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Dynamic Pricing of Perishable Assets Under Competition

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We study dynamic price competition in an oligopolistic market with a *mix* of substitutable and complementary perishable assets. Each firm has a fixed initial stock of items and competes in setting prices to sell them over a finite sales horizon. Customers sequentially arrive at the market, make a purchase choice, and then leave immediately with some likelihood of no purchase. The purchase likelihood depends on the time of purchase, product attributes, and current prices. The demand structure includes time-variant linear and multinomial logit demand models as special cases. Assuming deterministic customer arrival rates, we show that any equilibrium strategy has a simple structure, involving a finite set of *shadow prices* measuring capacity externalities that firms exert on each other: equilibrium prices can be solved from a one-shot price competition game under the *current-time* demand structure, taking into account capacity externalities through the *time-invariant* shadow prices. The former reflects the transient demand side at every moment, and the latter captures the aggregate supply constraints over the sales horizon. This simple structure sheds light on dynamic revenue management problems under competition, which helps capture the essence of the problems under demand uncertainty. We show that the equilibrium solutions from the deterministic game provide precommitted and contingent heuristic policies that are asymptotic equilibria for its stochastic counterpart, when demand and supply are sufficiently large.

Keywords: revenue management; oligopoly; dynamic pricing; open-loop; feedback strategy; differential game; stochastic game; approximate dynamic programming
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1. Introduction

Providers of perishable capacities, such as airlines, compete by setting prices to sell fixed capacities over a finite sales horizon. Online travel sites, such as Expedia, gather information and list flight fares in real time among competitive airlines. This price transparency enables customers to comparison shop among different products based on product attributes and prices. This trend requires that airlines respond in real time to competitors' pricing strategies. The real-time competitive pricing problem is further complicated by the fact that aggregate market demands and their elasticities evolve over time. For example, over the sales horizon, leisure-class customers tend to arrive earlier hunting for bargain tickets, and business-class customers tend to arrive later, willing to pay the full price.

Revenue management (RM) techniques help firms set the right price at the right time to maximize revenue. They have been successfully applied to airline and many other industries. Traditional RM models, however, typically assume a monopoly setting. The literature on competitive RM is scant. This is due, in part, to the challenges imposed by the complex game of capacitated intertemporal price competition, and the even more thorny problems of time-varying demands. A few stylized models have been built to examine the game from various perspectives. However, no *structural* results for the general problem have been discussed. Algorithmic approaches have also been implemented, hoping to capture the whole dynamics and a focus on the computation of equilibrium policies. Yet without the guidance of structural results on algorithm design, computational approaches can suffer from the curse of dimensionality.

We consider a setting where multiple capacity providers compete to sell their own fixed initial capacities of *differentiated*, substitutable or complementary, perishable items by varying prices over a common finite sales horizon. On the demand side, customers sequentially arrive at the market, make a purchase decision, and then leave immediately with some likelihood of no purchase. The customers' arrival rate and their price sensitivity can vary over time. The likelihood of purchase depends on the product attributes, prices, and the time of purchase. Each firm starts with initial inventories that cannot be replenished during the sales horizon.

We seek to uncover the strategic nature of this competitive RM game. To do this, we formulate this game as a differential game in continuous time. In this formulation, we obtain structural results that capture the nature of how transient market conditions and aggregate capacity constraints interact to jointly determine intertemporal pricing behavior in equilibrium. The structure revealed by the continuous time differential game would be lost if the formulation was instead in discrete time, or if we formulated the problem as a one-shot game. The structure of the differential game arises because (i) intertemporal sales share aggregate capacity constraints over the entire sales horizon, and (ii) demands (e.g., arrival rates and purchase likelihoods) are independent of inventory levels. Because of the structural result, the computation of continuous-time equilibrium policies can be cast as finite-dimensional optimization problems that can be efficiently solved. These solutions to the differential game can then be used to derive asymptotically-optimal heuristics for a formulation that takes into account demand uncertainty.

We provide two main contributions. First, we focus on the first-order effect in a market under demand uncertainty by assuming a deterministic arrival process. We show that the equilibrium strategy has a simple structure: There exists a finite set of *shadow prices* measuring aggregate capacity externalities that firms exert on each other; the equilibrium prices at any time can be solved from a one-shot price competition under the *current-time* demand structure, taking into account capacity externalities with time-invariant shadow prices. A firm with ample capacity does not exert any capacity externality on the price competition. A firm with limited capacity exerts capacity externality by alleviating price competition among substitutable products and by undercutting the prices of other complementary products. Because of the structure, the computation of infinite-dimensional continuous-time equilibrium pricing policies reduces to solving for finite-dimensional shadow prices.

Second, we show that insights from the deterministic problem are valuable in capturing the essence of the stochastic problem where customer arrivals follow certain random process. There is an active research stream of applying computational approaches, called approximate dynamic programming (ADP), to generate heuristics for the stochastic problem in RM. We show that the shadow prices obtained from the deterministic problem coincide with the solution obtained from an *affine* ADP approach to the stochastic game. Moreover, applying the efficiently computable solutions of the deterministic game, as committed or contingent (i.e., dynamic) pricing heuristics, to the stochastic game, sustains as an asymptotic equilibrium, when demand and supply are sufficiently large. In practice, our model suggests that in a competitive market environment under demand uncertainty, *re-solving heuristics* should perform well, in which firms constantly re-solve deterministic best-response problems with updated information about market conditions, such as firms' inventory levels and demand patterns.

1.1. Literature Review

There is a growing body of literature on competitive RM. Depending on the chosen decision variables, RM is categorized as quantity-based or price-based, or a mix of the two. Netessine and Shumsky (2005) examine one-shot, quantity-based RM duopoly games of setting booking limit controls under both horizontal and vertical competition. Jiang and Pang (2011) study an oligopolistic version in a network RM setting. These works ignore firms' intertemporal interactions.

Oligopoly pricing, common in the economics and marketing literature, is gaining traction within the RM community. Unlike a standard oligopoly pricing setting, firms in an RM model are capacity constrained and pricing decisions need to be made over time. One line of research is to use *variational inequalities* to characterize intertemporal price equilibrium under capacity constraints. Perakis and Sood (2006) study a discrete-time stochastic game of setting prices and protection levels by using variational inequalities and ideas from robust optimization. Mookherjee and Friesz (2008) consider a discrete-time combined pricing, resource allocation, overbooking RM problem under demand uncertainty over networks and under competition. Adida and Perakis (2010) consider a continuous-time deterministic differential game of joint pricing and inventory control where each firm's multiple products share production capacity. The authors also study the robust fluid model of the corresponding stochastic game. See also Friesz (2010, Chap. 10) for applications of finite or infinite dimensional quasi-variational inequality to various RM settings. This research stream seeks efficient algorithms to compute equilibrium prices. In contrast, we focus on structural properties of intertemporal equilibrium pricing behavior.

In the economics and marketing literature, many works resort to the differential game as a tool to study dynamic market interactions. Feichtinger and Dockner (1985) consider a differential game of price competition without capacity constraints in an oligopolistic market where the market shares evolve depending on the current market shares and prices posted by firms. Instead, we assume that demand depends on the current prices but not on the market shares, which conforms to the industries selling perishable assets over a short sales horizon. Moreover, we allow for more general demand structures and account for capacity constraints. Chintagunta and Rao (1996) consider a differential game of dynamic pricing in a duopolistic market with a logit demand and consumer preferences evolving over time. They focus on the steady state of the equilibrium open-loop price paths under uncapacitated price competition. In contrast, we study the differential game of dynamic pricing with capacity constraints and focus on structures of transient pricing behavior.

Some works assume that the competition is to sell a homogeneous product. Mantin (2008) analyzes a multiperiod duopoly pricing game where a homogeneous perishable good is sold to consumers who visit one of the retailers in each period. Talluri and Martínez de Albéniz (2011) study perfect competition of a homogeneous product in an RM setting under demand uncertainty and derive a closed-form solution to the equilibrium price paths. They show a structural property of the equilibrium policy such that the seller with the lower equilibrium reservation value sells a unit at a price equal to the competitor's equilibrium reservation value. This structural property is due to the nature of Bertrand competition of a homogeneous product that a seller is willing to undercut the competitor down to its own reservation value. The authors also show that the equilibrium sales trajectory is such that firms alternatively serve as a monopoly; the firm with less capacity sells out before the firm with more capacity. We complement Talluri and Martínez de Albéniz (2011) by studying price competition of differentiated products and exploring its structural nature. To customers who shop only for the lowest fares, the products can be viewed as more or less homogeneous. However, pricing transparency facilitated by third-party travel websites exposes the same price to various consumer segments with heterogeneous price sensitivities, e.g., loyal customers and bargain hunters. The aggregate demand structure is closer to the case of differentiated products, and the resulting equilibrium behavior is different from the case of a homogeneous product.

Strategic consumers have been examined in the competitive RM setting with various assumptions, such as what they know and how they behave. Levin et al. (2009) present a unified stochastic dynamic pricing game of multiple firms where differentiated goods are sold to finite segments of strategic customers who may time their purchases. The key insight is that firms may benefit from limiting the information available to consumers. Liu and Zhang (2013) study dynamic pricing competition between two firms offering vertically differentiated products to strategic consumers, where price skimming arises as a subgame perfect equilibrium. This model may be more applicable to the fashion industry, and less applicable to the airline industry where the average price trend is typically upward (see Pang et al. 2013). We do not take consumers' strategic waiting behavior into account and admit this as a limitation. One may argue that when the aggregate demand arrival process, as an input to our model, is calibrated from real data over repeated horizons, it should, to some extent, have captured the equilibrium waiting/purchase behavior of strategic consumers. Consequently, our model may provide a more practical approach to addressing strategic consumer behavior. Firms can repeatedly solve the same problem with updated time-varying demand patterns to address repeated interactions with strategic consumers.

Two papers closest to ours in the operations management literature are Lin and Sibdari (2009) and Xu and Hopp (2006). The former proves the existence of a pure-strategy subgame perfect Nash equilibrium in a discrete-time stochastic game with a stationary multinomial logit (MNL) demand. The main difference, apart from the demand structure and the choice of how to model time, is that we focus on the structural nature of the game and its implications, beyond the existence and uniqueness results. Similar to our paper, the latter studies a dynamic pricing problem under oligopolistic competition in a continuous-time setting. The authors establish a weak perfect Bayesian equilibrium of the pricing game. There are several notable differences. Most significantly, the latter obtains a cooperative *fixed-pricing* equilibrium strategy in a perfect competition of a homogeneous product. We obtain *time-varying* pricing strategies for imperfect competition with differentiated products. Furthermore, Xu and Hopp (2006) assume a quasi-linear consumer utility function. Our demand structure allows for a more general consumer utility function.

In the extension, we study Markovian pricing equilibrium in a continuous-time dynamic stochastic game over a finite horizon. In the economics literature, Pakes and McGuire (1994) develop an algorithm for computing Markovian equilibrium strategies in a discrete-time infinite-horizon dynamic game of selling differentiated products. Fershtman and Pakes (2000) apply the algorithm to a collusive framework with heterogeneity among firms' investment, entry, and exit. Borkovsky et al. (2010) discuss an application of the homotopy method to solving these dynamic stochastic games. Farias et al. (2012) introduce a new method to compute Markovian equilibrium strategies in large-scale dynamic oligopoly models by approximating the best-response value function with a linear combination of basis functions. (See references therein for comprehensive review of this line of development.) We show that a Markovian equilibrium of the continuous-time stochastic game is reduced to an equilibrium of the differential game if the value functions are approximated by affine functions. Moreover, instead of discretizing time to compute a Markovian equilibrium of the stochastic game like Lin and Sibdari (2009), we show that the heuristics suggested by the corresponding differential game are asymptotic equilibria with large supply and demand.

2. The Model

We introduce some notation: $\mathbb{R}_+ \equiv [0, +\infty)$ and $\mathbb{R}_{++} \equiv (0, +\infty)$, x_i denotes the ith component of vector $\vec{x}$, $\vec{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)^{\mathrm{T}}$ is a subvector of $\vec{x}$ with components other than i, and $\vec{e}_i$ denotes a vector with the ith element 1 and all other elements 0's. For notation simplicity, 0 can denote a scalar or a vector of any dimension with all entries being zeros. A function is said to be increasing (decreasing) when it is nondecreasing (nonincreasing).

We consider a market of m competing firms selling differentiated perishable assets over a finite horizon $[0, T]$. At time $t = 0$, each firm i has an initial inventory of C_i units of one product. (All results can be extended when a firm sells multiple products.) We count the time forwards, and use t for the elapsed time, and $s \equiv T - t$ for the remaining time.

2.1. Assumptions

Consumers sequentially arrive at the market and make a purchase choice based on attributes of the differentiated products and their current prices across the market. Both the arrival rate to the market and the purchase likelihood can be *time-dependent*. We specify the aggregate demand rate function in a general way: at any time $t \in [0, T]$, the vector of demand rates $\vec{d}(t, \vec{p}(t))$ for all firms is time-dependent and influenced by the current market price vector $\vec{p}(t)$. The general form of the demand rate functions can allow for general consumer utility functions and general time-varying arrival processes. In air-ticket selling, this demand rate function can be calibrated over repeated sales horizon from data of arrival rates to the market and intertemporal price elasticities for the same origin-destination "local" market. We further assume that the demand rate function is public information. In the airline industry, firms typically have access to the same sources of pricing/sales data and have very similar or sometimes even identical forecasting systems. We make the following general assumptions on the demand rate functions.

Assumption 1 (Demand Rate). *The following assumptions hold for all i:*

(a) (Differentiability). *$d_i(t, \vec{p})$ is continuously differentiable in $\vec{p}$ for all t;*

(b) (Pseudo-Convexity). *$\int_0^T d_i(t, \vec{p}(t))\, dt$ is pseudo-convex in $\{p_i(t), 0 \le t \le T\}$.*

As a technical remark, the pseudo-convexity assumption is a slight relaxation of convexity: a function f is pseudo-convex on a nonempty open set X if for any $x, y \in X$, $(y - x)^{\mathrm{T}} \cdot \nabla_x f(x) \ge 0 \Rightarrow f(y) \ge f(x)$, where ∇_x is the gradient operator. (Similarly, the pseudo-convexity of a functional in a normed space can be defined with the gradient replaced by the functional derivatives.) A function f is pseudo-concave if and only if $-f$ is pseudo-convex. Pseudo-convexity is stronger than quasi-convexity but weaker than convexity. Unlike standard oligopoly pricing problems without capacity constraints, we resort to this weaker version of convexity on the demand rate function to account for general demand functions and to address capacity constraints.

We denote the revenue rate function for any firm i at time t by $r_i(t, \vec{p}) \equiv p_i d_i(t, \vec{p})$.

