Supplement to "Distribution-Free Pricing"

A. The Rest of Proofs

Proof of Proposition 2. We can find the worst-case scenario that results in the largest standard deviation for all distributions that share the same mean μ and support $[v_L, v_H]$. It is easy to show that such a worst-case scenario is achieved by the following two-point distribution $V = \begin{cases} v_L \text{ with prob. } w, \\ v_U \text{ with prob. } 1-w, \end{cases}$ where $w = (v_U - \mu)/(v_U - v_L)$. As a result, the worst possible CV is $\bar{\delta} = \sqrt{\mu(v_L + v_U) - v_L v_U - \mu^2}/\mu$ among all distributions that share the same mean μ and support $[v_L, v_U]$. By Corollary 1(a), the performance guarantee is decreasing in the CV. Hence, the maximin price heuristic is the one corresponding to the largest possible CV and we obtain the desired result by applying Theorem 2. $\square$

Proof of Proposition 3. For any $F \in \mathscr{F}_l$, write one of its optimal prices as $p' = \mu - k'\sigma$ for some $k' \geq 0$. Consider two cases: (a) $k^* \geq k'$; (b) $k^* \leq k'$.

Case (a): $p^* \leq p'$, then $\bar{F}(p^{*-}) \geq \bar{F}(p'^{-})$, and thus

$$\frac{\pi(p^*;F)}{\pi(p';F)} = \frac{(p^*-c)\bar{F}(p^{*-})}{(p'-c)\bar{F}(p'^{-})} \geq \frac{p^*-c}{p'-c} \geq \frac{\mu - k^*\sigma - c}{\mu - c} = 1 - k^*\frac{\delta}{1-\gamma} = 1 - \frac{k^*}{\tau},$$

where the first inequality is due to $\bar{F}(p^{*-}) \geq \bar{F}(p'^{-})$, and the second inequality is due to $p' \leq \mu$.

Case (b): $p^* \geq p'$, then $\bar{F}(p^{*-}) \leq \bar{F}(p'^{-})$, and thus

$$\frac{\pi(p^*;F)}{\pi(p';F)} = \frac{(p^*-c)\bar{F}(p^{*-})}{(p'-c)\bar{F}(p'^{-})} \geq \frac{\bar{F}(p^{*-})}{\bar{F}(p'^{-})} \geq \bar{F}(p^{*-}) \geq 1 - \frac{1}{(k^*)^2+1},$$

where the first inequality is due to $p^* \geq p'$, the second inequality is due to $\bar{F}(p'^{-}) \leq 1$ and the last inequality is due to Lemma OA.1.

By (OA.4), $1 - \frac{k^*}{\tau} = 1 - \frac{2}{(k^*)^2+3} \in [\frac{1}{3}, 1)$. Moreover, $1 - \frac{1}{(k^*)^2+1} \in [0,1)$. As a result, for all $F \in \mathscr{F}^o$,

$$\tilde{\rho} = \frac{\pi(p^*;F)}{\pi(p';F)} \geq \min\{1 - \frac{k^*}{\tau}, 1 - \frac{1}{(k^*)^2+1}\} > (1 - \frac{k^*}{\tau})(1 - \frac{1}{(k^*)^2+1}) \geq \frac{(1-\frac{k^*}{\tau})(1-\frac{1}{(k^*)^2+1})}{1 + \frac{\gamma}{(1-\gamma)(1+\tau^2)}} = \rho, \tag{S.1}$$

where the first inequality is obtained by combining cases (a) and (b), and the last equality is due to (OA.5).

Moreover, $\frac{\tilde{\rho}-\rho}{\rho} = \frac{\tilde{\rho}}{\rho} - 1 \geq 1 + \frac{\gamma}{(1-\gamma)(1+\tau^2)} - 1 = \frac{\gamma}{(1-\gamma)(1+\tau^2)}$, where the inequality is due to the last inequality in (S.1). Thus the relative improvement of $\tilde{\rho}$ beyond ρ is no less than $\frac{\gamma}{(1-\gamma)(1+\tau^2)}$. $\square$

Proof of Corollary 4. It is easy to verify that the objective functions of problem (2) are symmetrical with respect to k_i if $\mu_1 = \mu_2 = \cdots = \mu_n$, $\sigma_1 = \sigma_2 = \cdots = \sigma_n$, $c_1 = c_2 = \cdots = c_n$, we thus have

$k_1^* = k_2^* = \cdots = k_n^*$. Moreover, the values of problems (2) are equal. Let $k_1 = k_2 = \cdots = k_n$, we can reduce u, T, $\bar{k}_i$ defined in Proposition 8 into: $u = \left(\frac{\sqrt{n(n-1)}}{\bar{k}_i} + \frac{1}{k_b}\right)^2$, $T = 1 - \frac{1}{(n+1)^2}\left(\frac{\sqrt{n(n-1)}}{\bar{k}_i} + \frac{1}{k_b} + \sqrt{n}\sqrt{\frac{2n}{\bar{k}_i^2} + \frac{n}{k_b^2} - \frac{2\sqrt{n(n-1)}}{k_b \bar{k}_i}}\right)^2$, and $k_i + \sqrt{n-1}\bar{k}_i = \sqrt{n}k_b$. Hence, we obtain the desired result. □

