r_i/\partial p_i = 0$, $(p_i - \mu) a_i' \sum_{j\neq i} a_j / (\sum_j a_j)^2 = -a_i / \sum_j a_j$, thus

$$\frac{\partial^2 r_i}{\partial {p_i}^2} = 2\frac{a_i' \sum_{j\neq i} a_j}{(\sum_j a_j)^2} - \frac{a_i}{\sum_j a_j}\left(\frac{a_i''}{a_i'} - \frac{2a_i'}{\sum_j a_j}\right) = \frac{2a_i' - a_i a_i''/a_i'}{\sum_j a_j} > (resp. <) 0,$$

if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$. By Lemma 1, $r_i(t,\vec{p}) - \mu d_i(t,\vec{p})$ is pseudo-convex (resp. pseudo-concave) in p_i if $2a_i' - a_i a_i''/a_i' > (resp. <) 0$. □

In the proof of Proposition 1, what we essentially show is that the demand rate function satisfies that at a stationary point the second order derivative of the function is always positive. Hence, by Lemmas 2 and 3, we immediately have the following results.

PROPOSITION 2. *The following functional pseudo-properties of general attraction models hold:*

(i) if $a_i'' >$ (resp. $<$)0 for all i, $\int_0^T d_i(t,\vec{p}(t))\,dt$ is pseudo-convex (resp. pseudo-concave) in $\{p_i(t), 0 \leqslant t \leqslant T\}$ for all i;

(ii) if $a_i > 0$, $a_i'' <$ (resp. $>$)0 for all i, $\int_0^T d_i(t,\vec{p}(t))\,dt$ is pseudo-convex (resp. pseudo-concave) in $\{p_j(t), 0 \leqslant t \leqslant T\}$ for all $j \neq i$;

(iii) if $2a_i' - a_i a_i''/a_i' >$ (resp. $<$)0 for all i, $\int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t))]\,dt$ is pseudo-convex (resp. pseudo-concave) in $\{p_i(t), 0 \leqslant t \leqslant T\}$ for all i and $\{\mu(t), 0 \leqslant t \leqslant T\}$.

The MNL demand assumes $a_i(t,p_i) = \beta_i(t)\exp(-\alpha_i(t)p_i)$, $\alpha_i(t), \beta_i(t) > 0$ for all i. Since $a_i'' = \alpha_i(t)^2 \beta_i(t)\exp(-\alpha_i(t)p_i) > 0$ and $2a_i' - a_i a_i''/a_i' = -\alpha_i(t)\beta_i(t)\exp(-\alpha_i(t)p_i) < 0$, we have the following corollary as an immediate result of Proposition 2.

COROLLARY 1. *For the MNL demand, $\int_0^T d_i(t,\vec{p}(t))\,dt$ for all i is pseudo-convex in $\{p_i(t), 0 \leqslant t \leqslant T\}$, is pseudo-concave in $\{p_j(t), 0 \leqslant t \leqslant T\}$ for all $j \neq i$, and $\int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t))]\,dt$ for all i and all $\{\mu(t), 0 \leqslant t \leqslant T\}$ is pseudo-concave in $\{p_i(t), 0 \leqslant t \leqslant T\}$.*

Linear Models

The demand rate function has the form of

$$d_i(t,\vec{p}) = a_i(t) - b_i(t)p_i + \sum_{j \neq i} c_{ij}(t) p_j,$$

where $a_i(t), b_i(t) > 0$ for all i and $c_{ij}(t) \in \mathbb{R}$ for all $j \neq i$, for all t. It is easy to see that $d_i(t,\vec{p})$ for any i is convex in p_j for all j and $r_i(t,\vec{p})$ for all i is strictly concave in p_i. Then $\int_0^T d_i(t,\vec{p}(t))\,dt$ is convex in $\{p_j(t), 0 \leqslant t \leqslant T\}$ for all j and $\int_0^T r_i(t,\vec{p}(t))\,dt$ is concave in $\{p_i(t), 0 \leqslant t \leqslant T\}$. We immediately have the following result.