Assumption 2 (Revenue Rate). *The following assumptions hold for all i:*

(a) (Pseudo-Concavity). *$\int_0^T r_i(t, \vec{p}(t))\, dt$ is pseudo-concave in $\{p_i(t), 0 \le t \le T\}$;*

(b) (Bounded Revenue). *There exists a function $\bar{R}_i(t)$ such that $r_i(t, \vec{p}) \le \bar{R}_i(t)$ for all t and $\int_0^T \bar{R}_i(t)\, dt < \infty$.*

The less used pseudo-convexity/concavity assumptions on demand and revenue rates are used to accommodate commonly used MNL demand functions:

$$d_i(t, \vec{p}) = \lambda(t) \frac{\beta_i(t) e^{-\alpha_i(t) p_i}}{a_0(t) + \sum_j \beta_j(t) e^{-\alpha_j(t) p_j}}, \tag{1}$$

where $\lambda(t)$, $a_0(t)$, $\alpha_i(t)$, $\beta_i(t) > 0$ for all i and t.

Lemma 1. *MNL demand functions* (1) *satisfy Assumptions* 1 *and* 2.

Proof of Lemma 1. See Corollary 1 in Electronic Companion A. (The electronic companion is available at http://ming.hu.) $\square$

Moreover, Assumptions 1 and 2 are also satisfied by linear demand functions where differentiated substitutable and complementary products *co-exist*, e.g., $d_i(t, \vec{p}) = a_i(t) - b_i(t) p_i + \sum_{j \ne i} c_{ij}(t) p_j$, where $a_i(t), b_i(t) > 0$ for all i, and $c_{ij}(t) \in \mathbb{R}$ for all $j \ne i$, for all t. These linear demand functions can arise when a representative consumer in the market maximizes a linear-quadratic utility function (see, e.g., Federgruen and Hu 2013).

Lemma 2. *Linear demand functions satisfy Assumptions* 1 *and* 2.

Proof of Lemma 2. See Proposition 3 in Electronic Companion A. $\square$

It is easy to see that in a linear demand model, each firm's feasible strategy set could depend on competitors' strategies. This is called *coupled strategy constraints* (coined by Rosen 1965). We make the following assumptions on the feasible strategy set of each firm.

Assumption 3 (Price Set). *The following assumptions hold for any competitors' prices $\vec{p}_{-i}$ for all i, t:*

(a) (Choke Price). *There exists a choke price $p_i^\infty(t, \vec{p}_{-i})$ such that $\lim_{p_i \to p_i^\infty(t, \vec{p}_{-i})} d_i(t, \vec{p}) = 0$ and $\lim_{p_i \to p_i^\infty(t, \vec{p}_{-i})} r_i(t, \vec{p}) = 0$, which is the only pricing option when a firm runs out of stock;*

(b) (Feasible Set). *Other than the choke price that is available at any time and is the only option on stockout, firm i chooses prices from the set $\mathcal{P}_i(t, \vec{p}_{-i})$ that is a nonempty, compact and convex subset of $\{p_i \in \mathbb{R}_+ \mid d_i(t, p_i, \vec{p}_{-i}) \ge 0\}$.*

Assumption 3(a) ensures that a firm immediately exits the market on a stockout. In this case, customers who originally prefer the stockout firm will *spill over* to the remaining firms that still have positive inventory. The spillover is endogenized from the demand model according to customers' preferences and product substitutability. For example, in any MNL demand function, ∞ is the choke price, yielding the attraction value of the stockout firm equal to zero. We further illustrate the spillover effect by the following example.

Example 1 (Demand Model with Spillover). For a duopoly with stationary linear demand rate functions $d_i(t, p_i, p_{-i}) = 1 - p_i + \gamma p_{-i}$, $i = 1, 2$, $\gamma \in [0, 1)$, firm 1 can post a choke price $p_1^\infty(t, p_2) = 1 + \gamma p_2$, which is solved from $d_1(t, p_1, p_2) = 1 - p_1 + \gamma p_2 = 0$, to shut down its own demand. The resulting demand rate function for firm 2 with the spillover effect is $d_2(t, p_1^\infty(t, p_2), p_2) = 1 - p_2 + \gamma p_1^\infty(t, p_2) = (1+\gamma) - (1-\gamma^2)p_2$. The spillover-adjusted demand for firm 2 has a higher potential market size (higher intercept, i.e., $1+\gamma \geq 1$), and is less price sensitive to firm 2's own price (smaller linear coefficient of p_2, i.e., $1-\gamma^2 \leq 1$), as compared to before firm 1 posts the choke price.

In view of Assumption 3(b), we do not exclude the possibility of shutting down demand by posting a choke price in a firm's strategy before its stockout. It is possible for firms to do so in equilibrium (see Example 2 in §3.5.1). The joint feasible price set at any time t is denoted by $\mathcal{P}(t) \equiv \{\vec{p} \mid p_i \in \mathcal{P}_i(t, \vec{p}_{-i}) \cup \{p_i^\infty(t, \vec{p}_{-i})\}, \forall\, i\}$.

We also assume that the salvage value of the asset at the end of the horizon is zero and that all other costs are sunk. We can always transform a problem with positive salvage cost c_i for firm i to the zero-salvage-cost case by changing variables from p_i to $p_i - c_i$ in the demand rate function.

2.2. The Differential Game

We formulate a finite-horizon noncooperative differential game, where demand is a deterministic fluid process (Dockner et al. 2000). In the extension, we consider its stochastic counterpart where demand follows a random process. Firms compete in influencing demand rates by adjusting prices. At any time $t \in [0, T]$, firm i sets its own price $p_i(t)$. We assume the following information structure throughout the paper.

Assumption 4 (Information Structure). *All firms have perfect knowledge about each other's inventory levels at any time.*

This assumption is standard in game theory for seeking subgame perfect equilibrium. It used to be unrealistic, but now inventory information in real time may be considered as being revealed in some way; almost all online travel agencies and major airlines offer a feature of previewing seat availability from their websites.

We denote by $\vec{x}(t)$ the joint inventory level at time t, which is assumed to be a *continuous* quantity in the differential game. Let $\mathcal{X} \equiv \times_i [0, C_i]$ denote the state space of inventory in the market. (The inventory level will be discrete in the stochastic extension; see §4.) We differentiate the following two types of pricing strategies. In an open-loop strategy, firms make an irreversible *precommitment* to a future course of action at the beginning of the game. Alternatively, feedback strategies designate prices according to the current time and joint inventory level, which capture the *feedback* reaction of competitors to the firm's chosen course of action.

Definition 1 (Open-Loop Strategy). A joint open-loop strategy $\vec{p}(t)$ depends only on time t and the given initial joint inventory level $\vec{x}(0) = \vec{C}$.

Definition 2 (Feedback Strategy). A joint feedback strategy $\vec{p}(t, \vec{x}(t))$ depends on time t and the current joint inventory level $\vec{x}(t)$.

The set of all joint open-loop strategies such that $\vec{p}(t) \in \mathcal{P}(t)$ for all t, is denoted by $\mathcal{P}_O$. The set of all joint feedback strategies such that $\vec{p}(t, \vec{x}(t)) \in \mathcal{P}(t)$ for all t, is denoted by $\mathcal{P}_F$. Let $D[0, T]$ denote the space of all right-continuous real-valued functions with left limits defined on interval $[0, T]$, where the left discontinuities accommodate price jumps after a sale in a pricing strategy. Given joint pricing control path $\vec{p} \in (D[0, T])^m$ (i.e., $\{\vec{p}(t), 0 \leq t \leq T\}$ for open-loop strategies and $\{\vec{p}(t) = \vec{p}(t, \vec{x}(t)), 0 \leq t \leq T\}$ for feedback strategies), we denote the total profit for any firm i by $J_i[\vec{p}] \equiv \int_0^T r_i(t, \vec{p}(t))\, dt$. Inventory depletes at the demand rate, hence the inventory evolves according to the following kinematic equation: for all i,

$$\dot{x}_i(t) = -d_i(t, \vec{p}(t)), \quad 0 \leq t \leq T, \quad \text{and} \quad x_i(0) = C_i. \tag{2}$$

Any firm i's objective is to maximize its own total revenue over the sales horizon subject to all capacity constraints at any time, i.e.,

problem (P_i)

$$\max_{\{p_i(t),\, 0 \leq t \leq T\}} \int_0^T r_i(t, \vec{p}(t))\, dt$$
$$\text{s.t.} \quad x_j(t) = C_j - \int_0^t d_j(v, \vec{p}(v))\, dv \geq 0, \quad 0 \leq t \leq T, \ \forall\, j. \tag{3}$$

Firms simultaneously solve their own revenue maximization problems subject to a joint set of constraints, giving rise to a game with coupled strategy constraints (3) for all i, i.e., any firm's feasible strategy set depends on competitors' strategies through these capacity constraints. For this type of game, Rosen (1965) coined the term a *generalized* Nash game with coupled constraints; see also Topkis (1998) for a treatment of such generalized games. In the differential game, the pricing strategies are simultaneously presented by all firms before the game starts. If some pricing policy is not jointly feasible such that one firm may have negative inventory at some time, then it will be eliminated from the joint feasible strategy space. In other words, all firms face a joint set of constraints, $\vec{x}(t) \geq 0$ for all t, in selecting feasible strategies such that their pricing strategies remain *credible*. This explains why any firm i is constrained by all firms' capacity constraints in its own revenue maximization problem (P_i).

The definitions of generalized Nash equilibrium for open-loop (OLNE) and feedback strategies (FNE) follow immediately. A generalized (omitted hereafter) OLNE (respectively, FNE) $\vec{p}^* \in \mathscr{P}_O$ (respectively, $\in \mathscr{P}_F$) is an m-tuple of open-loop (respectively, feedback) strategies such that its control path $\vec{p}^*(t) \in (D[0, T])^m$ and $\{p_i^*(t), 0 \leq t \leq T\}$ is a solution to problem (P_i) for all i. In a non-zero-sum differential game, open-loop and feedback strategies are generally different, in form or in terms of generated price path and inventory trajectory. However, we demonstrate in §3.4 that re-solving OLNE with the *current* time and inventory level continuously over time results in an FNE, which generates the same price path and inventory trajectory as those of the OLNE with the same *initial* time and inventory level. Because of this relationship between OLNE and FNE, for convenience, we may loosely call an OLNE, an equilibrium strategy, in the following discussion.

3. Equilibrium

In this section, we show equilibrium existence, and its uniqueness in some sense. We fully explore equilibrium structural properties by deriving necessary conditions for an equilibrium, and illustrate them with examples. We also establish sufficient conditions of an equilibrium for some special equilibrium concepts. These investigations help build insights and appropriate heuristics for the intractable stochastic problem.

3.1. Existence

To show the existence of an infinite-dimensional OLNE, we invoke an infinite-dimensional version of Kakutani's fixed-point theorem (Bohnenblust and Karlin 1950).

Proposition 1 (Existence of OLNE). *The following equilibrium existence results hold*:

(i) *If* $p_i^\infty(t, \vec{p}_{-i}) \in \mathscr{P}_i(t, \vec{p}_{-i})$, *there exists an OLNE.*

(ii) *For MNL demands, there exists an OLNE where firms do not use the choke price* ∞ *at any time.*

We list MNL demand models separately. Because any feasible price set containing MNLs choke price ∞ will not be compact and convex, we need to treat them differently.

3.2. Characterization

3.2.1. Necessary Condition: Equilibrium Structure. We follow the *maximum principle* of the differential game with constrained state space (see, e.g., Hartl et al. 1995) to derive the set of necessary conditions for OLNE. Then, under additional assumptions on demand and revenue rate functions, we verify that the set of necessary conditions can also be sufficient. The following necessary conditions capture the structure that any OLNE has to satisfy.

Proposition 2 (Necessary Condition of OLNE). *If the open-loop pricing policy* $\{\vec{p}^*(t)\colon 0 \leq t \leq T\}$, *with its corresponding inventory trajectory* $\{\vec{x}^*(t)\colon 0 \leq t \leq T\}$, *is an OLNE, then there exists a matrix of nonnegative shadow prices* $M \equiv [\mu_{ij}]_{m\times m} \geq 0$ *such that the following conditions are satisfied for all* i:

(i) (Equilibrium Prices). *For any time* t *such that* $x_i^*(t) > 0$,

$$p_i^*(t) = \operatorname*{arg\,max}_{p_i \in \mathscr{P}_i(t, \vec{p}^*_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}^*_{-i}(t))\}} \Big\{ r_i(t, p_i, \vec{p}^*_{-i}(t)) - \overbrace{\sum_j \mu_{ij} d_j(t, p_i, \vec{p}^*_{-i}(t))}^{\text{capacity externality}} \Big\}; \tag{4}$$

(ii) (Market Exit). *If the period with zero-inventory* $E_i = \{t \in [0, T] \mid x_i^*(t) = 0\}$ *is nonempty, then during this period, there exist decreasing shadow price processes* $\underline{\mu}_{ij}(t) \in [0, \mu_{ij}]$ *for all* j *that shut down firm* i*'s demand, i.e., for* $t \in [\bar{t}_i, T]$ *where* $\bar{t}_i \equiv \inf E_i$,

$$p_i^*(t) = p_i^\infty(t, \vec{p}^*_{-i}(t)) = \operatorname*{arg\,max}_{p_i \in \mathscr{P}_i(t, \vec{p}^*_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}^*_{-i}(t))\}} \Big\{ r_i(t, p_i, \vec{p}^*_{-i}(t)) - \sum_j \underline{\mu}_{ij}(t) d_j(t, p_i, \vec{p}^*_{-i}(t)) \Big\};$$

(iii) (Complementary Slackness). $\mu_{ij} x_j^*(T) = 0$ *for all* j.

The OLNE has a simple structure. First, there exists a *finite* set of shadow prices, independent of time, measuring capacity externalities that firms exert on each other. Second, the *intertemporal* equilibrium prices at any time can be solved from a *one-shot* price competition game under the *current-time* demand structure, taking into account capacity externalities with *time-invariant* shadow prices: That is, at any time t, each firm i in the set of those firms who still have positive inventory, denoted by $\mathscr{S}(t)$, simultaneously solves the following one-shot price competition game:

$$\max_{p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}} \Big\{ r_i(t, \vec{p}(t)) - \sum_{j \in \mathscr{S}(t)} \mu_{ij} d_j(t, \vec{p}(t)) \Big\}, \tag{5}$$

with the rest of firms in $\overline{\mathscr{S}(t)}$ posting choke prices.

We illustrate this structure in detail. First, we focus on the shadow prices. The structure states that in the differential game, at equilibrium the shadow prices that measure the externalities of any firm's capacity exerted on all firms are *constant* over time before the firm runs out of stock. Intuitively, the time-invariant shadow prices are due to the fact that capacity constraints are imposed on the *total* sales over the entire sales horizon, and that the demand rate is independent of current inventory levels. Technically, the time-invariant structure comes from our specific adjoint equations when the maximum principle is applied. Furthermore, the complementary slackness condition indicates when and when not to expect capacity externalities. If at the end of the sales horizon, some firm, along the equilibrium inventory path, still has positive inventory, then this firm's capacity exerts no externalities. Otherwise, nonzero externalities may be exerted. Whenever a firm's inventory level hits zero before the end of the horizon, the firm has to post an appropriate choke price to exit the market, which is the only option to avoid taking orders but being unable to fulfill them. The demand system among the remaining firms with positive inventory will be adjusted to account for spillover, and the firm that has run out of stock no longer exerts any further capacity externality on all other firms under the spill-over adjusted demand system.