Proof of Theorem 3. We first characterize $\bar{p}$. Note $\min_p \max_{F\in\mathscr{F}} \left\{1 - \frac{\pi(p;F)}{\max_z\{\pi(z;F)\}}\right\} = 1 - \max_p \min_{F\in\mathscr{F}} \frac{\pi(p;F)}{\max_z\{\pi(z;F)\}}$. Then the minimax relative regret criterion is equal to maximize $\rho(p) = \min_{F\in\mathscr{F}} \frac{\pi(p;F)}{\max_z\{\pi(z;F)\}}$. Then we should solve the following optimization problem to obtain $\bar{p}$,

$$\rho^* = \max_p \rho(p) = \max_p \min_{F\in\mathscr{F}} \frac{\pi(p;F)}{\max_z\{\pi(z;F)\}} = \max_p \min_{F\in\mathscr{F}} \min_z \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V \geq z)}. \tag{S.2}$$

Thus, the worst relative regret for $\bar{p}$ is equal to $1 - \rho(\bar{p})$. We define

$$g(x,t) = \begin{cases} 0, & x < t, \\ t - c, & x \geq t. \end{cases}$$

Since $\pi(z;F) = (z-c)\mathsf{P}(V \geq z)$ and $\pi(p;F) = (p-c)\mathsf{P}(V \geq p)$, by inverting the order of minimization, we can rewrite (S.2) into:

$$\max_p \rho(p) = \max_p \min_z \min_{F\in\mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V \geq z)}.$$

Since $\pi(p;F) < 0$ for $p < c$, and $\pi(z;F) < 0$ for $z < c$, then we just need to consider $p \geq c$ and $z \geq c$. When p and z are given, we consider the following optimization problem:

$$\begin{aligned} \min_F \quad & \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V \geq z)} \\ \text{s.t.} \quad & \mathsf{E}_F[1] = 1, \quad \mathsf{E}_F[V] = \mu, \quad \mathsf{E}_F[V^2] = \mu^2 + \sigma^2, \quad V \geq 0. \end{aligned} \tag{S.3}$$

First, if F is a discrete distribution, suppose its support $S = \{v_1, v_2, \ldots, v_m\}$. We denote $x = \frac{1}{\mathsf{P}(V \geq z)}$ and $y_i = \frac{\mathsf{P}(V = v_i)}{\mathsf{P}(V \geq z)}$, where $i = 1, 2, \ldots, m$. Then we can rewrite (S.3) into:

$$\begin{aligned} \min_{x,y_i} \quad & \sum_{i=1}^m \frac{g(v_i,p)y_i}{z-c} \\ \text{s.t.} \quad & \sum_{i=1}^m y_i - x = 0, \quad \sum_{i=1}^m v_i y_i - \mu x = 0, \quad \sum_{i=1}^m v_i^2 y_i - (\mu^2+\sigma^2)x = 0, \quad \sum_{i=inf\{j|v_j \geq z\}}^m y_i = 1, \quad v_i \geq 0, \end{aligned} \tag{S.4}$$

whose dual problem is:

$$\begin{aligned} \max_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \quad & -\alpha_3 \\ \text{s.t.} \quad & \alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \leq 0, \\ & \alpha_0 + \alpha_1 v + \alpha_2 v^2 + \frac{g(v,p)}{z-c} \geq 0, \quad if\ 0 \leq v < z, \\ & \alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_3 + \frac{g(v,p)}{z-c} \geq 0, \quad if\ v \geq z. \end{aligned} \tag{S.5}$$

It is easy to see problems (S.4) and (S.5) are finite linear programming (since $v \in S = \{v_1, v_2, \ldots, v_m\}$, there are finite constraints in problem (S.5)), then the strong duality holds.

Next, we consider F is not a discrete distribution. Denote $B = \{-1\} \cup [0, \infty)$, following the notations in Hettich and Kortanek (1993), we let $M^+(B)$ be the space of nonnegative Borel measures on B, and

$$\mathbb{R}_+^{(B)} = \{\eta \in M^+(B) | supp(\eta) \ is \ finite\}.$$

Let

$$\eta(v) = \begin{cases} \frac{1}{\mathsf{P}(V \geq z)}, & \text{if } v = -1, \\ \frac{f(v)}{\mathsf{P}(V \geq z)}, & \text{if } v \geq 0, \end{cases}$$

and $\eta \in \mathbb{R}_+^{(B)}$. We rewrite problem (S.3) in the space of generalized finite sequences (GFS) by (the same formulation on page 398 of Hettich and Kortanek 1993)

$$\begin{aligned} v(D) = \min_{\eta} \quad & \sum_{v \in B} b(v)\eta(v) \\ \text{s.t.} \quad & \sum_{v \in B} a(v)\eta(v) = C, \quad \eta \in \mathbb{R}_+^{(B)}, \end{aligned} \tag{S.6}$$