PROPOSITION 3 (LINEAR MODEL). *For the linear demand model, $\int_0^T d_i(t,\vec{p}(t))\,dt$ for all i is pseudo-convex in $\{p_i(t), 0 \leqslant t \leqslant T\}$ and $\int_0^T r_i(t,\vec{p}(t))\,dt$ for all i is pseudo-concave in $\{p_i(t), 0 \leqslant t \leqslant T\}$.*

Note that these pseudo-properties do not use the signs of cross-price elasticity term c_{ij}'s, hence a linear demand model of complementary products also satisfies Assumptions 1(b) and 2(a).

B. The Fixed Point Theorem

THEOREM 1 (BOHNENBLUST AND KARLIN (1950, THEOREM 5)). *Let X be a weakly separable Banach space with S a convex, weakly closed set in X. Let $\mathcal{B}: S \to 2^S \backslash \{\emptyset\}$ be a set-valued mapping satisfying the following:*

(a) $\mathcal{B}(x)$ is convex for each $x \in S$;

(b) The graph of $\mathcal{B}, \{(x,y) \in S \times S : y \in \mathcal{B}(x)\}$, is weakly closed in $X \times X$. That is, if $\{x_n\}$ and $\{y_n\}$ are two sequences in S such that $x_n \to x$, $y_n \to y$, weakly in X with $x_n \in \mathcal{B}(y_n)$, then necessarily we have $x \in \mathcal{B}(y)$;

(c) $\bigcup_{x \in S} \mathcal{B}(x)$ is contained in a sequentially weakly compact set;

Then there exists $x^* \in S$ such that $x^* \in \mathcal{B}(x^*)$.

C. HJB Equivalence

We establish the equivalency between the HJB equation (6) and the optimization problem by showing that any feasible solution $\vec{V}(t,\vec{n})$ to the optimization problem is an upper bound of the value function $\vec{V}^*(t,\vec{n})$ satisfying the HJB equation (6). We prove it by induction on the value of $\vec{e}^{\mathrm{T}}\vec{n}$, where $\vec{e}$ denotes a vector with all entries being ones. As an initial step, for $\vec{n} = 0$ such that $\vec{e}^{\mathrm{T}}\vec{n} = 0$, by the boundary conditions, $\vec{V}(t,\vec{n}) = \vec{V}^*(t,\vec{n}) = 0$ for all t. Now suppose for all $\vec{n}$ such that $\vec{e}^{\mathrm{T}}\vec{n} = l_o$, we have $\vec{V}(t,\vec{n}) \geqslant \vec{V}^*(t,\vec{n})$ for all t. Let us consider any $\vec{n}_o$ such that $\vec{e}^{\mathrm{T}}\vec{n}_o = l_o + 1$. We further show by induction on time. As an initial step, for $s = 0$, again by the boundary conditions, we have $\vec{V}(0,\vec{n}_o) = \vec{V}^*(0,\vec{n}_o) = 0$. Suppose for some $s_o \geqslant 0$, we have $\vec{V}(s,\vec{n}_o) \geqslant \vec{V}^*(s,\vec{n}_o)$ for all $s \in [0, s_o]$. For any i, there exists $h > 0$ small enough such that