Second, we discuss how the intertemporal equilibrium prices emerge from the interaction between the current-time demand structure and aggregate supply constraints. To build intuition, we start with a one-shot monopoly problem to illustrate the *self-inflicted* capacity externality. Suppose in a monopoly market with a continuous downward-sloping demand curve $d(p)$, a revenue-maximizing firm with capacity C faces a one-shot pricing decision. The revenue maximization problem can be written as $\max_p pd(p)$, such that $d(p) \le C$. The optimal solution is the maximum between the market-clearing price $p^c = \inf\{p \mid d(p) \le C\}$, and the revenue-maximizing price $p^* = \arg\max_p pd(p)$. The first-order condition of this problem is $\partial[pd(p) + \mu(C - d(p))]/\partial p = \partial[(p-\mu)d(p)]/\partial p = 0$, where $\mu \ge 0$ is the shadow price of capacity. If the firm has ample capacity such that $d(p^*) \le C$, then the optimal price is the revenue-maximizing price p^* and the capacity constraint exerts no externality on setting the price (i.e., $\mu = 0$). If the firm has limited capacity such that $d(p^*) > C$, then the optimal price is the market-clearing price p^o and the capacity constraint exerts an externality to boost the optimal price to be higher than p^* (i.e., $\mu > 0$).

Returning to the intertemporal price competition game, we have already explained that the shadow prices, measuring a firm's capacity externalities exerted on all firms, are time-invariant. If shadow prices are known or can be approximated by good proxies, e.g., oversales penalty costs, the price equilibrium at any time is simply to solve a one-shot price competition game under the current-time demand structure that has been adjusted for spillover, taking into account capacity externalities with time-invariant shadow prices (see problem (5)). Now we illustrate how capacity externalities influence the equilibrium pricing intertemporally. We fix an arbitrary time t and focus on the first-order conditions of problem (5) of any firm i that has taken into account capacity externalities. On one hand, if firms i and j offer substitutable products, then $\mu_{ij}\, \partial d_j(t, \vec{p})/\partial p_i \ge 0$, and firm j's scarce capacity exerts an externality on firm i by pushing up firm i's price: Since firm j has limited capacity, it has a tendency to increase its own price due to the self-inflicted capacity externality (which we have illustrated for the monopoly case). Because of the substitutability between products from firms i and j, the price competition between the two firms will be alleviated so that firm i can also post a higher price. On the other hand, if firms i and j offer complementary products, then $\mu_{ij}\, \partial d_j(t, \vec{p})/\partial p_i \le 0$, and firm j's scarce capacity exerts an externality on firm i by pushing down firm i's price: while firm j has a tendency to increase its own price as just explained, due to the complementarity between products from firms i and j, firm i has to undercut its price to compensate for the price increase of firm j. By a similar reasoning, on stockout, a product's market exit by posting choke prices will be a boon for its substitutable products and a bane for its complementary products.

We close the discussion on the equilibrium structure by applying Proposition 2 to the specific setting of stationary demand systems.

Corollary 1 (Stationary Demand System). *Suppose the demand system is stationary, i.e., $d_i(t, \vec{p})$ is independent of time t for all i. Then an OLNE has the following structure: The price trajectories and the available products in the market remain constant before the first stockout event, between any two consecutive stockout events, and after the final stockout event until the end of the sales horizon.*

3.2.2. Sufficient Conditions. First, motivated by firms' self-interested behavior, we propose a special notion, *bounded rational equilibrium*, whose shadow price matrix is diagonal.

Definition 3 (Bounded Rational OLNE). A bounded rational OLNE has its matrix of constant shadow prices to satisfy $\mu_{ij} = 0$ for all $i \ne j$; namely, $M \equiv [\mu_{ij}]_{m\times m}$ is a diagonal matrix with the diagonal $[\mu_{ii}]_{m\times 1} \in \mathbb{R}^m_+$.

The bounded rational equilibrium may arise if in the best-response problem of each firm, only the firm's own capacity constraint is taken into account. The bounded rational equilibrium can be a relevant equilibrium concept, if firms do not have competitors' inventory information and equilibrium outcomes emerge from repeated best responses. Moreover, it can also arise when firms assume that the competitors have sufficiently large capacities as if they would never stock out.

Proposition 2 provides the necessary conditions of OLNE; next we show that under additional assumption, these conditions can be sufficient.

Proposition 3 (Sufficient Condition of OLNE). *The following sufficient conditions of OLNE hold*:

(i) *If $r_i(t,\vec{p})$ is concave in p_i and $d_i(t,\vec{p})$ is convex in p_j for all j, i, t, which is satisfied by the linear demand models, then necessary conditions in Proposition 2 are also sufficient for an OLNE.*

(ii) *If $\int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t))]\,dt$ is pseudo-concave in $\{p_i(t), 0 \le t \le T\}$ for all $\{\mu(t) \ge 0, 0 \le t \le T\}$ and all i, which is satisfied by the MNL demands (1), then necessary conditions in Proposition 2 together with $\mu_{ij}(t) = 0$ for all $i \ne j$ and t are sufficient for a bounded rational OLNE.*

3.2.3. Comparative Statics. In view of how capacity externalities influence price competition depending on the nature of product differentiation, we can obtain the following comparative statics of equilibrium prices in bounded rational OLNE with respect to initial capacity levels.

Proposition 4 (Comparative Statics of Bounded Rational OLNE in Capacity). *Suppose at any time each firm's feasible price set is a lattice. If all products are substitutable such that the price competition is (log-) supermodular, i.e., $\partial d_i(t,\vec{p})/\partial p_i < 0$, $\partial^2(\log)[r_i(t,\vec{p}) - \mu d_i(t,\vec{p})]/\partial p_i \partial p_j \ge 0$ for all i, $j \ne i$, t and $\mu \ge 0$, then a decrease in any firm's initial capacity level leads to higher equilibrium prices at any time for all firms in a bounded rational OLNE. In a duopoly selling complementary products such that the price competition is (log-)submodular, i.e., $\partial d_i(t,\vec{p})/\partial p_i < 0$, $\partial^2(\log)[r_i(t,\vec{p}) - \mu d_i(t,\vec{p})]/\partial p_i \partial p_{-i} \le 0$ for all $i = 1, 2$, t and $\mu \ge 0$, then a decrease in one firm's initial capacity level leads to higher equilibrium prices at any time for the firm itself and lower equilibrium prices at any time for the other firm in a bounded rational OLNE.*

Proposition 4 states that the lower any firm's initial capacity level is, the higher the bounded rational OLNE equilibrium prices are at any time for all firms in a competition of selling substitutable products. This decreasing monotonicity of equilibrium prices in capacities is driven by the decreasing monotonicity of bounded rational shadow prices in initial capacity levels, as a natural extension of the monopoly case.

3.3. Uniqueness

In our game, one firm's strategy set depends on competitors' strategies. This is referred to as *generalized* Nash game in the literature. Rosen (1965) investigates the notion of *normalized* Nash equilibrium in the context of finite-dimensional generalized Nash games. In a series of papers (e.g., Carlson 2002), Carlson extends the idea to infinite-dimensional generalized Nash games. Similarly, for our differential game, we can define a normalized Nash equilibrium that has the constant shadow prices related in a specific way, and provide a sufficient condition to guarantee its uniqueness.

Definition 4 (Normalized OLNE). A normalized OLNE has its matrix of constant shadow prices specified by one vectors $\vec{\xi} \in \mathbb{R}_+^m$ as $\mu_{ij} = \xi_j$ for all i and j; namely, $M \equiv [\mu_{ij}]_{m\times m}$ is a matrix with all rows being equal. (See Adida and Perakis 2010 for an application of the same notion.)

Recall that in any firm i's revenue maximization, the shadow price μ_{ij}, for all j, measures how much externality firm j's capacity exerts on firm i. The normalized Nash equilibrium can be interpreted qualitatively as follows: All firms use the *same* set of shadow prices for a firm's capacity constraint in their best-response problems. It may be reasonable to argue that in the airline industry, each firm infers the same set of shadow prices from the commonly observed capacity levels across firms. This is because there are common business practices across the airline industry, e.g., most airlines use very similar or sometimes even identical RM systems, and they use common external sources of data. If the normalized Nash equilibrium is applicable, its uniqueness can be guaranteed under the commonly used *strict diagonal dominance* (SDD) condition.

Proposition 5 (Unique Normalized OLNE). *Suppose $d_i(t,\vec{p})$ is twice continuously differentiable in $\vec{p}$ for all i, t. If $d_i(t,\vec{p})$ is convex in p_j for all i, j, t and*

$$\frac{\partial^2 r_i(t,\vec{p})}{\partial p_i^2} + \sum_{j\ne i} \left| \frac{\partial^2 r_i(t,\vec{p})}{\partial p_i \partial p_j} \right| < 0 \qquad \text{(SDD)}$$

for all i, t, then there exists a unique normalized OLNE.

To accommodate MNL demand models where $d_i(t,\vec{p})$ is not convex in p_j, we provide a sufficient condition to guarantee the uniqueness of a bounded rational OLNE.

Proposition 6 (Unique Bounded Rational OLNE). *Suppose $d_i(t,\vec{p})$ is twice continuously differentiable in $\vec{p}$ for all i, t. If $\partial d_i(t,\vec{p})/\partial p_i < 0$ for all i, t, and the Jacobian and Hessian matrix of the demand function $\vec{d}(t,\vec{p})$ with respect to $\vec{p}$ are negative semidefinite for all $\vec{p} \in \mathcal{P}(t)$ and all t, then there exists at most one bounded rational OLNE for any vector of diagonal shadow prices $[\mu_{ii}]_{m\times 1} \in \mathbb{R}_+^m$. Moreover, there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.*

We show that for any vector of nonnegative diagonal shadow prices, there exists a unique price equilibrium at any time, arising from the uncapacitated one-shot price competition game, with the diagonal shadow prices as the marginal supply costs. However, an arbitrary vector of diagonal shadow prices may not necessarily result in a bounded rational OLNE. Only if the entire price path satisfies the equilibrium characterization does this vector of diagonal shadow prices correspond to an equilibrium, with a unique joint equilibrium pricing policy. By the result of Proposition 1, which essentially shows the existence of a bounded rational OLNE, we know that there exists *at least one* vector of diagonal shadow prices such that its corresponding bounded rational OLNE is unique. As an immediate result of Propositions 5 and 6, we can provide the following sufficient conditions for linear demand models to guarantee the uniqueness of bounded rational OLNE.

Corollary 2. *For any linear demand model with* $\vec{d}(t,\vec{p}) = \vec{a}_t - B_t\vec{p}$, *where* $\vec{a}_t \in \mathbb{R}^m_{++}$, $B_t \in \mathbb{R}^{m\times m}$ *is a diagonally dominant matrix with diagonal entries positive and off-diagonal entries nonpositive,*

(i) *there exists a unique normalized OLNE;*

(ii) *there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.*

Moreover, by Gallego et al. (2006), for a one-shot unconstrained price competition under the MNL demand model and with constant marginal supply costs, there exists a unique Nash equilibrium. Combining the existence result of bounded rational OLNE for the MNL demand (see the proof of Proposition 1), we have the following corollary.

Corollary 3. *For the MNL demand* (1), *there exists a unique bounded rational OLNE for some vector of diagonal shadow prices.*

3.4. Feedback Nash Equilibrium

So far we have characterized OLNE. In general, FNE specifies prices for any time and joint inventory levels at the time, while OLNE only specifies prices as a function of time. Hence, they may differ in form, even though they can generate the same inventory trajectory and price path. Next we establish a connection between OLNE and FNE for our differential game. Given time t with a joint inventory level $\vec{x}(t)$, firms can solve a differential game, denoted by $\mathrm{P}(t,\vec{x}(t))$, with a remaining sales horizon $[t,T]$ and a current inventory level $\vec{x}(t)$, as the initial condition. We denote by $\vec{p}^f(t,\vec{x})$ the mapping from the initial condition $(t,\vec{x})$ of the differential game $\mathrm{P}(t,\vec{x}(t)=\vec{x})$, to the equilibrium prices $\vec{p}^*(t)$ of an OLNE at its initial time t. Intuitively, $\vec{p}^f(t,\vec{x})$ is *re-solving* OLNE for any initial condition $(t,\vec{x})$. If OLNE is unique in some sense as discussed in §3.3, the designation of the mapping $\vec{p}^f(t,\vec{x})$ is unambiguous. In some other scenarios, even OLNE may not be unique; a natural *focal* point can be the Pareto-dominant equilibrium. For example, for any diagonal shadow prices $[\mu_{ii}]_{m\times 1} \in \mathbb{R}^m_+$, if the revenue rate function $r_i(t,p_i,\vec{p}_{-i})$ for all i has increasing differences in $(p_i,\vec{p}_{-i})$ for any t, multiple bounded rational OLNE may arise, but the largest one is preferred by all firms (see, e.g., Bernstein and Federgruen 2005, Theorem 2).

Open-loop strategy is a static concept. For a given initial time and initial capacity levels, it specifies a time-dependent control path. Feedback strategy is a dynamic concept and specifies reactions to all possibilities of current time and joint inventory levels. However, in our RM differential game, a joint feedback strategy that solves an OLNE at every time with the current joint inventory level is an FNE and can generate the same equilibrium price path and inventory trajectory as an OLNE.

Again, this is due to the structural nature of our RM differential game. By the characterization of OLNE, the set of shadow prices to determine an OLNE for a differential game $\mathrm{P}(t,\vec{x}(t))$ depends only on the game's initial time and capacity levels, i.e., time t and the joint inventory level $\vec{x}(t)$. Hence, the prices for the current time t in the re-solving feedback strategy are uniquely determined by the current time t and the current joint inventory level $\vec{x}(t)$, and are independent of future inventory levels. Successively re-solving the open-loop game for the current time will result in updated prices solved from the current shadow prices that have fully captured the capacity externalities over the remaining horizon. Hence, the re-solving mapping $\vec{p}^f(t,\vec{x})$ is an FNE by definition (Starr and Ho 1969); the existence of OLNE also guarantees the existence of FNE. In the extreme case when all firms have ample capacities and there is no capacity externality, both the re-solving feedback strategy and open-loop equilibrium reduce to a one-shot price competition with zero marginal supply costs at any time, independent of any inventory levels.

Starting from any given initial time and capacity levels, there exists an OLNE by Proposition 1. Because the shadow prices along the equilibrium inventory trajectory in *this* OLNE are *constant* by Proposition 2, the re-solving FNE's prices determined by those shadow prices evolve along the same price path and result in the same inventory trajectory as predicted by the *very* OLNE. The same type of behavior has been observed in the monopoly RM problem (Maglaras and Meissner 2006).