and problem (S.5) is equal to

$$\begin{aligned} v(P) = \max_{Z} \quad & C^T Z \\ \text{s.t.} \quad & a^T(v) Z \leq b(v), \quad \forall v \in B, \end{aligned} \tag{S.7}$$

where $C = (0, 0, 0, -1)^T$, $Z = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)^T$,

$$a(v) = \begin{cases} (1, \mu, \mu^2 + \sigma^2, 0)^T, & \text{if } v = -1, \\ (-1, -v, -v^2, 0)^T, & \text{if } 0 \leq v < z, \\ (-1, -v, -v^2, -1)^T, & \text{if } v \geq z, \end{cases} \text{ and } \quad b(v) = \begin{cases} 0, & \text{if } V = -1, \\ \frac{g(v,p)}{z-c}, & \text{if } v \geq 0. \end{cases}$$

According to Theorem 6.5(ii) of Hettich and Kortanek (1993), if $v(P)$ is finite and M_{n+1} is closed, $v(P) = v(D)$, where

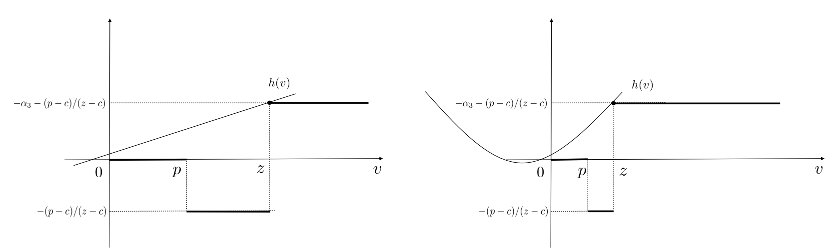
$$\begin{aligned} M_{n+1} = co\left(\left\{\left(\begin{array}{c} a(v) \\ b(v) \end{array}\right) \middle| v \in B\right\}\right) &= \left\{\omega = \left(\begin{array}{c} \sum\limits_{v \in B} a(v)\eta(v) \\ \sum\limits_{v \in B} b(v)\eta(v) \end{array}\right) \middle| \eta \in \mathbb{R}_+^{(B)}\right\} \\ &= \left(\begin{array}{c} \eta(-1) - \sum\limits_{v \in [0,\infty)} \eta(v) \\ \mu\eta(-1) - \sum\limits_{v \in [0,\infty)} v\eta(v) \\ (\mu^2 + \sigma^2)\eta(-1) - \sum\limits_{v \in [0,\infty)} v^2\eta(v) \\ -\sum\limits_{v \in [z,\infty)} \eta(v) \\ \sum\limits_{v \in [0,\infty)} \frac{g(v,p)}{z-c}\eta(v) \end{array}\right). \end{aligned}$$

Let $\bar{\eta} = supp(\eta)$, which is finite since $\eta \in \mathbb{R}_+^{(B)}$, and then $\eta \in [0, \bar{\eta}]$. We have $\eta(-1) - \sum_{v\in[0,\infty)} \eta(v) \in (-\infty, \bar{\eta}]$, $\mu\eta(-1) - \sum_{v\in[0,\infty)} v\eta(v) \in (-\infty, \bar{\eta}\mu]$, $(\mu^2+\sigma^2)\eta(-1) - \sum_{v\in[0,\infty)} v^2\eta(v) \in (-\infty, \bar{\eta}(\mu^2+\sigma^2)]$, $-\sum_{v\in[z,\infty)} \eta(v) \in (-\infty, -\bar{\eta}\mu]$, $\sum_{v\in[0,\infty)} \frac{g(v,p)}{z-c}\eta(v) \in [0,\infty)$. Therefore, M_{n+1} is closed. Next, we will solve problem (S.5) and verify its optimal value is finite, i.e., $v(P)$ is finite. Consequently, the optimal value of problem (S.3) is equal to that of problem (S.5).

We consider problem (S.5) with two cases: $z \geq p$ and $z \leq p$. Let $G^+(p) = \min_{z\geq p} \min_{F\in\mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V\geq z)}$ and $G^-(p) = \min_{z\leq p} \min_{F\in\mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V\geq z)}$.

Case 1: $z \geq p$. The right hand side of the second and third constraints in problem (S.5) is piecewise linear. A feasible function is any function $h(v) = \alpha_0 + \alpha_1 v + \alpha_2 v^2$ that lies above 0 for for $0 \leq v < p$, above $-(p-c)/(z-c)$ for $p \leq v < z$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \geq z$. We consider $h(v)$ is a straight line ($\alpha_2 = 0$) or quadratic function ($\alpha_2 > 0$ due to $h(v)$ needs to lie above 0 for $0 \leq v < p$, above $-(p-c)/(z-c)$ for $p \leq v < z$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \geq z$). Denote $x_0 = -\alpha_1/(2\alpha_2)$ as the minimum of the quadratic function $h(v)$ for $\alpha_2 > 0$. Note that $h(z) \geq -\alpha_3 - (p-c)/(z-c)$ due to $h(v)$ needs to lie above $-\alpha_3 - (p-c)/(z-c)$, i.e., $h(v)$ passes through $(z, -\alpha_3 - (p-c)/(z-c))$. If $h(z) > -\alpha_3 - (p-c)/(z-c)$, we can fix α_0, α_1, α_2, then decrease α_3 (increase $-\alpha_3$). Hence problem (S.5) is maximized with $h(z) = -\alpha_3 - (p-c)/(z-c)$. If $x_0 > z$, then $h(z) > h(x_0) \geq -\alpha_3 - (p-c)/(z-c)$, where the first inequality is due to $h(v)$ is minimized at x_0. Thus, we just need to consider $x_0 \leq z$ for $\alpha_2 > 0$. There are four possible cases that we will analyze as follows.