$$
\begin{aligned}
& V_i(s_o + h, \vec{n}_o) \\
= {} & V_i(s_o, \vec{n}_o) + \frac{\partial V_i(s_o, \vec{n}_o)}{\partial s} h + o_1(h) \\
\geqslant {} & V_i(s_o, \vec{n}_o) + \{r_i(\vec{p}(T - s_o, \vec{n}_o)) - \vec{d}(\vec{p}(T - s_o, \vec{n}_o))^{\mathrm{T}} \nabla \vec{V}_i(s_o, \vec{n}_o)\} h + o_1(h) \\
\geqslant {} & V_i(s_o, \vec{n}_o) + \{r_i(\vec{p}^*(T - s_o, \vec{n}_o)) - \vec{d}(\vec{p}^*(T - s_o, \vec{n}_o))^{\mathrm{T}} \nabla \vec{V}_i(s_o, \vec{n}_o)\} h + o_1(h) \\
= {} & [1 - \vec{e}^{\mathrm{T}} \vec{d}(\vec{p}^*(T - s_o, \vec{n}_o)) h] V_i(s_o, \vec{n}_o) \\
& \quad + \vec{d}(\vec{p}^*(T - s_o, \vec{n}_o))^{\mathrm{T}} (V_i(s_o, \vec{n}_o - \vec{e}_1), V_i(s_o, \vec{n}_o - \vec{e}_2), \ldots, V_i(s_o, \vec{n}_o - \vec{e}_m)) h + o_1(h) \\
\geqslant {} & [1 - \vec{e}^{\mathrm{T}} \vec{d}(\vec{p}^*(T - s_o, \vec{n}_o)) h] V_i^*(s_o, \vec{n}_o) \\
& \quad + \vec{d}(\vec{p}^*(T - s_o, \vec{n}_o))^{\mathrm{T}} (V_i^*(s_o, \vec{n}_o - \vec{e}_1), V_i^*(s_o, \vec{n}_o - \vec{e}_2), \ldots, V_i^*(s_o, \vec{n}_o - \vec{e}_m)) h + o_1(h) \\
= {} & V_i^*(s_o + h, \vec{n}_o) + o_2(h),
\end{aligned}
$$

where the first inequality is due to the feasibility of $\vec{V}(s,\vec{n})$ to the optimization problem, the second inequality is due to the inequality constraints in the optimization problem hold for all pricing strategies, and the third inequality is due to the induction hypothesis. Therefore, there exists a neighborhood $[s_o, s_o + h_o]$ with $h_o > 0$ such that $V_i(s,\vec{n}_o) \geqslant V_i^*(s,\vec{n}_o)$ for all $s \in [s_o, s_o + h_o]$. □

D. Computation of OLNE

Friesz (2010, Chapter 10) formulates the equilibrium problem as an infinite-dimensional differential quasi-variational inequality and computes the generalized differential Nash equilibrium by a gap function algorithm. Adida and Perakis (2010) discretize the time horizon and solve for the finite-dimensional generalized Nash equilibrium by a relaxation algorithm. Instead, we explore the structural property of our differential game and cast the computation of OLNE as a much smaller size of finite-dimensional nonlinear complementarity problem (NCP).

By Proposition 2, the OLNE is equivalently characterized by the following m^2-dimensional NCP:

$$\mu_{ij}\left(C_j - \int_0^T d_j(t, \vec{p}^*(t; [\mu_{ij}]_{m\times m}))\, dt\right) = 0, \text{ for all } i, j,$$
$$C_j - \int_0^T d_j(t, \vec{p}^*(t; [\mu_{ij}]_{m\times m}))\, dt \geqslant 0, \text{ for all } i, j, \quad \mu_{ij} \geqslant 0, \text{ for all } i, j,$$

with appropriate ancillary decreasing shadow price processes $\mu_{ij}^-(t) \in [0, \mu_{ij}]$ for all i, j that can shut down demand upon a stockout, where $\vec{p}^*(t; [\mu_{ij}]_{m\times m})$ is the solution of (4) for any given matrix of shadow prices $[\mu_{ij}]_{m\times m} \geqslant 0$ at any time t that may have closed-form solutions in some cases, e.g., under linear demand models. The process of computing the equilibrium candidate $\{\vec{p}^*(t; [\mu_{ij}]_{m\times m}), 0 \leqslant t \leqslant T\}$ involves solving the one-shot price competition game (5) at any time on an on-going basis from $t = 0$ while keeping checking whether firms have run out of inventory; whenever a firm's inventory process hits zero, we can check if there exists decreasing shadow price processes of shutting down demand: if so, the firm exits the market and the price competition afterwards only involves remaining firms of positive inventory with an updated demand function taking consideration of spillover; otherwise, the matrix of shadow prices does not sustain as equilibrium shadow prices. If a bounded rational OLNE is sought after, we can restrict $\mu_{ij}(t) = 0$ for all t and all $i \neq j$ and further reduce the NCP to an m-dimension problem. Upon a stockout, the checking of whether there exist appropriate decreasing shadow price processes to shut down demand is also much simplified for computation of bounded rational OLNE. For many commonly used demand models, e.g., MNL and linear, there exists a unique equilibrium candidate $\{\vec{p}^*(t; [\mu_{ij}]_{m\times m}), 0 \leqslant t \leqslant T\}$ for any set of nonnegative shadow prices $[\mu_{ii}]_{m\times 1}$ with $\mu_{ij} = 0$ for all $i \neq j$. Mature computation algorithms for NCP with (i) a sub-loop of computing the equilibrium candidate and (ii) upon a stockout a sub-loop of checking whether choke prices can be generated by decreasing shadow price processes, can be applied to identify OLNE that indeed satisfies the complementarity condition.