Proposition 7 (Feedback Equilibrium). *The re-solving strategy* $\vec{p}^f(t,\vec{x})$ *is an FNE of the differential game. For an initial condition* $(t_0,\vec{x}(t_0))$, *the equilibrium price path and inventory trajectory under the re-solving FNE are the same as those under its corresponding OLNE with the same initial condition* $(t_0,\vec{x}(t_0))$.

We have characterized OLNE and identified an FNE in a feedback form that results in coincidental price path and inventory trajectory as its corresponding OLNE. We caution that this coincidence holds only for the deterministic problem. For problems with random demand, the price path and inventory trajectory under open-loop and re-solving feedback strategies, in general, are different. However, one can surmise that because a deterministic problem provides the first-order approximation to the corresponding stochastic problem, the feedback strategies obtained from the deterministic problem should serve as a reasonably good heuristic for the stochastic problem. We will provide more rigorous arguments for this claim in §4.

3.5. Applications

From the structural characterization of OLNE, we know that the intertemporal equilibrium prices are jointly determined by the current-time market condition (on the intertemporal demand side) and time-independent shadow prices reflecting capacity externality (on the aggregate supply side). Next we illustrate with two examples how these two-sided influences interact to determine the *intertemporal* equilibrium pricing behavior. Each example comes with its own theme, set to illustrate a set of managerial insights under the framework. These insights cannot be gained by analyzing a one-shot capacitated competition model. Though the analysis is conducted for OLNE, the same type of intertemporal behavior can also be sustained at an FNE, by Proposition 7.

3.5.1. Alternating Monopoly. In a dynamic Bertrand-Edgeworth competition of selling a homogeneous product (Talluri and Martínez de Albéniz 2011), along the trajectory of a noncooperative subgame perfect equilibrium, firms may avoid head-to-head competition and take turns acting as monopolists. In other words, at any time there is only one firm who sells at its monopoly price and all other firms post choke prices. Is this phenomenon unique to price competition of a homogeneous product? For an RM game of differentiated products, can such an outcome be sustained in equilibrium? The answer is *yes*, but it depends on the intertemporal demand structure.

First, we show that for any MNL demand model, it is *impossible* to have an alternating monopoly in equilibrium. We prove this result by contradiction. Suppose an alternating monopoly sustains in equilibrium. In an MNL demand model, for any finite price of a product, no matter how high it is, there always exists a positive demand rate. Because of this nature of MNL, we can show that it is beneficial for any firm to deviate by evening out a sufficiently small amount of inventory from its own monopoly period to a competitor's monopoly period. Hence, we can reach the following conclusion. (See the appendix for a rigorous proof.)

Proposition 8. *In a differential game with a time-varying MNL demand* (1) *and the strategy space for any firm being the full price space* $\mathbb{R}_+$, *an alternating monopoly cannot be sustained in equilibrium.*

Next, in the following example, we show that for a time-varying linear demand model, it is possible to have an alternating monopoly. Because of the nature of a linear demand model, demand can be zero when price is sufficiently high. Hence, if a firm is exerted a sufficiently high externality by the competitor's capacity, then it is possible for the firm to optimally post the choke price even before its stockout.

Example 2 (Alternating Monopoly). Consider a duopoly with a time-varying linear demand rate function:

$$\begin{cases} d_1(t, p_1, p_2) = 1 - p_1 + \gamma_H p_2 \\ d_2(t, p_1, p_2) = 1 - p_2 + \gamma_L p_1 \end{cases} \quad \text{for } t \in [0, T/2),$$

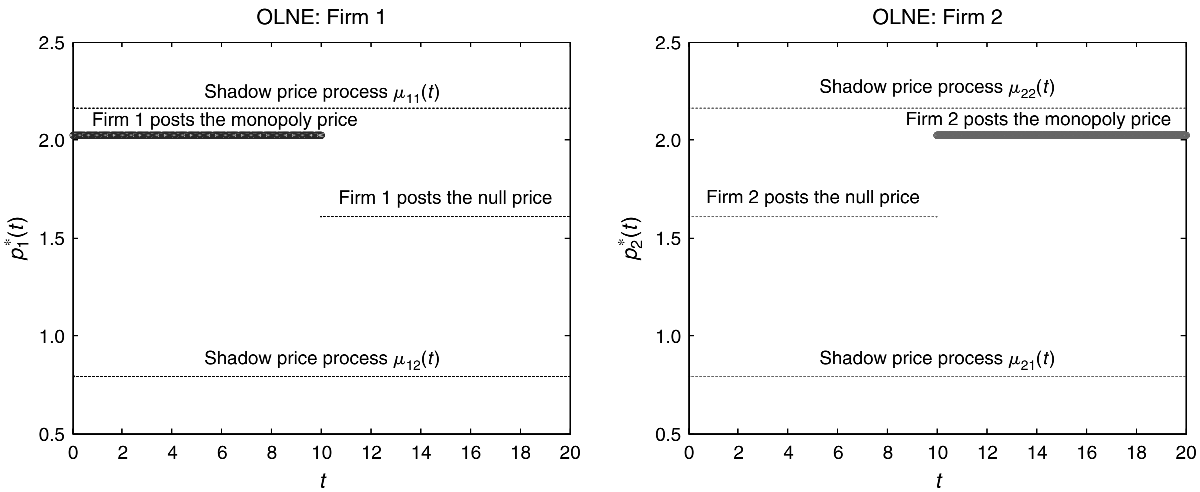
$$\begin{cases} d_1(t, p_1, p_2) = 1 - p_1 + \gamma_L p_2 \\ d_2(t, p_1, p_2) = 1 - p_2 + \gamma_H p_1 \end{cases} \quad \text{for } t \in [T/2, T],$$

where $0 < \gamma_L < \gamma_H < 1$. The feasible price set $\mathcal{P}_i(t, p_{-i}) = \{p_i \geq 0 \mid d_i(t, p_i, p_{-i}) \geq 0\}$. In this demand model, firm i, $i = 1, 2$, is more sensitive to the competitor's price in the ith period $[(i-1)T/2, iT/2)$. If firms can be protected from competition and sell as a monopoly in one period, firm i would prefer to sell as a monopoly in period i: without capacity constraints, the monopoly price that firm i can charge in period i is $(1+\gamma_H)/(2(1-\gamma_L\gamma_H))$, while firm i can only charge a lower price $(1+\gamma_L)/(2(1-\gamma_L\gamma_H))$ in period $j \neq i$ as a monopoly.

Suppose both firms have limited capacity $C_1 = C_2 = 1$ relative to the sales horizon T that is assumed to be sufficiently large. (See Electronic Companion E for the lower bound on T such that the equilibrium result holds.) We propose a joint policy under which two firms alternately sell as a monopoly for one half of the sales horizon: $p_i^*(t) = p^* \equiv ((1+\gamma_H) - 2/T)/(1-\gamma_H\gamma_L)$, $p_{-i}^*(t) = p_{-i}^\infty(t, p_i^*(t)) = 1 + \gamma_L p^*$ for $t \in [(i-1)T/2, iT/2)$, $i = 1, 2$. The proposed joint policy is such that firm i, $i = 1, 2$ will be a monopoly to sell off its capacity $C_i = 1$ using the ith half of the sales horizon $[0, T]$. (See Figure 1 for an illustration.)

One intuitive way to verify the proposed joint policy as an OLNE is to examine if there is an incentive for any firm to unilaterally deviate from this policy (see Electronic Companion E for such a verification). By Proposition 3(i), an alternative way of verifying OLNE is to check against the maximum principle when it is also sufficient. It is easily verified that the proposed joint pricing policy and the shadow price processes $\mu_{ii}(t) = ((2\gamma_H - \gamma_L\gamma_H{}^2 - \gamma_L{}^2)p^* - (\gamma_L + \gamma_H + \gamma_H{}^2))/(\gamma_H - \gamma_L)$ and $\mu_{i,-i}(t) = ((2 - \gamma_L\gamma_H - \gamma_L)p^* - (2+\gamma_H))/(\gamma_H - \gamma_L)$ for $i = 1, 2$ and all $t \in [0, T)$, indeed satisfy the sufficient conditions. From the behavior of firm 2 in the 1st half horizon, we can immediately make the following observation.

Figure 1 **Alternating Monopoly**



Note. This figure illustrates OLNE for $\gamma_L = 0.3$, $\gamma_H = 0.7$, and $T = 20$ with the shadow price processes $\mu_{ii}(t) = 2.1636$, $\mu_{i,-i} = 0.7943$, $i = 1, 2$, for all $t \in [0, 20)$.

Observation 1. At an OLNE, a firm may post a choke price and temporarily exit the market even with positive on-hand inventory.

By the example illustrated in Figure 1, shadow prices can be strictly larger than posted prices, e.g., $\mu_{ii}(t) > p_i^*(t)$, $t \in [0, T)$, $i = 1, 2$, not necessarily like the monopoly problem where shadow prices are always no larger than posted prices. This is summarized below.

Observation 2. At an OLNE (except for bounded rational OLNE), shadow prices can be larger than posted prices.

In the oligopoly problem, the posted price of any firm is jointly determined by shadow prices posted by all firms. In a market with substitutable products, positive shadow prices posted by the competitors put an upward pressure on the firm's own posted prices to keep them high. In Figure 1, consider firm i. It is the sufficiently high positive shadow prices $\mu_{i,-i}(t)$ due to firm $-i$'s capacity constraint that sustain sufficiently high positive shadow prices $\mu_{ii}(t)$ $(> p_i^*(t))$ so that firm i can glean high profit in its own monopoly period and shut down its demand in the competitor's monopoly period.

From the perspective of shadow prices, it is not only easy to verify OLNE, but it is also intuitive to see how capacity externalities interact with the intertemporal demand structure to determine equilibrium behavior. We see from Example 2 that the externality exerted by the competitor's scarce capacity can lead a firm to shut down its demand before stockout.

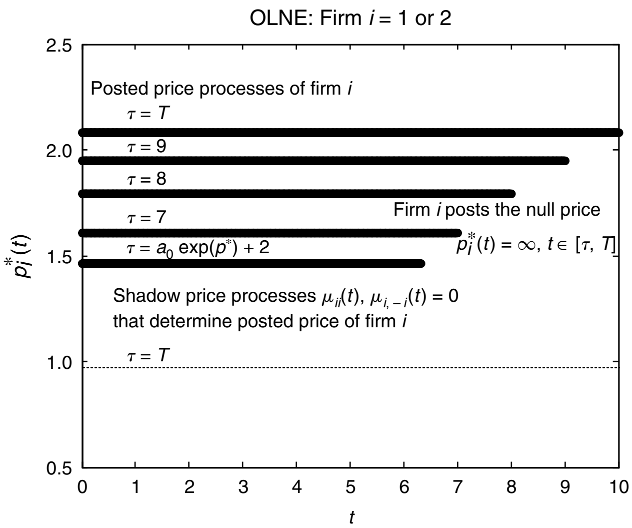
3.5.2. Effective Sales Horizon. We illustrate by an example that in equilibrium (all) firms may not fully use the nominal sales horizon $[0, T]$ due to competitors' limited capacities, even under a *stationary* demand structure. This poses a stark contrast to the monopoly case (see, e.g., Gallego and van Ryzin 1994) where the full sales horizon is always used in the optimal solution under a stationary demand model.

Example 3 (Head-to-Head). Consider a duopoly with a stationary and symmetric MNL demand rate function: $d_i(t, p_i, p_{-i}) = \exp(-p_i)/(a_0 + \exp(-p_i) + \exp(-p_{-i}))$, $t \in [0, T]$, $a_0 > 0$, $i = 1, 2$. Other than the choke price ∞, firms choose price from the set $\mathcal{P}_i(t, p_{-i}) = [0, L]$, where L is sufficiently large. Suppose both firms have limited capacity $C_1 = C_2 = 1$ relative to the sales horizon T that is assumed to be sufficiently large. We show that the following joint open-loop policy:

$$p_i(t) = p_{-i}(t) = \begin{cases} \ln\left(\dfrac{\tau - 2}{a_0}\right) & t \in [0, \tau), \\ \infty & t \in [\tau, T], \end{cases}$$

is an OLNE for *any* $\tau \in [a_0 \exp(p^*) + 2, T]$, where p^* is the price equilibrium without capacity constraints and is characterized by the equation $a_0(1 - p) + 2\exp(-p) = 0$. With the proposed joint policy, both firms price at $\ln((\tau - 2)/a_0)$ $(\geq p^*)$ until the sellout at time $t = \tau \leq T$; both firms earn a total revenue $\ln((\tau - 2)/a_0)$ that is increasing in τ. We verify that the proposed policy is indeed an OLNE by determining whether there is any incentive to deviate given that joint feasibility is maintained. First, given the competitor's strategy fixed as in the proposed equilibrium, it is not beneficial for any firm to shorten its effective sales horizon within period $[0, \tau)$. Second, it seems that a firm may improve its profit by evening out a small amount of capacity from period $[0, \tau)$ and selling it as

Figure 2 Head-to-Head Competition



Note. This figure illustrates OLNE for $a_0 = 1$, $T = 10$.

a monopoly in period $[\tau, T]$. Such a deviation would be profitable if the amount was made sufficiently small and if it was jointly feasible. However, it is clear that such a deviation of evening out some capacity by a price increase in period $[0, \tau)$ will make its competitor sell more than its capacity in period $[0, \tau)$. Thus, such a deviation is not in the jointly feasible strategy space of the generalized game, though it is unilaterally feasible. Therefore, the proposed joint open-loop policy is indeed an OLNE, where both firms do not fully use the whole sales horizon (see Figure 2 for an illustration).

Observation 3. Even under a stationary demand structure, at an OLNE, firms may run out of stock before the end of the sales horizon.

There are *infinite* numbers of OLNE differing in the length of the effective sales horizon. Among all such equilibria, the one using the whole sales horizon Pareto-dominates all others, and is the *unique* bounded rational OLNE. By Proposition 3(ii), we can also verify the bounded rational OLNE by checking against the sufficient condition. It is easy to see that the proposed joint pricing policy $p_i(t) = p_{-i}(t) = \ln((T-2)/a_0)$ for $t \in [0, T)$ and the shadow price process for all i, $\mu_{ii}(t) = \ln((T-2)/a_0) - T/(T-1)$, $\mu_{i,-i}(t) = 0$ for $t \in [0, T)$, satisfy the sufficient condition.

The discrepancy of Observation 3 from the monopoly case is due to that in the oligopoly. As firms precommit to OLNE, they also take into account competitors' capacity constraints to make the pre-commitment *credible*. However, in practice, the precise mechanism of maintaining credibility can be elusive, and the dominating bounded rational OLNE that uses the full sales horizon is most likely to be sustained.