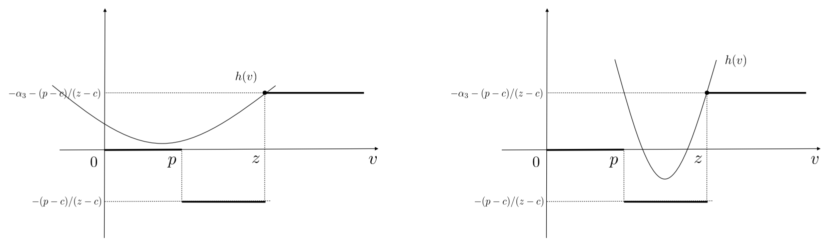
Figure S.1 Graphical illustration of functions satisfying feasibility conditions of the dual problem (Case 1)



(a) Case 1.1 (b) Case 1.2

Case 1.1: $h(v)$ is a straight line. Since the straight line $h(v) = \alpha_0 + \alpha_1 v$ has to stay above 0 for $0 \leq v < p$, above $-(p-c)/(z-c)$ for $p \leq v < z$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \geq z$, as illustrated in Figure S.1(a), all feasible straight lines should satisfy $h(0) = \alpha_0 \geq 0$, $\alpha_1 \geq 0$ and $h(z) = -\alpha_3 - (p-c)/(z-c)$.

Figure S.2 Graphical illustration of functions satisfying feasibility conditions of the dual problem (Case 1 cont'd)



(a) Case 1.3 (b) Case 1.4

If $\alpha_1 = 0$, i.e., $h(v)$ is a horizontal line, then $h(z) = \alpha_0 = -\alpha_3 - (p-c)/(z-c)$. Since the first constraint in problem (S.5) that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \le 0$, then $\alpha_0 \le 0$ due to $\alpha_1 = \alpha_2 = 0$. Recall $\alpha_0 \ge 0$, then we must have $\alpha_0 = 0$. The dual objective value $-\alpha_3$ is equal to $\alpha_0 + (p-c)/(z-c) = (p-c)/(z-c)$. If $\alpha_1 > 0$, $h(z) = \alpha_0 + \alpha_1 z = -\alpha_3 - (p-c)/(z-c)$. Recall $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \le 0$, then $\alpha_0 + \alpha_1\mu \le 0$ due to $\alpha_2 = 0$. However, $\alpha_0 + \alpha_1\mu > 0$ due to $\alpha_0 \ge 0$ and $\alpha_1 > 0$. Thus any straight line with $\alpha_1 > 0$ is not a feasible function.

Case 1.2: $x_0 = -\alpha_1/(2\alpha_2) \le 0$. Since $h(v)$ has to stay above 0 for $0 \le v < p$, as illustrated in Figure S.2(b), all feasible function should satisfy $h(0) = \alpha_0 \ge 0$. Note that the first constraint in problem (S.5) that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \le 0$, which implies $\alpha_1 < 0$, i.e., $x_0 = -\alpha_1/(2\alpha_2) > 0$ due to $\alpha_0 \ge 0$ and $\alpha_2 > 0$. Thus this case is not feasible.

Case 1.3: $0 < x_0 = -\alpha_1/(2\alpha_2) \le p$. Since $h(v)$ has to stay above 0 for $0 \le v < p$, as illustrated in Figure S.2(a), the minimum value of the quadratic function $h(x_0) \ge 0$. Note that the first constraint in problem (S.5) that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \le 0$, which shows $h(\mu) = \alpha_0 + \alpha_1\mu + \alpha_2\mu^2 < 0$. This contradicts to $h(v)$ has a minimum value that is greater than zero. Thus this case is not feasible.

Case 1.4: $p < x_0 = -\alpha_1/(2\alpha_2) \le z$. Since $h(v)$ has to stay above 0 for for $0 \le v < p$, above $-(p-c)/(z-c)$ for $p \le v < z$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \ge z$, as illustrated in Figure S.2(b), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \ge -(p-c)/(z-c)$ and $h(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 \ge 0$. We first show this case is not feasible for $p \ge \mu$ by contradiction. We suppose $p \ge \mu$, and recall $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \le 0$, then $h(\mu) = \alpha_0 + \alpha_1\mu + \alpha_2\mu^2 < 0$ due to $\sigma > 0$. Note that $h(v)$ is decreasing in $v \le x_0$, and thus $h(p) \le h(\mu) < 0$ due to $x_0 > p \ge \mu$, which contradicts to $h(p) \ge 0$. Next we consider $p < \mu$.