E. Verification of OLNE in Example 2

Since firms have limited capacity relative to the sales horizon, their revenues depend on how high prices can be set to sell the capacity. It is definitely worse off for any firm i to sell faster in its monopoly period by setting a price lower than the market-clearing price p^* that sells off capacity over the half horizon. This rules out the possibility that firms want to have a monopoly sales horizon shorter than $T/2$. What about setting a price higher than p^*? Suppose firm i deviates by evening out ϵ amount of inventory from its monopoly period and competing in selling the ϵ amount with the competitor in firm $-i$'s originally monopoly period. First, we check if such a deviation is jointly feasible. It is obviously feasible for firm i as its total sales volume remains unchanged. To see the feasibility for firm $-i$, we check the derivative $(\partial d_{-i}(t,p_i,p_{-i})/\partial p_i)(\partial p_i^{-1}(t,d_i,p_{-i})/\partial d_i)$, where $p_i^{-1}(t,d_i,p_{-i})$ is the inverse function of $d_i(t,p_i,p_{-i})$. This derivative captures the impact, on firm $-i$'s sales, of firm i's small change in its sales by varying its price p_i while the competitor's price p_{-i} is fixed.

$$\frac{\partial d_{-i}(t,p_i,p_{-i})}{\partial p_i}\frac{\partial p_i^{-1}(t,d_i,p_{-i})}{\partial d_i} = \begin{cases} -\gamma_L & \text{if } t \in [(i-1)T/2, iT/2)\,, \\ -\gamma_H & \text{otherwise.} \end{cases}$$

The deviation will cause the sales of firm $-i$ to increase by $\gamma_L\epsilon$ amount in firm i's monopoly period and to decrease by $\gamma_H\epsilon$ amount in firm $-i$'s originally monopoly period. The total sales of firm $-i$ will decrease by $(\gamma_H-\gamma_L)\epsilon$ amount under firm i's deviation, which remains feasible for firm $-i$ for all $\epsilon\in[0,1)$. Next, we fix firm $-i$'s policy at $\{p^*_{-i}(t), 0\leqslant t\leqslant T\}$ to see firm i's payoff under the deviation of evening out the ϵ amount. The highest price $\bar{p}$ firm i can sell the ϵ amount is $\bar{p}$ such that $(1-\bar{p}+\gamma_L p^*)T/2=\epsilon$. We solve $\bar{p}=1+\gamma_L p^*-2\epsilon/T$. The highest price firm i can sell the $1-\epsilon$ amount in its monopoly period is $\tilde{p}$ such that $[1-\tilde{p}+\gamma_H(1+\gamma_L p^*)]T/2=1-\epsilon$. We solve $\tilde{p}=p^*+2\epsilon/T$. The profit firm i can earn under the deviation is

$$\tilde{p}(1-\epsilon)+\bar{p}\epsilon = p^* + \epsilon\left[\frac{\gamma_L-\gamma_H}{1-\gamma_H\gamma_L}+\frac{2(2-\gamma_L-\gamma_H\gamma_L)}{T(1-\gamma_H\gamma_L)}-\frac{4\epsilon}{T}\right] < p^*$$

for all $\epsilon\in(0,1]$, provided that $T>\frac{2(2-\gamma_L-\gamma_H\gamma_L)}{\gamma_H-\gamma_L}$ (note that $\gamma_L<\gamma_H$). Hence if T is sufficiently large, the proposed joint policy is indeed an OLNE where firms are alternating monopolies. □

References

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