4. The Stochastic Game

We extend the differential game to account for demand uncertainty by considering its stochastic-game counterpart in continuous time. We show that the solutions suggested by the differential game capture the essence and provide a good approximation to the stochastic game. The stochastic game formulation can be viewed as a game version of the optimal dynamic pricing problem considered in Gallego and van Ryzin (1994) with time-varying demand structures. Firms compete in influencing stochastic demand intensity by adjusting prices. More specifically, demand for a product is assumed to be a nonhomogeneous Poisson process with Markovian intensities, instead of deterministic rates. Let $N_i^{\vec{u}}(t)$ denote the number of items sold up to time t for firm i under joint pricing policy $\vec{u}$. A demand for any firm i is realized at time t if $dN_i^{\vec{u}}(t) = 1$. We denote the joint Markovian allowable pricing policy space by $\mathscr{P}$, where any joint allowable pricing policy $\vec{u} = \{\vec{p}(t, \vec{n}(t)), 0 \le t \le T\}$ satisfies $\vec{p}(t, \vec{n}(t)) \in \mathscr{P}(t)$ for all t and $\int_0^T dN_i^{\vec{u}}(t) \le C_i$ for all i. By the Markovian property of $\mathscr{P}$, we mean that the pricing policy offered by any firm is a function of the elapsed time and current joint inventory level; that is, $\vec{p}(t, \vec{n}(t)) = \vec{p}(t, C_1 - N_1^{\vec{u}}(t), C_2 - N_2^{\vec{u}}(t), \ldots, C_m - N_m^{\vec{u}}(t))$ for all t. We want to analyze strategies with Markovian properties, and again assume the same information structure as in Assumption 4.

Given pricing policy $\vec{u} \in \mathscr{P}$, we denote the expected profit for any firm i by $G_i[\vec{u}] \equiv E[\int_0^T p_i(t, \vec{n}(t))\, dN_i^{\vec{u}}(t)]$. The goal of any firm i is to maximize its total expected profit over the sales horizon. A joint pricing policy $\vec{u}^* \in \mathscr{P}$ constitutes a Nash equilibrium if, whenever any firm modifies its policy away from the equilibrium, its own payoff will not increase. More precisely, $\vec{u}^*$ is called a Markovian equilibrium strategy if $G_i[u_i, \vec{u}^*_{-i}] \le G_i[\vec{u}^*]$ for $[u_i, \vec{u}^*_{-i}] \in \mathscr{P}$ and all i. By applying Brémaud (1980, Theorem VII.T1) to the context of the RM stochastic game, we show that the following set of Hamilton–Jacobi–Bellman (HJB) equations is a sufficient condition for Markovian equilibrium strategies.

Proposition 9 (Stochastic RM Game: HJB). *If functions $V_i(s, \vec{n})$: $[0, T] \times \{\mathbb{Z}^m \cap \mathscr{X}\} \mapsto \mathbb{R}_+$ for all i are differentiable in remaining time $s \equiv T - t$ and simultaneously satisfy the following set of HJB equations*:

$$-\frac{\partial V_i(s, \vec{n})}{\partial t} = \sup_{p_i \in \mathscr{P}_i(t, \vec{p}_{-i}) \cup \{p_i^\infty(t, \vec{p}_{-i})\}} \{r_i(t, \vec{p}) - \nabla \vec{V}_i(s, \vec{n})^{\mathrm{T}} \vec{d}(t, \vec{p})\}, \quad n_i > 0, \tag{6}$$

where $\nabla \vec{V}_i(s, \vec{n}) \equiv (\Delta V_{i,1}(s, \vec{n}), \Delta V_{i,2}(s, \vec{n}), \ldots, \Delta V_{i,m}(s, \vec{n}))^{\mathrm{T}}$ and $\Delta V_{i,j}(s, \vec{n}) \equiv V_i(s, \vec{n}) - V_i(s, \vec{n} - \vec{e}_j)$, with boundary conditions as for all i, (i) *$V_i(0, \vec{n}) = 0$ for all $\vec{n}$ and* (ii) *$V_i(s, \vec{n}) = 0$ if $n_i \le 0$ for all s, and $p_i^*(t, \vec{n})$: $[0, T] \times \{\mathbb{Z}^m \cap \mathscr{X}\} \mapsto \mathbb{R}_+$ achieves the supremum in the HJB equations* (6) *for any* *firm i at all $(t, \vec{n})$, then $\vec{u}^* = \{\vec{p}^*(t, \vec{n})\} \in \mathscr{P}$ is a Markovian equilibrium strategy.*

For a discrete-time version of the stochastic game under a stationary MNL demand model, Lin and Sibdari (2009) demonstrate the existence of a Markovian equilibrium strategy by backward-inductively solving the set of HJB equations. However, other than the existence result, no further structural results are known. For the continuous-time stochastic game, we focus on exploring its natural links to the differential game that has a simple and intuitive structural characterization.

4.1. Affine Functional Approximations

The set of HJB equations, as a sufficient condition for Markovian equilibrium strategies, has two components: the set of differential and difference Equations (6) that governs the dynamics, and the boundary conditions. We focus on applying the affine functional approximation approach to the dynamics (6) while *temporarily* ignoring the feedback-type boundary conditions (see Adelman 2007 for the same approach). The obtained solution can be used to obtain pricing heuristics, in which the boundary conditions are *re-imposed* by posting choke prices when running out of stock or time. This approach is a coarse approximation, but it allows us to establish a natural and novel link between the stochastic game and its differential counterpart.

To be specific, in the stochastic game, we adopt an *affine* functional approximation to the value functions and temporarily ignore the boundary conditions: $V_i(s = T - t, \vec{n}) \approx W_i(s = T - t, \vec{n}) \equiv \int_t^T \theta_i(v)\, dv + \vec{w}_i(t)^{\mathrm{T}} \cdot \vec{n}$ for all i, t and $\vec{n}$, and $\vec{w}_i(t) \geq 0$ is a piecewise continuously differentiable function. If we restrict $\vec{w}_i(t) = \vec{w}_i$ for all $t \in [0, T)$ and $\vec{w}_i(T) = 0$, the approximation is called a *quasi-static affine* functional approximation (Adelman 2007). The term $\theta_i(t)$ approximates the marginal value of time-to-go and $\vec{w}_i(t)$ approximates the marginal value of capacity at time t. We can see that the affine functional approximation $W_i(s, \vec{n})$ does not necessarily satisfy the boundary conditions, e.g., $W_i(s, n_i = 0, \vec{n}_{-i})$ is not necessarily zero because the value of time-to-go $\theta_i(t) = \theta_i(T - s)$ may not be zero for $s > 0$. Omitting the boundary conditions, we show that the first-order approximated capacity marginal value process $\vec{w}_i(t)$ is exactly equal to the shadow price process $\vec{\mu}_i(t)$ in the differential game. By Proposition 2, in any OLNE the shadow price processes are constant before stockout events, hence we do not lose generality by restricting the approximation to a quasi-static affine approximation.

Proposition 10 (Affine Approximation to Stochastic Game). *A joint strategy satisfies the conditions obtained from an affine or quasi-static affine functional approximation to the value functions in the set of HJB equations* (6) *with boundary conditions temporarily omitted if and only if it is an OLNE in the differential game.*

4.2. Heuristics as Asymptotic Equilibria

The differential game can be analyzed to derive tractable and efficiently computable heuristics for its stochastic counterpart. Next we propose heuristics suggested by OLNE and FNE of the differential game, and show that they are equilibria in an asymptotic sense for the stochastic game, in the limiting regime where the potential demand and capacity are proportionally scaled up. Specifically, using k as an index, we consider a sequence of problems with demand rate function $\vec{d}^k(t, \vec{p}) = k\vec{d}(t, \vec{p})$ and capacity $\vec{C}^k = k\vec{C}$, and let k increase to infinity; hereafter, a superscript k will denote quantities that scale with k.

Definition 5 (Asymptotic Nash Equilibrium). In the stochastic game, $\vec{u}^* \in \mathscr{P}$ is called an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games, if for any $\epsilon > 0$ and all i, there exists l such that for all $k > l$, $(1/k)G_i^k[u_i, \vec{u}^*_{-i}] \leq (1/k)G_i^k[\vec{u}^*] + \epsilon$ for all $(u_i, \vec{u}^*_{-i}) \in \mathscr{P}$.

The quantity ϵ here refers to a small amount *relative* to the profit under an asymptotic equilibrium, more than which a firm's profit cannot be improved by a unilateral deviation.

4.2.1. Precommitment. Even under the assumed information structure that competitors' inventory levels are known in real time, firms may precommit to open-loop policies up to the point of stockout. This decision of precommitment can arise when firms trade off between price precommitment and pricing flexibility. As contingent or dynamic pricing in response to demand uncertainty may intensify competition, price precommitment can result in higher revenues for firms than pricing flexibility under competition; see Xu and Hopp (2006) for a discussion on this in the context of a homogeneous product and Wang and Hu (2014) for the case of differentiated products. We assume that firms implement any open-loop *heuristic* by precommitting to the open-loop policy up to the (random) time of running out of stock and posting choke prices afterwards. We show that any OLNE heuristic is asymptotically optimal under competition.

Proposition 11 (OLNE Heuristic as Asymptotic Nash Equilibrium). *Any OLNE heuristic corresponding to an OLNE of the differential game is an asymptotic Nash equilibrium* (*among all pre-committed open-loop heuristics corresponding to jointly allowable open-loop strategies*), *in the limiting regime of the sequence of scaled stochastic games.*

4.2.2. Contingent Pricing. Under the assumed information structure, the inventory level of any firm in real time is public information. The re-solving feedback strategy $\vec{p}^f(t, \vec{x})$ in the differential game provides a heuristic in feedback form for the stochastic game.

Because of the differentiability of the demand function in prices at any time (Assumption 1(a)), by Proposition 2 and implicit function theorem, we can show that the re-solving FNE $\vec{p}^f(t, \vec{x})$ is piecewise continuous in the current inventory level $\vec{x}$ for all t. By extending Maglaras and Meissner (2006) to the competition context, we show that this feedback heuristic is an asymptotic Nash equilibrium in the limiting regime as demand and supply grow proportionally large.

Proposition 12 (FNE as Asymptotic Nash Equilibrium). *The re-solving FNE heuristic $\vec{p}^f(t, \vec{x})$ is an asymptotic Nash equilibrium in the limiting regime of the sequence of scaled stochastic games.*

5. Conclusion

Current RM practice of legacy airlines is carried out with a pricing team designing fares and an operations team allocating capacity to fare classes. This flaw is exacerbated by low cost carriers offering fares with few or no restrictions and by Internet-enabled price transparency. RM researchers and practitioners are trying to integrate pricing and capacity allocation into a single system that takes into account pricing and quality attributes of the products available to customers at the time of purchase. The challenge is the complexity of solving such systems.

We have shown that such intertemporal pricing problems under competition, formulated as a differential game, has a simple structure in nature. The structure sheds light on how transient market conditions and aggregate supply constraints interact to determine intertemporal equilibrium pricing behavior. It is encouraging that the existence and uniqueness (in the notion of normalized or bounded rational equilibrium), of the equilibrium can be established for two commonly used demand rate functions—MNL and linear demand functions. Moreover, by the structural characterization, the infinite-dimensional time-varying equilibrium pricing policy can be determined by the finite set of shadow prices measuring capacity externalities. Because of this structure, the equilibrium computation can be significantly facilitated, and be cast as a finite-dimensional nonlinear complementarity problem. Last, we show that the equilibrium solutions from the differential game can provide precommitted or contingent heuristic policies, capturing the first-order effect for its stochastic counterpart. The re-solving feedback heuristic, which is dynamically easy to implement and asymptotically optimal, should be of practical interest to RM managers.

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Appendix. Proofs

Proof of Proposition 1. We will apply the infinite-dimensional fixed-point theorem (Bohnenblust and Karlin 1950, Theorem 5; see Electronic Companion B). To apply the theorem, we consider the following set-valued function $\mathcal{B}[\vec{p}] = \prod_i \mathcal{B}_i[\vec{p}_{-i}]$, for $\vec{p} \in S = \prod_i S_i$, where $\mathcal{B}_i[\vec{p}_{-i}] = \arg\max_{p_i \in S_i} J_i[p_i, \vec{p}_{-i}]$, for $\vec{p}_{-i} \in \prod_{j\neq i} S_j$ and $S_i \equiv \{p_i(t), 0 \le t \le T \mid p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)), \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \le C_i, \forall\, t, \vec{p}_{-i} \in (D[0, T])^{m-1}\}$. For $p_i^\infty(t, p_{-i}) \in \mathcal{P}_i(t, \vec{p}_{-i})$, $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t))$ is equivalent to $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}$. For MNL demand models, in a best response, it is not beneficial for a firm to use a choke price at any time if only constrained with its own capacity: Suppose in the best response there is a period of time where a firm uses a choke price, then the firm can even out a small amount of capacity from other time to sell in this period; the capacity constraint for the firm is not violated; as long as the feasible set $\mathcal{P}_i(t, \vec{p}_{-i}(t))$, $\forall\, t$ is sufficiently large to contain the prices to sell the sufficiently small amount so that the total profit is improved, we reach a contradiction. Hence, we can use the unilateral feasible set as defined in S_i where $p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t))$ also for MNL. For MNL demands, the OLNE, of which we show the existence in this proof as a solution to a fixed-point problem, does not use the choke price (i.e., ∞) at any time. But there can be other OLNE that indeed uses the choke price for positive measurable set of time (see Example 3).

Step 1. We show that S is convex. It suffices to show that S_i is convex for all i. By Assumption 1(b), $\int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv$ is pseudo-convex, hence quasi-convex, in $\{p_i(v), 0 \le v \le t\}$. Then its lower level set $\{p_i(v), 0 \le v \le t \mid \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \le C_i\}$ is convex. Since $\mathcal{P}_i(t, \vec{p}_{-i}(t))$ is convex for all t (Assumption 3(b)), S_i is convex by the fact that the intersection of any collection of convex sets is convex.

Step 2. We show that S is weakly closed. It suffices to show that S_i is (strongly) closed for all i. Since $\mathcal{P}_i(t, \vec{p}_{-i})$ is compact in $\mathbb{R}$ for all t (Assumption 3(b)), hence $\mathcal{P}_i(t, \vec{p}_{-i})$ is closed for all t and $\vec{p}_{-i}$ by the fact that in an Euclidean space every compact set is closed. Then $\{p_i(t), 0 \le t \le T \mid p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t)), \forall\, t\}$ is closed for all $\{\vec{p}_{-i}(t), 0 \le t \le T\}$ by the fact that the product of closed sets is closed. Since $d_i(v, \vec{p})$ is continuous in $\vec{p}$ (Assumption 1(a)), by Cesari (1983, Theorem 10.8.i), $\int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv$ is a lower semicontinuous functional in $\{p_i(v), 0 \le v \le t\}$. By an equivalent definition of lower semicontinuity (Royden 1988, problem 2.50(c)), the integral functional's lower level set $\{p_i(v), 0 \le v \le t \mid \int_0^t d_i(v, p_i(v), \vec{p}_{-i}(v))\, dv \le C_i\}$ is closed. Therefore, S_i is closed by the fact that the intersection of any collection of closed sets is closed.