Note that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2 + \sigma^2]$ is increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2 by fixing $\alpha_1/(2\alpha_2)$. Recall $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 = -\alpha_3 - (p-c)/(z-c)$.

The dual objective value $-\alpha_3 = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + (p-c)/(z-c) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + z]^2 + (p-c)/(z-c)$, which is increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2 by fixing $\alpha_1/(2\alpha_2)$. If $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) < 0$, we can always fix $\alpha_1/(2\alpha_2)$ and increase $\alpha_0 - \alpha_1^2/(4\alpha_2)$ or α_2 to increase the dual objective value $-\alpha_3$. Hence, the dual objective value is maximized with $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = 0$. Therefore, $\alpha_0 - \alpha_1^2/(4\alpha_2) = -\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2 + \sigma^2]$ due to $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$.

Consequently, we have $-\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2] \geq -(p-c)/(z-c)$, i.e., $\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2 + \sigma^2] \leq (p-c)/(z-c)$ due to $\alpha_0 - \alpha_1^2/(4\alpha_2) \geq -(p-c)/(z-c)$. The dual objective value $-\alpha_3 = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + (p-c)/(z-c) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2)+z]^2 + (p-c)/(z-c) = \alpha_2[\alpha_1/(2\alpha_2) + z]^2 - \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2] + (p-c)/(z-c) = \alpha_2[(z-\mu)(\alpha_1/\alpha_2 + z + \mu) - \sigma^2] + (p-c)/(z-c)$. When $z \leq \mu$, $\alpha_2[(z-\mu)(z+\mu-2x_0) - \sigma^2] < 0$ due to $x_0 \leq z \leq \mu$ and $\sigma > 0$. When $z > \mu$, $\alpha_2[(z-\mu)(z+\mu-2x_0)-\sigma^2] > 0$ is increasing in α_2 and decreasing in x_0. If $\alpha_2[(x_0-\mu)^2+\sigma^2] < (p-c)/(z-c)$, we can always increase α_2 to increase $-\alpha_3 = \alpha_2[(z-\mu)(z+\mu-2x_0)-\sigma^2]$. Hence, the dual objective value is maximized with $\alpha_0 - \alpha_1^2/(4\alpha_2) = -\alpha_2[(\alpha_1/(2\alpha_2)-\mu)^2+\sigma^2] = -(p-c)/(z-c)$. Thus the dual objective value $-\alpha_3 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2)+z]^2 + (p-c)/(z-c) = \alpha_2[\alpha_1/(2\alpha_2)+z]^2 = (p-c)[\alpha_1/(2\alpha_2)+z]^2/[(z-c)((\alpha_1/(2\alpha_2)-\mu)^2+\sigma^2)] = (p-c)(z-x_0)^2/[(z-c)((x_0-\mu)^2+\sigma^2)]$.

Moreover, $h(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)) + p]^2 = \alpha_2[(\alpha_1/(2\alpha_2)) + p]^2 - \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2] = \alpha_2[(p-\mu)(\alpha_1/\alpha_2 + p + \mu) - \sigma_2] = \alpha_2[(\mu-p)(2x_0-p-\mu)-\sigma^2] \geq 0$, where the second equality is due to $\alpha_0 - \alpha_1^2/(4\alpha_2) = -\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$, and the inequality is due to $h(p) \geq 0$. Since $\alpha_2 > 0$ and $\mu > p$, $\alpha_2[(\mu-p)(2x_0-p-\mu)-\sigma^2] \geq 0$ implies that $x_0 \geq (\mu^2+\sigma^2-p^2)/2(\mu-p)$.

We denote $L(x_0) = (z-x_0)^2/[(x_0-\mu)^2+\sigma^2]$, consider the following FOC condition,

$$\frac{\partial L(x_0)}{\partial x_0} = \frac{-2(z-x_0)[(x_0-\mu)(z-\mu)+\sigma^2]}{[(x_0-\mu)^2+\sigma^2]^4} = 0.$$

Then we have $x_0^* = \mu - \frac{\sigma^2}{z-\mu}$, and $\frac{\partial L(x_0)}{\partial x_0} \geq 0$ for $x \leq x_0^*$, $\frac{\partial L(x_0)}{\partial x_0} \leq 0$ for $x \geq x_0^*$. Recall we should maximize $-\alpha_3 = (p-c)(z-x_0)^2/[(z-c)((x_0-\mu)^2+\sigma^2)] = (p-c)/(z-c)L(x_0)$ under the constraint $x_0 \geq (\mu^2+\sigma^2-p^2)/2(\mu-p)$. Thus, the dual objective value $-\alpha_3$ is maximized at $x_0^* = \mu - \frac{\sigma^2}{z-\mu}$ and the corresponding dual objective value is $[(p-c)((z-\mu)^2+\sigma^2)]/[(z-c)\sigma^2] > (p-c)/(z-c)$ if $(\mu^2+\sigma^2-p^2)/2(\mu-p) \leq x_0^* = \mu - \frac{\sigma^2}{z-\mu}$, i.e., $z \geq \mu + 2\sigma^2(\mu-p)/[(\mu-p)^2-\sigma^2]$; the dual objective value $-\alpha_3$ is maximized at $x_0 = (\mu^2+\sigma^2-p^2)/2(\mu-p)$ and the corresponding dual objective value is $[(p-c)(2z(\mu-p)-(\mu^2+\sigma^2-p^2))]/[(z-c)((\mu-p)^2+\sigma^2)]$ if $(\mu^2+\sigma^2-p^2)/2(\mu-p) \geq x_0^* = \mu - \frac{\sigma^2}{z-\mu}$, i.e., $z \leq \mu + 2\sigma^2(\mu-p)/[(\mu-p)^2-\sigma^2]$.