Step 3. We show that S is compact. It suffices to show that S_i is compact for all i. Since $\mathcal{P}_i(t, \vec{p}_{-i}(t))$ is a compact set for any fixed $\vec{p}_{-i} \in \prod_{j\neq i} S_j$ (Assumption 3(b)), the set $\{p_i(t), 0 \le t \le T \mid p_i(t) \in \mathcal{P}_i(t, \vec{p}_{-i}(t))\}$ is compact by Tychonoff's theorem. Since S_i is closed (Step 2) and is a subset of the compact set $\{p_i(t), 0 \le t \le T \mid p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t))\}$, S_i is compact by the fact that a closed subset of a compact set is compact.

Step 4. We show that for any $\vec{p} \in S$, $\mathscr{B}[\vec{p}]$ is nonempty. It suffices to show that for any $\vec{p}_{-i} \in \prod_{j\neq i} S_j$, $\mathscr{B}_i[\vec{p}_{-i}]$ is nonempty. Under Assumptions 1(a) and 2(b), $J_i[p_i, \vec{p}_{-i}]$ is a weakly continuous functional in p_i on S_i for any fixed $\vec{p}_{-i} \in \prod_{j\neq i} S_j$ by Cesari (1983, Theorem 10.8.v). Hence, the continuous functional $J_i[p_i, \vec{p}_{-i}]$ that is bounded above (by Assumption 2(b)) can attain its maximum on the compact set S_i by an infinite-dimensional version of the extreme value theorem (Luenberger 1968, Theorem 2.13.1). Therefore, $\mathscr{B}_i[\vec{p}_{-i}]$ is nonempty.

Step 5. We show that for any $\vec{p} \in S$, $\mathscr{B}[\vec{p}]$ is convex. It suffices to show that for any $\vec{p}_{-i} \in \prod_{j\neq i} S_j$, $\mathscr{B}_i[\vec{p}_{-i}] = \arg\max_{p_i \in S_i} J_i[p_i, \vec{p}_{-i}]$ is convex. Since $\mathscr{B}_i[\vec{p}_{-i}]$ is nonempty, let p_i^* denote an element of the set. By Assumption 2(a), the integral functional $J_i[p_i, \vec{p}_{-i}] = \int_0^T r_i(t, p_i(t), \vec{p}_{-i}(t))\,dt$ is pseudoconcave in $p_i = \{p_i(t), 0 \le t \le T\}$, and hence is quasi-concave in p_i. By one of equivalent definitions of quasi-concavity, $\{p_i \mid J_i[p_i, \vec{p}_{-i}] \ge J_i[p_i^*, \vec{p}_{-i}]\}$ is convex for any $\vec{p}_{-i} \in \prod_{j\neq i} S_j$. Hence, $\mathscr{B}_i[\vec{p}_{-i}] = \{p_i \mid J_i[p_i, \vec{p}_{-i}] \ge J_i[p_i^*, \vec{p}_{-i}]\} \cap S_i$ is convex since S_i is convex (Step 1).

Step 6. We show that the graph $\mathscr{B}$ is weakly closed. Let $\{(\vec{x}^n, \vec{y}^n)\}_{n=1}^\infty$ be a sequence in $S \times S$ that converges weakly to $(\vec{x}, \vec{y}) \in S \times S$ such that $\vec{x}^n \in \mathscr{B}[\vec{y}^n]$, i.e., $J_i[p_i, \vec{y}^n_{-i}] \le J_i[x_i^n, \vec{y}^n_{-i}]$ for all $p_i \in S_i$ and all i. Under Assumptions 1(a) and 2(b), $J_i[\vec{p}]$ is weakly continuous in $\vec{p}$ by Cesari (1983, Theorem 10.8.v), hence $J_i[p_i, \vec{y}_{-i}] = \lim_{n\to\infty} J_i[p_i, \vec{y}^n_{-i}] \le \lim_{n\to\infty} J_i[x_i^n, \vec{y}^n_{-i}] = J_i[x_i, \vec{y}_{-i}]$ for all $p_i \in S_i$ and all i. Then $\vec{x} \in \mathscr{B}[\vec{y}]$.

Step 7. Note that $\bigcup_{\vec{p}\in S} \mathscr{B}(\vec{p})$ is a subset of S, which is compact by Step 3. This ensures that $\bigcup_{\vec{p}\in S} \mathscr{B}(\vec{p})$ is contained in a sequentially weakly compact set.

Step 8. Combining all of the above steps, we are ready to apply Bohnenblust and Karlin (1950, Theorem 5). Thus, $\mathscr{B}[\vec{p}]$ has a fixed point on S, namely, there exists an OLNE to the following differential game with *relaxed* constraints: given competitors' open-loop price policies $\{\vec{p}_{-i}(t), 0 \le t \le T\}$, each player i is to simultaneously $\max_{p_i \in D[0,T]} \int_0^T r_i(t, \vec{p}(t))\,dt$ such that $p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t))$ for all t, $d_i(t, \vec{p}(t)) \ge 0$ for all t, $C_i - \int_0^t d_i(v, \vec{p}(v))\,dv \ge 0$ for all t. In contrast with the original differential game, the firms in the game with relaxed constraints have bounded rationality and ignore the nonnegative demand and capacity constraints of competitors in their best responses.

Step 9. We argue that any OLNE of the game with relaxed constraints is one of the original game. Suppose $\{\vec{p}^*(t), 0 \le t \le T\}$ is an OLNE of the game with relaxed constraints, namely, given $\{\vec{p}^*_{-i}(t), 0 \le t \le T\}$, $\{p_i^*(t), 0 \le t \le T\}$ for all i maximizes $\int_0^T r_i(t, p_i(t), \vec{p}^*_{-i}(t))\,dt$ subject to $p_i(t) \in \mathscr{P}_i(t, \vec{p}^*_{-i}(t))$, $d_i(t, p_i(t), \vec{p}^*_{-i}(t)) \ge 0$ and $\int_0^t d_i(v, p_i(v), \vec{p}^*_{-i}(v))\,dv \le C_i$ for all t. Thus, OLNE satisfies the joint constraints, i.e., $p_j^*(t) \in \mathscr{P}_j(t, \vec{p}^*_{-j}(t))$, $d_j(t, \vec{p}^*(t)) \ge 0$ and $\int_0^t d_j(v, \vec{p}^*(v))\,dv \le C_j$ for all t and all j. Therefore, given $\{\vec{p}^*_{-i}(t), 0 \le t \le T\}$, $\{p_i^*(t), 0 \le t \le T\}$ for all i also maximizes $\int_0^T r_i(t, p_i(t), \vec{p}^*_{-i}(t))\,dt$ subject to $(p_i(t), \vec{p}^*_{-i}(t)) \in \mathscr{P}(t)$, $d_j(t, p_i(t), \vec{p}^*_{-i}(t)) \ge 0$ and $\int_0^t d_j(v, p_i(v), \vec{p}^*_{-i}(v))\,dv \le C_j$ for all t and all j. $\square$

Proof of Proposition 2. Introducing piecewise continuously differentiable costate variable $\vec{\mu}_i(t) = (\mu_{ij}(t), \forall j)$ for all i and t, we define the Hamiltonians H_i: $[0, T] \times \mathscr{X} \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ by $H_i(t, \vec{x}, \vec{p}(t), \vec{\mu}_i(t)) \equiv r_i(t, \vec{p}(t)) - \sum_j \mu_{ij}(t) d_j(t, \vec{p}(t))$ for all i and t. In addition, we have the state constraint $\vec{x}(t) \ge 0$. Hence, we define the Lagrangians L_i: $[0, T] \times \mathscr{X} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}$ by $L_i(t, \vec{x}, \vec{p}(t), \vec{\mu}_i(t), \vec{\eta}_i(t)) \equiv r_i(t, \vec{p}(t)) - \sum_j \mu_{ij}(t) d_j(t, \vec{p}(t)) + \sum_j \eta_{ij}(t) x_j(t)$ for all i and t, where Lagrangian multipliers $\eta_{ij}(t)$ for all i, j are piecewise continuous. Any OLNE $\{\vec{p}(t), 0 \le t \le T\}$, its corresponding costate trajectory $\{\vec{\mu}_i(t), 0 \le t \le T\}$ for all i, its corresponding Lagrange multiplier trajectory $\{\vec{\eta}_i(t), 0 \le t \le T\}$ for all i and its equilibrium state trajectory $\{\vec{x}(t), 0 \le t \le T\}$ need to satisfy the following set of *necessary* conditions (namely, the maximum principle for problems with inequality constraints under the direct adjoining method; see Hartl et al. 1995, §4): For all t and all i, j,

$$p_i(t) = \underset{p_i \in \mathscr{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}}{\arg\max} H_i(t), \tag{7}$$

$$-\frac{\partial \mu_{ij}(t)}{\partial t} = \frac{\partial L_i}{\partial x_j} = \eta_{ij}(t), \tag{8}$$

$$\mu_{ij}(T) x_j(T) = 0, \quad \mu_{ij}(T),\ x_j(T) \ge 0, \tag{9}$$

$$\eta_{ij}(t) x_j(t) = 0, \quad \eta_{ij}(t),\ x_j(t) \ge 0, \tag{10}$$

together with the jump conditions that $\mu_{ij}(t)$ for all i, j may jump down at the junction time when $x_j(t)$ hits zero, and the kinematic equation $\partial x_i(t)/\partial t = -d_i(t, \vec{p}(t))$, $x_i(0) = C_i$ for all i that is obvious from the context. Note that $\eta_{ij}(t) \ge 0$, $\mu_{ij}(T) \ge 0$, by Equation (8), we have $\mu_{ij}(t) \ge 0$ for all t.

Next we equivalently simplify the set of conditions (7)–(10) together with the jump conditions, by eliminating $\eta_{ij}(t)$. First, for all $t < \sup\{v \in [0, T] \mid x_j(v) > 0\}$, we have $x_j(t) > 0$, thus $\eta_{ij}(t) = 0$ for all i by the complementary slackness condition (10). Consider ordinary differential equation (8): The piecewise continuously differentiable costate trajectory $\mu_{ij}(t) \ge 0$ with derivative equal to zero everywhere (not almost everywhere) must be constant, which we can denote as $\mu_{ij} \ge 0$, before the inventory of firm j hits zero (if it happens). For t such that $x_i(t) > 0$ and $x_j(t) = 0$ for some j, in the Hamiltonian maximization (7) of firm i we can still set $\mu_{ij}(t) = \mu_{ij}$ even though $x_j(t)$ has hit zero, since at such a time firm j is simultaneously forced to post a choke price; hence, the term $\mu_{ij}(t) d_j(t, \vec{p}(t))$ is zero regardless of choices of $\mu_{ij}(t)$. Second, in Equation (7), it is understood that for $t \in [\bar{t}_i, T]$ (if the set E_i is nonempty), an appropriate costate variable process, denoted by $\mu_{ij}^-(t)$ to distinguish from μ_{ij}, can be chosen such that a choke price is the optimal solution to the Hamiltonian H_i; hence, the state $x_i(t)$ stays at zero. By Equation (8), such a piecewise continuously differentiable process $\mu_{ij}^-(t)$ is a decreasing process, which may have a jump-down discontinuity at the junction time $\bar{t}_i$. Last, consider the transversality condition (9). If $x_j(T) > 0$, then $\mu_{ij}(T) = \mu_{ij}$ for all i, hence condition (9) is equivalent to $\mu_{ij} x_j(T) = 0$. If $x_j(T) = 0$, condition (9) and $\mu_{ij} x_j(T) = 0$ always hold regardless of choices of shadow prices. Therefore, we can reach the equivalent set of simplified necessary conditions as described in the proposition. $\square$

Proof of Corollary 1. All firms start with positive inventory. By Proposition 2, all shadow prices μ_{ij} for all i, j stay constant until some firm, say i_o, first runs out of stock and exits the market (i.e., the first stockout event). (It is possible that multiple firms run out of stock simultaneously.) Because of the stationary demand structure, the equilibrium price trajectories solved from (4) and the resulting product assortment available in the market remain constant before the first stockout. After the first stockout, firm i_o posts choke prices and the demand system within the remaining firms who still have positive demand will reset to account for spillovers from firm i_o. Because the remaining firms still have positive inventory, by Proposition 2, the shadow prices they charge to each other are the same constant as before the first stockout. Again because of the stationary demand structure, the equilibrium price trajectories and the resulting product assortment available in the market stay constant until another stockout event. The same argument can be repeatedly applied for the remaining sales horizon. Between the final stockout event and the end of the horizon, all firms post constant choke prices to stay out of the market. □

Proof of Proposition 3. (i) Under the concavity of revenue rate functions $r_i(t,\vec{p})$ in p_i and the convexity of demand rate functions $d_i(t,\vec{p})$ in p_i for all t, the Hamiltonians H_i, for all i, are concave in p_i for all t. Moreover, the Hamiltonians H_i, for all i, are independent of $\vec{x}$, hence they are jointly concave in $(p_i, \vec{x})$. By Hartl et al. (1995, Theorem 8.3), the maximum principle is also a *sufficient* condition for OLNE.

(ii) We only consider bounded rational OLNE and hence let costate variables $\mu_{ij}(t) = 0$ for all $j \neq i$ and t. Since $\int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t))]\,dt$ for any $\{\mu(t) \geq 0, 0 \leq t \leq T\}$ is pseudo-concave in $\{p_i(t), 0 \leq t \leq T\}$ and the Lagrangian L_i at any time is linear in $\vec{x}$, then $\int_0^T L_i(t)\,dt = \int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t)) + \sum_j \eta_{ij}(t) x_j(t)]\,dt$ is jointly pseudo-concave in $(\{p_i(t), 0 \leq t \leq T\}, \{\vec{x}(t), 0 \leq t \leq T\})$. Because pseudo-concavity is stronger than invexity, by Arana-Jiménez et al. (2008, Theorem 2), the maximum principle is also a *sufficient* condition for any bounded rational OLNE where $\mu_{ij}(t) = 0$ for all $j \neq i$ and t. See Corollary 1 in Electronic Companion A for the proof that MNL demands satisfy the pseudo-concavity of $\int_0^T [r_i(t,\vec{p}(t)) - \mu(t) d_i(t,\vec{p}(t))]\,dt$ in $\{p_i(t), 0 \leq t \leq T\}$. □

Proof of Proposition 4. First, consider the competition of substitutable products. For an arbitrary firm, given its competitors' fixed price paths, we consider its own best-response problem. In this firm's constrained optimization problem, let us tighten its capacity constraint, which is the only constraint the firm is facing, because we focus on the bounded rational OLNE. This tightened capacity constraint increases the *constant* shadow price corresponding to this constraint in the best-response problem. At any time t, $p_i^*(t, \vec{p}_{-i}(t); \mu_{ii}) = \arg\max_{p_i \in \mathcal{P}_i(t,\vec{p}_{-i}(t)) \cup \{p_i^\infty(t,\vec{p}_{-i}(t))\}} \{r_i(t,\vec{p}) - \mu_{ii} d_i(t,\vec{p})\}$ is increasing in μ_{ii}, because $r_i(t,\vec{p}) - \mu_{ii} d_i(t,\vec{p})$ is supermodular in (p_i, μ_{ii}) due to $\partial d_i(t,\vec{p})/\partial p_i < 0$. Therefore, the tightened capacity constraint will lead to a pointwise higher best-response price path. Because the best-response correspondences of all firms are increasing in competitors' price paths, due to the (log-)supermodularity of profit functions at any time taking into account capacity constraints, the resulting equilibrium prices are higher (see Topkis 1998). Second, consider the competition of two complementary products. The result can be obtained by reversing the order of the strategy set of one firm and then applying the obtained result for substitutable products (see Vives 1999, Remark 2.20). □

Proof of Proposition 5. Define $\Gamma[\vec{p},\vec{q}] \equiv \int_0^T \sum_i r_i \cdot (t, p_1(t), \ldots, p_{i-1}(t), q_i(t), p_{i+1}(t), \ldots, p_m(t))\,dt$. It is a commonly used technique of applying the appropriate fixed-point theorem to the set-valued mapping $\arg\max_{\vec{q}}\{\Gamma[\vec{p},\vec{q}] \mid C_i - \int_0^T d_i(t,\vec{q}(t))\,dt \geq 0, \forall\, i\}$ to show the existence of a Nash equilibrium of the original problem. Following the same procedures as in the proof of Proposition 1 and noting that $d_i(t,\vec{q})$ is convex in q_j for all i, j, t, we can verify the existence of a fixed point $\vec{p}^*$ such that $\Gamma[\vec{p}^*,\vec{p}^*] = \max_{\vec{q}}\{\Gamma[\vec{p}^*,\vec{q}] \mid C_i - \int_0^T d_i(t,\vec{q}(t))\,dt \geq 0, \forall\, i\}$. Such a fixed point $\vec{p}^*$ is a Nash equilibrium with shadow prices satisfying $\mu_{ij} = \xi_j$ for all i, j, where $\vec{\xi}$ is the Lagrangian multipliers in the maximization problem. Following the same procedures as in the proof of Rosen (1965, Theorem 4) and noting that condition (SDD) at any time is sufficient for pointwise strict diagonal concavity and hence for integrally strict diagonal concavity, we can obtain the desired result. □

Proof of Proposition 6. Let $\Pi_i(t,\vec{p}) \equiv r_i(t,\vec{p}) - \mu_{ii} d_i(t,\vec{p})$. The first-order derivative is $\partial \Pi_i/\partial p_i = d_i(t,\vec{p}) + (p_i - \mu_{ii})\partial d_i(t,\vec{p})/\partial p_i$. The second-order derivatives are $\partial^2 \Pi_i/\partial p_i{}^2 = 2\partial d_i(t,\vec{p})/\partial p_i + (p_i - \mu_{ii})\partial^2 d_i(t,\vec{p})/\partial p_i{}^2$ and $\partial^2 \Pi_i/\partial p_i \partial p_j = \partial d_i(t,\vec{p})/\partial p_j + (p_i - \mu_{ii})\partial^2 d_i(t,\vec{p})/\partial p_i \partial p_j$, $\forall\, j \neq i$. Under the assumption that the Jacobian and Hessian matrix of the demand function $\vec{d}(t,\vec{p})$ with respect to $\vec{p}$ are negative semidefinite, the Hessian of $\vec{\Pi}(t,\vec{p})$, and hence its leading principal submatrices, are all negative definite at $\vec{p} = \vec{p}^*$ satisfying $\partial \Pi_i(t,\vec{p})/\partial p_i = 0$ for all i such that $x_i(t) > 0$. This is because (i) $\partial d_i(t,\vec{p})/\partial p_i < 0$; (ii) $(p_i - \mu_{ii})|_{\vec{p}=\vec{p}^*} = -d_i(t,\vec{p})/(\partial d_i(t,\vec{p})/\partial p_i)|_{\vec{p}=\vec{p}^*} \geq 0$ for all i such that $x_i(t) > 0$; and (iii) that the negative semidefiniteness is preserved under additivity. By the Poincaré–Hopf index theorem (see Vives 1999, §2.5), for any set of shadow prices $\{\mu_{ii}, \forall\, i\}$ with $\mu_{ij} = 0$ for all $i \neq j$ at any time t, there exists a unique price vector satisfying $\partial \Pi_i(t,\vec{p})/\partial p_i = 0$ for any set of firms $\{i \mid x_i(t) > 0\}$ together with the rest of firms posting the choke prices. (Depending on the demand structures, there may exist multiple choke prices for a firm that effectively shut down its demand, but they are unique in the sense of generating the same zero demand and revenue.) Moreover, by the proof of Proposition 1, there exists a bounded rational OLNE; hence, for its diagonal shadow prices, the corresponding bounded rational OLNE is unique. □

Proof of Proposition 8. We prove by contradiction. Suppose at the joint open-loop equilibrium $\vec{p}^*$, there exist two intervals $(b, b+\delta)$ and $(\tilde{b}, \tilde{b}+\delta)$ such that for some firm i, $p_i^*(t)$ is infinite on $(b, b+\delta)$ and $p_i^*(t)$ is finite on $(\tilde{b}, \tilde{b}+\delta)$, and for any firm $j \neq i$, $p_j^*(t)$ is infinite on $(\tilde{b}, \tilde{b}+\delta)$, namely, firm i is the monopoly on $(\tilde{b}, \tilde{b}+\delta)$ and some firm other than firm i can be the monopoly on $(b, b+\delta)$. Let $\bar{p}_i(t,\epsilon) \equiv \inf\{p_i \geq 0 \mid d_i(t, p_i, \vec{p}^*_{-i}(t)) = \epsilon/\delta\}$, which is finite for MNL demand rate functions. Let $\tilde{p}_i(t,\epsilon) \equiv \inf\{p_i \geq 0 \mid d_i(t, p_i, \vec{p}^*_{-i}(t)) = d_i(t,\vec{p}^*(t)) - \epsilon/\delta\}$. It is easily verified that $\tilde{p}_i(t,\epsilon) > p_i^*(t)$ for $\epsilon > 0$ since $\tilde{p}_i(t,\epsilon)$ is increasing in ϵ. Furthermore, since $\bar{p}_i(t,\epsilon)$ is decreasing in ϵ and $\lim_{\epsilon \to 0} \bar{p}_i(t,\epsilon) = \infty$, there exists sufficiently small $\bar{\epsilon}(t) > 0$ for any $t \in (\tilde{b}, \tilde{b}+\delta)$ such that $\bar{p}_i(t + b - \tilde{b}, \bar{\epsilon}(t)) > p_i^*(t)$. We construct a price policy for firm i,

$$p_i(t) = \begin{cases} \bar{p}_i(t, \bar{\epsilon}(t + \tilde{b} - b)) & \text{if } t \in (b, b+\delta), \\ \tilde{p}_i(t, \bar{\epsilon}(t)) & \text{if } t \in (\tilde{b}, \tilde{b}+\delta), \\ p_i^*(t) & \text{otherwise.} \end{cases}$$

Under the above policy, firm i evens out a small quantity $\tilde{\epsilon}(t)$ at any time $t \in (\tilde{b}, \tilde{b}+\delta)$ to the corresponding time $t+b-\tilde{b}$ in $(b, b+\delta)$.

First, we determine whether the constructed policy is jointly feasible. It is obviously feasible for firm i as its total sales remains unaltered. To see the feasibility for other firms, we check the derivative $(\partial d_j(t, a_i, \vec{a}_{-i})/\partial a_i)(\partial a_i^{-1}(t, d_i, \vec{a}_{-i})/\partial d_i)$, where $a_i = \beta_i(t)e^{-\alpha_i(t)p_i}$ is the attraction value of firm i, $a_i^{-1}(t, d_i, \vec{a}_{-i}) = d_i \sum_{k\neq i} a_k(t)/\lambda(t) - d_i$ is the inverse function of $d_i(t, a_i, \vec{a}_{-i}) = (\lambda(t)a_i(t))/(a_i(t)+\sum_{k\neq i} a_k(t))$. This derivative captures the impact on firm j's sales volume due to a small change in firm i's sales volume by changing its price p_i while the competitor's price $\vec{p}_{-i}$ is fixed. It is easily verified that $(\partial d_j(t, a_i, \vec{a}_{-i})/\partial a_i)(\partial a_i^{-1}(t, d_i, \vec{a}_{-i})/\partial d_i) = -a_j(t)/\sum_{k\neq i} a_k(t)$. The constructed deviation of firm i will cause any firm j's sales to stay the same for $t \in (\tilde{b}, \tilde{b}+\delta)$ since $a_j(t) = 0$ for $t \in (\tilde{b}, \tilde{b}+\delta)$, and to decrease by $a_j(t)\epsilon(t)/\sum_{k\neq i} a_k(t)$ for $t \in (b, b+\delta)$. As long as $\epsilon(t)$ is sufficiently small, firm i's deviation does not violate competitors' capacity constraints and state positivity. Thus, the deviation is feasible in the generalized Nash game with coupled constraints.

Next, we compare the profit before and after the deviation. Under the original policy $\vec{p}^*$, firm i earns $p_i^*(t)d_i(t, \vec{p}^*(t))$ for any time $t \in (\tilde{b}, \tilde{b}+\delta)$ and 0 for any time $t \in (b, b+\delta)$. Under the constructed policy, firm i earns $\tilde{p}_i(t, \tilde{\epsilon}(t))d_i(t, \tilde{p}_i(t, \tilde{\epsilon}(t)), \vec{p}^*_{-i}(t)) = \tilde{p}_i(t, \tilde{\epsilon}(t))(d_i(t, \vec{p}^*(t)) - \tilde{\epsilon}(t)/\delta) > p_i^*(t)(d_i(t, \vec{p}^*(t)) - \tilde{\epsilon}(t)/\delta)$ for any time $t \in (\tilde{b}, \tilde{b}+\delta)$ and $\bar{p}_i(t+b-\tilde{b}, \tilde{\epsilon}(t))(\tilde{\epsilon}(t)/\delta)$ in the corresponding time in $(b, b+\delta)$. Since $\bar{p}_i(t+b-\tilde{b}, \tilde{\epsilon}(t)) > p_i^*(t)$, the constructed policy has a *positive* profit improvement for firm i over p_i^* given $\vec{p}^*_{-i}$ is fixed. We see a contradiction to the assumption that $\vec{p}^*$ is an equilibrium. □

Proof of Proposition 9. First, to generalize the HJB as a sufficient condition of the optimal control strategy for a single firm to the game setting, we require a Markovian equilibrium strategy that satisfies the set of HJB equations simultaneously. Second, to apply Brémaud (1980, Theorem VII.T1) to each firm, the only condition that needs to be verified is the boundedness of the value functions that are guaranteed by Assumption 2(b). Last, the boundary conditions that $V_i(s, \vec{n}) = 0$ if $n_i = 0$ for all s enforce that on a stockout a choke price is the only option. Hence, the joint strategy satisfying the set of HJB equations must be in $\mathscr{P}$. Therefore, the proof is complete. □

Proof of Proposition 10. Under the affine functional approximation with boundary conditions omitted, the set of HJB equations (6) becomes

$$\theta_i(t) - \left(\frac{\partial w_{i1}(t)}{\partial t}, \ldots, \frac{\partial w_{im}(t)}{\partial t}\right)^{\mathrm{T}} \vec{n} = \sup_{p_i \in \mathscr{P}_i(t, \vec{p}_{-i}) \cup \{p_i^\infty(t, \vec{p}_{-i})\}} \{r_i(t, \vec{p}) - \vec{w}_i(t)^{\mathrm{T}} \vec{d}(t, \vec{p})\}, \quad \vec{n} \in \mathbb{Z}^m \cap \mathscr{X}, \quad (11)$$

for all i and t. Taking the difference between Equation (11) evaluated at $(t, \vec{n})$ and at $(t, \vec{n} - \vec{e}_j)$ for all j, we obtain $\partial w_{ij}(t)/\partial t = 0$ for all i, j and t. Since $w_{ij}(t)$ is piecewise continuously differentiable, $w_{ij}(t)$ must be a constant, which we denote by w_{ij}. Hence, we do not lose generality by restricting the functional approximation to a quasi-static affine functional approximation with boundary conditions omitted.

It has been shown that the HJB equation for a discrete-time monopolistic RM problem can be equivalently stated as an optimization problem (Adelman 2007). In Electronic Companion D, we show that it is also true for continuous-time problems. Specifically, we show that if $\vec{V}^*(s, \vec{n})$ solves the set of HJB equations (6) for the continuous-time stochastic game, and that a differentiable function $\vec{V}(s, \vec{n})$ is a feasible solution to a game where any firm i simultaneously solves the following functional optimization problem given competitors' strategy $\vec{p}_{-i}(t, \vec{n})$:

$$\min_{\{V_i(\cdot,\,\cdot)\}} \quad V_i(T, \vec{C})$$
$$\text{s.t.} \quad -\frac{\partial V_i(s, \vec{n})}{\partial t} \geq \{r_i(t, \vec{p}(t, \vec{n})) - \nabla \vec{V}_i(s, \vec{n})^{\mathrm{T}} \vec{d}(t, \vec{p}(t, \vec{n}))\}, \quad \forall \vec{p}(t, \vec{n}) \in \mathscr{P}(t), \ \forall (t, \vec{n}).$$

Hence, the equilibrium value function $\vec{V}(T, \vec{C})$ at the initial time $t = 0$ and state $\vec{n} = \vec{C}$ can be obtained by solving the functional optimization game. Under the affine functional approximation, we can approximate the functional minimization problem for any firm i as follows:

$$(\mathrm{D}_i) \quad \min_{\vec{w}_i \geq 0} \int_0^T \theta_i(t)\,dt + \vec{w}_i^{\mathrm{T}} \vec{C}$$
$$\text{s.t.} \quad \theta_i(t) \geq r_i(t, \vec{p}(t)) - \vec{w}_i^{\mathrm{T}} \vec{d}(t, \vec{p}(t)), \quad \forall \vec{p}(t) \in \mathscr{P}(t), \forall t.$$

Since (D_i) is a minimization problem, it is optimal to set

$$\theta_i(t) = \max_{p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}} \{r_i(t, \vec{p}(t)) - \vec{w}_i^{\mathrm{T}} \vec{d}(t, \vec{p}(t))\}, \quad \forall t$$

in the objective function. Then the objective of any firm i becomes

$$\min_{\vec{w}_i \geq 0} \left[\max_{p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}, \forall t} \left\{ \int_0^T r_i(t, \vec{p}(t))\,dt + \vec{w}_i^{\mathrm{T}} \cdot \left(\vec{C} - \int_0^T \vec{d}(t, \vec{p}(t))\,dt \right) \right\} \right].$$

This is equivalent to the maximization problem $\max_{p_i(t) \in \mathscr{P}_i(t, \vec{p}_{-i}(t)) \cup \{p_i^\infty(t, \vec{p}_{-i}(t))\}, \forall t} \int_0^T r_i(t, \vec{p}(t))\,dt$ with capacity constraints $\vec{C} - \int_0^T \vec{d}(t, \vec{p}(t))\,dt \geq 0$ dualized by the vector $\vec{w}_i \geq 0$. Strong duality holds here because this continuous-time maximization primal problem has pseudo-concave objective function (Assumption 2(a)) and quasi-convex constraints (the left-hand sides of "≤" constraints are quasi-convex by Assumption 1(b)), and both primal and dual are feasible (see Zalmai 1985). □