Now we will compare the four cases above to derive the optimal value of problem (S.5). If $p \geq \mu$, only Case 1.1 is feasible, thus the optimal dual objective value is $(p-c)/(z-c)$. If $p \leq z \leq \mu$, the dual objective value in Case 1.4 is less than zero, thus the optimal dual objective value is $(p-c)/(z-c)$. Consider $p < \mu < z$. We denote $H_1(z) = [(p-c)((z-\mu)^2+\sigma^2)]/[(z-c)\sigma^2]$ and $H_2(z) = [(p-c)(2z(\mu-p)-(\mu^2+\sigma^2-p^2))]/[(z-c)((\mu-p)^2+\sigma^2)]$, and take the derivative of $H_2(z)$ with respect to z,

$$\frac{\partial H_2(z)}{\partial z} = \frac{(p-c)[(\mu+p)(\mu+p-2c)+\sigma^2]}{(z-c)^2((\mu-p)^2+\sigma^2)} > 0,$$

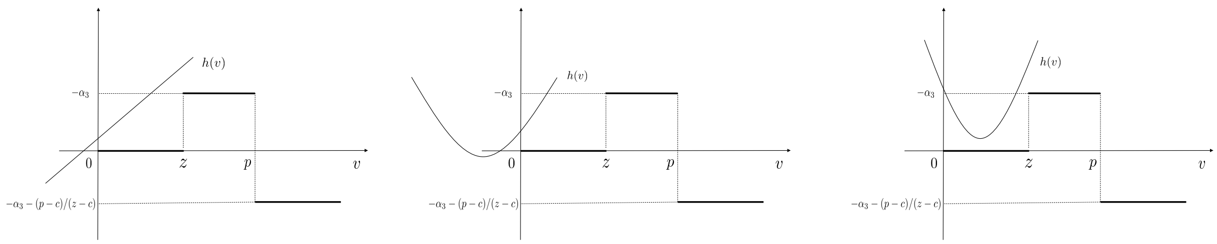
where the inequality is due to $\mu > p \geq c$ and $\sigma > 0$. Solving $(p-c)/(z-c) = H_2(z)$, we obtain $z = (\mu^2+\sigma^2-\mu p)/(\mu-p) \equiv \bar{z} > \mu$. Note that $(p-c)/(z-c)$ is decreasing in z and $H_2(z)$ is increasing in z, and thus if $z \leq \bar{z}$, $(p-c)/(z-c) \geq H_2(z)$, and otherwise, $(p-c)/(z-c) \leq H_2(z)$. In sum, if $p < \mu$, and $\bar{z} \leq z \leq \mu + 2\sigma^2(\mu-p)/[(\mu-p)^2-\sigma^2]$, the optimal dual objective value is equal to $H_2(z)$; if $p < \mu$, and $z \geq \mu + 2\sigma^2(\mu-p)/[(\mu-p)^2-\sigma^2]$, the optimal dual objective value is equal to $H_1(z)$; otherwise, it is equal to $(p-c)/(z-c)$. Thus, the optimal value of problem (S.5) is finite in this case.

Recall $G^+(p) = \min\limits_{z \geq p} \min\limits_{F \in \mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V \geq z)}$ and $\min\limits_{F \in \mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V \geq z)}$ is the primal problem of (S.5). Now we will minimize the optimal value of problem (S.5) for $z \geq p$ to obtain $G^+(p)$. Because $H_1(z)$ is equal to $L(x_0^*)$ and $H_2(z)$ is equal to $L((\mu^2+\sigma^2-p^2)/2(\mu-p))$, $H_1(z) \geq H_2(z)$ for any z. Recall $(p-c)/(z-c)$ is decreasing in z and $H_2(z)$ is increasing in z, and $(p-c)/(\bar{z}-c) = H_2(\bar{z})$. Hence, $(p-c)/(z-c) \geq (p-c)/(\bar{z}-c)$ for $z \leq \bar{z}$, $H_2(z) \geq H_2(\bar{z}) = (p-c)/(\bar{z}-c)$ for $\bar{z} \leq z \leq \mu + 2\sigma^2(\mu - p)/[(\mu-p)^2-\sigma^2]$, $H_1(z) \geq H_2(z) \geq H_2(\bar{z}) = (p-c)/(\bar{z}-c)$ for $z \geq \mu + 2\sigma^2(\mu-p)/[(\mu-p)^2-\sigma^2]$. Therefore, if $p < \mu$, the minimum of the optimal value of problem (S.5) is $(p-c)/(\bar{z}-c)$. From the analysis above, we obtain:

$$G^+(p) = \begin{cases} (p-c)/(\bar{z}-c) = [(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2], & \text{if} \quad p < \mu, \\ \min\limits_{z \geq p > c} (p-c)/(z-c) = 0, & \text{if} \quad p \geq \mu. \end{cases} \tag{S.8}$$