Proof of Proposition 11. Suppose $\vec{p} = \{\vec{p}(t), 0 \leq t \leq T\} \in \mathscr{P}_O$ is any arbitrary joint open-loop policy subject to coupled constraints (3). Under policy $\vec{p}$, we denote by τ_i^k the minimum time of T and the random stopping time when the total sales process of firm i reach its original capacity in the kth system. In the deterministic differential game, we denote by $\bar{t}_i$ the minimum time of T and the time when the total sales of firm i reach its original capacity in the unscaled system, which is also such a time in the scaled regimes without demand uncertainty. As dictated by $\vec{p}$, any firm i implements the open-loop policy $p_i(t)$ up to the time either $\bar{t}_i$ or τ_i^k whichever comes earlier, and posts only a choke price afterwards. Without loss of generality, we index firms such that their deterministic stockout times are ordered as $0 \le \bar{t}_1 \le \bar{t}_2 \le \cdots \le \bar{t}_n \le T$. Let $N_i(\cdot)$ for all i denote independent unit rate Poisson processes. The functional strong law of large numbers for the Poisson process and composition convergence theorem assert that as $k \to \infty$, for any $\vec{p}(t)$,

$$\frac{N_i(kd_i(t, \vec{p}(t)))}{k} \to d_i(t, \vec{p}(t)) \quad \text{a.s. uniformly in } t \in [0, T]. \tag{12}$$

This suggests that in the stochastic system, the random stopping time τ_i^k should be close to its deterministic counterparts $\bar{t}_i$, at least the relative order, as k goes to infinity. For the time being, we suppose these stopping times are ordered almost surely (a.s.) as $\tau_1^k \le \tau_2^k \le \cdots \le \tau_n^k$, as k becomes sufficiently large for the simplicity of exposure, which we will confirm shortly. Up to time τ_1^k, all firms implement $\vec{p}(t)$. Arguing by contradiction and applying (12) to firm 1, one can easily conclude that $\tau_1^k \to \bar{t}_1$ a.s. as $k \to \infty$. The revenues of firm 1 extracted under the open-loop policy are $R_1^k[\vec{p}] \equiv \int_0^{\min\{\tau_1^k, \bar{t}_1\}} p_1(t)\, d(N_1(kd_1(t, \vec{p}(t))))$ as k is sufficiently large, and $(1/k)R_1^k[\vec{p}] \to \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$ as $k \to \infty$. Recall the fact established in Gallego and van Ryzin (1994) that the solution of the deterministic pricing problem serves as an upper bound for the revenues extracted in the stochastic system and by Assumption 2(b), we have $G_1^k[\vec{p}] = E(R_1^k[\vec{p}]) \le k \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$. By the bounded convergence theorem, $(1/k)G_1^k[\vec{p}] \to \int_0^{\bar{t}_1} r_1(t, \vec{p}(t))\, dt$.

The sales of firm 2 is

$$D_2^k[\vec{p}] \equiv \begin{cases} \displaystyle\int_0^{\tau_1^k} dN_2(kd_2(t, \vec{p}(t))) + \int_{\tau_1^k}^{\bar{t}_1} dN_2(kd_2(t, p_1^\infty(\vec{p}_{-1}(t)), \vec{p}_{-1}(t))) \\ \displaystyle\quad + \int_{\bar{t}_1}^{\min\{\tau_2^k, \bar{t}_2\}} dN_2(kd_2(t, \vec{p}(t))) \quad \text{if } \tau_1^k < \bar{t}_1, \\ \displaystyle\int_0^{\min\{\tau_2^k, \bar{t}_2\}} dN_2(kd_2(t, \vec{p}(t))) \qquad \text{otherwise,} \end{cases}$$

as k is sufficiently large. Since $\tau_1^k \to \bar{t}_1$ a.s. as $k \to \infty$, the term $\int_{\tau_1^k}^{\bar{t}_1} dN_2(kd_2(t, p_1^\infty(\vec{p}_{-1}(t)), \vec{p}_{-1}(t)))$ is asymptotically negligible. By applying (12) to firm 2 and arguing by contradiction, one can conclude that $\tau_2^k \to \bar{t}_2$ a.s. as $k \to \infty$. The revenues of firm 2 extracted under the open-loop policy are

$$R_2^k[\vec{p}] \equiv \begin{cases} \displaystyle\int_0^{\tau_1^k} p_2(t)\, dN_2(kd_2(t, \vec{p}(t))) \\ \displaystyle\quad + \int_{\tau_1^k}^{\bar{t}_1} p_2(t)\, dN_2(kd_2(t, p_1^\infty(\vec{p}_{-1}(t)), \vec{p}_{-1}(t))) \\ \displaystyle\quad + \int_{\bar{t}_1}^{\min\{\tau_2^k, \bar{t}_2\}} p_2(t)\, dN_2(kd_2(t, \vec{p}(t))) \quad \text{if } \tau_1^k < \bar{t}_1, \\ \displaystyle\int_0^{\min\{\tau_2^k, \bar{t}_2\}} p_2(t)\, dN_2(kd_2(t, \vec{p}(t))) \qquad \text{otherwise,} \end{cases}$$

as k is sufficiently large, and $(1/k)R_2^k[\vec{p}] \to \int_0^{\bar{t}_2} r_2(t, \vec{p}(t))\, dt$ as $k \to \infty$. Moreover, $E(R_2^k[\vec{p}]) \le k \int_0^{\bar{t}_2} r_2(t, \vec{p}(t))\, dt + \delta$, where a random variable δ bounds the revenue over $[\tau_1^k, \bar{t}_1]$ from the spillover sales from firm 1 and δ/k is asymptotically negligible. By the bounded convergence theorem, $(1/k)G_2^k[\vec{p}] = (1/k)E(R_2^k[\vec{p}]) \to \int_0^{\bar{t}_2} r_2(t, \vec{p}(t))\, dt$. Repeating the same argument, we conclude that $(1/k)G_i^k[\vec{p}] \to \int_0^{\bar{t}_i} r_i(t, \vec{p}(t))\, dt$ for all i, as $k \to \infty$. Applying this convergence result to an OLNE $\vec{p}^*$ and to any of its unilateral deviation $(p_i, \vec{p}_{-i}^*)$, we have $(1/k)G_i^k[\vec{p}^*] \to \int_0^{\bar{t}_i} r_i(t, \vec{p}^*(t))\, dt$ and $(1/k)G_i^k[p_i, \vec{p}_{-i}^*] \to \int_0^{\bar{t}_i} r_i(t, p_i(t), \vec{p}_{-i}^*(t))\, dt$ for all i, as $k \to \infty$. In other words, for any $\epsilon > 0$, there exists l such that for all $k > l$, $\left|(1/k)G_i^k[\vec{p}^*] - \int_0^{\bar{t}_i} r_i(t, \vec{p}^*(t))\, dt\right| < \epsilon/2$ and $\left|(1/k)G_i^k[p_i, \vec{p}_{-i}^*] - \int_0^{\bar{t}_i} r_i(t, p_i(t), \vec{p}_{-i}^*(t))\, dt\right| < \epsilon/2$. Since $\int_0^{\bar{t}_i} r_i(t, p_i(t), \vec{p}_{-i}^*(t))\, dt \le \int_0^{\bar{t}_i} r_i(t, \vec{p}_i^*(t))\, dt$, then $(1/k)G_i^k[p_i, \vec{p}_{-i}^*] \le (1/k)G_i^k[\vec{p}^*] + \epsilon$. $\square$

Proof of Proposition 12. Suppose $\vec{p}^c = \{\vec{p}^c(t, \vec{x}), 0 \le t \le T\} \in \mathscr{P}_F$ is any arbitrary joint feedback policy in the differential game subject to coupled constraints (3) and is piecewise continuous in $\vec{x}$. The cumulative demand for firm i up to time t driven by such a policy in the kth system is denoted by $N_i(A_i^{c,k}(t))$, where $A_i^{c,k}(t) = \int_0^t kd_i(v, \vec{p}^c(v, (1/k)\vec{X}^{c,k}(v)))\, dv$, and $X_i^{c,k}(t) = \max(0, kC_i - N_i(A_i^{c,k}(t)))$ for all i denotes the remaining inventory for firm i at time t. Note that $A_i^{c,k}(0) = 0$, $A_i^{c,k}(t)$ is nondecreasing and $A_i^{c,k}(t_2) - A_i^{c,k}(t_1) \le k\int_{t_1}^{t_2} d_{i,\max}(v)\, dv$, where $d_{i,\max}(v) \equiv \max_{\vec{p}(v) \in \mathscr{P}(v)} d_i(v, \vec{p}(v))$. This implies that the family of process $\{(1/k)A_i^{c,k}(t)\}$ for all i is equicontinuous, and therefore relatively compact. By the Arzelà–Ascoli theorem, we can obtain a converging subsequence $\{k_m\}$ of the sequence $\{(1/k)A_i^{c,k}(t)\}$ such that $(1/k_m)A_i^{c,k_m}(t) \to \bar{A}_i^c(t)$ for all i in the following way: For $i = 1$, there exists a converging subsequence $\{k_1\}$, such that $(1/k_1)A_1^{c,k_1}(t) \to \bar{A}_1^c(t)$; for $i = 2$, along sequence $\{k_1\}$, there exists a converging subsequence of $\{k_2\}$, such that $(1/k_2)A_1^{c,k_2}(t) \to \bar{A}_1^c(t)$ and $(1/k_2)A_2^{c,k_2}(t) \to \bar{A}_2^c(t)$. We can repeat the process until we have a subsequence $\{k_m\}$ satisfying the desired property. Let $N_i(\cdot)$ for all i denote independent unit rate Poisson processes. Recall that the functional strong law of large numbers for the Poisson process asserts that $(1/k)N_i(kt) \to t$, a.s. uniformly in $t \in [0, T]$ as $k \to \infty$. By composition convergence theorem, along the subsequence $\{k_m\}$ we get that $(1/k_m)N_i(A_i^{c,k_m}(t)) \to \bar{A}_i^c(t)$ for all i, and therefore that $\bar{X}_i^{c,k_m}(t) \equiv (1/k_m)X_i^{c,k_m}(t)$ converges to a limit $\bar{x}_i^c(t)$ for all i; the two converging results hold a.s. uniformly in $t \in [0, T]$. Using the continuity of $\vec{d}(t, \vec{p})$ in $\vec{p}$ and the piecewise continuity of $\vec{p}^f(t, \vec{x})$ in $\vec{x}$, by Dai and Williams (1995, Lemma 2.4) we get that as $k_m \to \infty$, for all i, $(1/k_m)A_i^{c,k_m}(t) = \int_0^t d_i(v, \vec{p}^c(v, (1/k_m)\vec{X}^{c,k_m}(v)))\, dv \to \int_0^t d_i(v, \vec{p}^c(v, \vec{\bar{x}}^c(v)))\, dv$, a.s. uniformly in $t \in [0, T]$. Thus, we get that as $k_m \to \infty$, for all i,

$$\bar{X}_i^{c,k_m}(t) = C_i - \frac{1}{k_m}N_i(A_i^{c,k_m}(t)) \to C_i - \int_0^t d_i(v, \vec{p}^c(v, \vec{\bar{x}}^c(v)))\, dv = \bar{x}_i^c(t), \tag{13}$$

a.s. uniformly in $t \in [0, T]$. This shows that the limiting state trajectories do not depend on the selection of the converging subsequence itself. Hence, in the sequel we denote the converging sequence by k to simplify notation. The last equality in (13) shows that $\{\vec{\bar{x}}^c(t), 0 \le t \le T\}$ is the state trajectory generated by the feedback policy $\vec{p}^c$ in the differential game. By the piecewise continuity of $\vec{p}^c(t, \vec{x})$ in $\vec{x}$, we have as $k \to \infty$, $\vec{p}^c(t, \vec{\bar{X}}^{c,k}(t)) \to \vec{p}^c(t, \vec{\bar{x}}^c(t))$, a.s.

uniformly in $t \in [0, T]$. Again by Dai and Williams (1995, Lemma 2.4), the revenue extracted under the feedback strategy $\vec{p}^c$ after normalization is, for all i, as $k \to \infty$, $(1/k)\int_0^T p_i^c(t, \bar{\vec{X}}^{c,k}(t))dN_i(A_i^{c,k}(t)) \to \int_0^T r_i(t, \vec{p}^c(t, \bar{\vec{x}}^c(t)))\,dt$, a.s. By Assumption 2(b) and the bounded convergence theorem, we have

$$\begin{aligned}\frac{1}{k}G_i^k[\{\vec{p}^c(t, \bar{\vec{X}}^{c,k}(t)), 0 \le t \le T\}] &= \frac{1}{k}E\left(\int_0^T p_i^c(t, \bar{\vec{X}}^{c,k}(t))dN_i(A_i^{c,k}(t))\right)\\ &\to \int_0^T r_i(t, \vec{p}^c(t, \bar{\vec{x}}^c(t)))\,dt. \qquad (14)\end{aligned}$$

We apply the convergence result (14) to the FNE $\vec{p}^f$: For any $\epsilon > 0$, there exists l_1 such that for all $k > l_1$, $\left|(1/k)G_i^k[\{\vec{p}^f(t, \bar{\vec{X}}^{f,k}(t)), 0 \le t \le T\}] - \int_0^T r_i(t, \vec{p}^f(t, \bar{\vec{x}}^f(t)))\,dt\right| < \epsilon/2$. We apply the convergence result (14) to any unilateral deviation $\vec{p}^c = (p_i^c, \vec{p}_{-i}^f) \in \mathscr{P}_F$ as a feedback policy in the differential game subject to coupled constraints (3): for the same $\epsilon > 0$, there exists l_2 such that for all $k > l_2$, $\left|(1/k)G_i^k[\{\vec{p}^c(t, \bar{\vec{X}}^{c,k}(t)), 0 \le t \le T\}] - \int_0^T r_i(t, \vec{p}^c \cdot (t, \bar{\vec{x}}^c(t)))\,dt\right| < \epsilon/2$. Since $\vec{p}^c$ is a unilateral deviation from the FNE $\vec{p}^f$ in the differential game, we have $\int_0^T r_i(t, \vec{p}^c(t, \bar{\vec{x}}^c(t)))\,dt \le \int_0^T r_i(t, \vec{p}^f(t, \bar{\vec{x}}^f(t)))\,dt$. Then for all $k > \max(l_1, l_2)$, $(1/k)G_i^k[\{\vec{p}^c(t, \bar{\vec{X}}^{c,k}(t)), 0 \le t \le T\}] \le (1/k) \cdot G_i^k[\{\vec{p}^f(t, \bar{\vec{X}}^{f,k}(t)), 0 \le t \le T\}] + \epsilon$. □

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