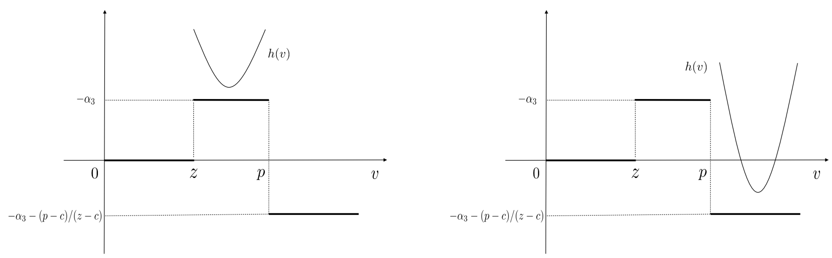
Case 2: $z \leq p$. The right hand side of the second and third constraints in problem (S.5) is piecewise linear. A feasible function is any function $h(v) = \alpha_0 + \alpha_1 v + \alpha_2 v^2$ that lies above 0 for for $0 \leq v < z$, above $-\alpha_3$ for $z \leq v < p$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \geq z$. We consider $h(v)$ is a straight line ($\alpha_2 = 0$) or quadratic function ($\alpha_2 > 0$ due to $h(v)$ needs to lie above 0 for $0 \leq v < z$, above $-\alpha_3$ for $p \leq v < z$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \geq z$). Denote $x_0 = -\alpha_1/(2\alpha_2)$ as the minimum of the quadratic function $h(v)$ for $\alpha_2 > 0$. There are five possible cases that we will analyze as follows.

Figure S.3 Graphical illustration of functions satisfying feasibility conditions of the dual problem (Case 2)



(a) Case 2.1 (b) Case 2.2 (c) Case 2.3

Figure S.4 Graphical illustration of functions satisfying feasibility conditions of the dual problem(Case 2 cont'd)



(a) Case 2.4 (b) Case 2.5

Case 2.1: $h(v)$ is a straight line. Since the straight line $h(v) = \alpha_0 + \alpha_1 v$ has to stay above 0 for $0 \le v < z$, above $-\alpha_3$ for $z \le v < p$, and above $-\alpha_3 - (p-c)/(z-c)$ for $v \ge z$, as illustrated in Figure S.3(a), all feasible straight lines should satisfy $h(0) = \alpha_0 \ge 0$, $\alpha_1 \ge 0$ and $h(z) \ge -\alpha_3$. If $h(z) > -\alpha_3$, then we can always decrease α_3 (increase $-\alpha_3$). Thus, the dual objective value is maximized with $h(z) = \alpha_0 + \alpha_1 z = -\alpha_3$.

If $\alpha_1 = 0$, i.e., $h(v)$ is a horizontal line. Then $-\alpha_3 = h(z) = \alpha_0 + \alpha_1 z = \alpha_0$. Since the first constraint in problem (S.5) that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) \le 0$, then $\alpha_0 \le 0$ due to $\alpha_1 = \alpha_2 = 0$. Recall $\alpha_0 \ge 0$, then we must have $\alpha_0 = 0$. The dual objective value $-\alpha_3$ is equal to $\alpha_0 = 0$. If $\alpha_1 > 0$, $h(z) = \alpha_0 + \alpha_1 z = -\alpha_3$. Recall $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) \le 0$, then $\alpha_0 + \alpha_1\mu \le 0$ due to $\alpha_2 = 0$. However, $\alpha_0 + \alpha_1\mu > 0$ due to $\alpha_0 \ge 0$ and $\alpha_1 > 0$. Thus any straight line with $\alpha_1 > 0$ is not a feasible function.

Case 2.2: $x_0 = -\alpha_1/(2\alpha_2) \le 0$. Since $h(v)$ has to stay above 0 for $0 \le v < z$, as illustrated in Figure S.3(b), all feasible function should satisfy $h(0) = \alpha_0 \ge 0$. Note that the first constraint in problem (S.5) that $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) \le 0$, which implies $\alpha_1 < 0$, i.e., $x_0 = -\alpha_1/(2\alpha_2) > 0$ due to $\alpha_0 \ge 0$ and $\alpha_2 > 0$. Thus this case is not feasible.

Case 2.3: $0 < x_0 = -\alpha_1/(2\alpha_2) \le z$. Since $h(v)$ has to stay above 0 for $0 \le v < z$, as illustrated in Figure S.3(c), the minimum value of the quadratic function $h(x_0) \ge 0$. Note that the first constraint in problem (S.5) that $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)\le 0$, which shows $h(\mu)=\alpha_0+\alpha_1\mu+\alpha_2\mu^2<0$. This contradicts to that $h(v)$ has a minimum value that is greater than zero. Thus this case is not feasible.

Case 2.4: $z<x_0=-\alpha_1/(2\alpha_2)\le p$. Since $h(v)$ has to stay above $-\alpha_3$ for $0\le v<z$, as illustrated in Figure S.4(a), then $h(x_0)\ge-\alpha_3$. Note that the first constraint in problem (S.5) that $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)\le 0$, which shows $h(\mu)=\alpha_0+\alpha_1\mu+\alpha_2\mu^2<0$. Since $h(v)$ is minimized at x_0, then $-\alpha_3\le h(x_0)\le h(\mu)<0$.

Case 2.5: $x_0=-\alpha_1/(2\alpha_2)>p$. Since $h(v)$ has to stay above 0 for for $0\le v<z$, above $-\alpha_3$ for $z\le v<p$, and above $-\alpha_3-(p-c)/(z-c)$ for $v\ge p$, as illustrated in Figure S.4(b), we have the value of $h(v)$ at x_0 satisfies $h(x_0)=\alpha_0-\alpha_1^2/(4\alpha_2)\ge-\alpha_3-(p-c)/(z-c)$ and $h(p)=\alpha_0+\alpha_1p+\alpha_2p^2\ge-\alpha_3$. We first show the dual objective value $-\alpha_3$ is less than 0 for $p\ge\mu$. We suppose $p\ge\mu$, and recall $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)\le 0$, and then $h(\mu)=\alpha_0+\alpha_1\mu+\alpha_2\mu^2<0$ due to $\sigma>0$. Note that $h(v)$ is decreasing in $v\le x_0$, and thus $-\alpha_3\le h(p)\le h(\mu)<0$ due to $x_0>p\ge\mu$. Next we consider $p<\mu$.

If $h(x_0)=\alpha_0-\alpha_1^2/(4\alpha_2)>-\alpha_3-(p-c)/(z-c)$ and $h(p)=\alpha_0+\alpha_1p+\alpha_2p^2>-\alpha_3$, we can always increase $-\alpha_3$ by fixing α_0, α_1 and α_2. Then we have the dual objective value $-\alpha_3$ is maximized with $h(x_0)=\alpha_0-\alpha_1^2/(4\alpha_2)=-\alpha_3-(p-c)/(z-c)$ or $h(p)=\alpha_0+\alpha_1p+\alpha_2p^2=-\alpha_3$. First, we consider $h(x_0)=\alpha_0-\alpha_1^2/(4\alpha_2)=-\alpha_3-(p-c)/(z-c)$, i.e., $-\alpha_3=\alpha_0-\alpha_1^2/(4\alpha_2)+(p-c)/(z-c)$. Then $h(p)=\alpha_0+\alpha_1p+\alpha_2p^2=\alpha_0-\alpha_1^2/(4\alpha_2)+\alpha_2[\alpha_1/(2\alpha_2)+p)]^2\ge-\alpha_3=\alpha_0-\alpha_1^2/(4\alpha_2)+(p-c)/(z-c)$, which implies $\alpha_2[\alpha_1/(2\alpha_2)+p)]^2\ge(p-c)/(z-c)$. Recall $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)\le 0$. If $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)<0$, we can always increase α_0 and fix α_1 and α_2 to increase $-\alpha_3$. Hence, the dual objective value $-\alpha_3$ is maximized with $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)=0$. Note that $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)=\alpha_0-\alpha_1^2/(4\alpha_2)+\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]=0$, and then $-\alpha_3=\alpha_0-\alpha_1^2/(4\alpha_2)+(p-c)/(z-c)=-\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]+(p-c)/(z-c)$ is decreasing in α_2 by fixing $\alpha_1/(2\alpha_2)$. Recall $\alpha_2[\alpha_1/(2\alpha_2)+p)]^2\ge(p-c)/(z-c)$, so if $\alpha_2[\alpha_1/(2\alpha_2)+p)]^2>(p-c)/(z-c)$, we can always fix $\alpha_1/(2\alpha_2)$ and decrease α_2 to increase the dual objective value $-\alpha_3$. Thus, the dual objective value is maximized with $\alpha_2[\alpha_1/(2\alpha_2)+p]^2=(p-c)/(z-c)$, i.e., $\alpha_2=(p-c)/[(z-c)[\alpha_1/(2\alpha_2)+p)]^2]$. Therefore, $-\alpha_3=-\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]+(p-c)/(z-c)=(p-c)/(z-c)-(p-c)[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]/[(z-c)[\alpha_1/(2\alpha_2)+p)]^2]=(p-c)/(z-c)-(p-c)[(x_0-\mu)^2+\sigma^2]/[(z-c)(x_0-p)^2]$. Second, we consider $h(p)=\alpha_0+\alpha_1p+\alpha_2p^2=-\alpha_3$. If $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)<0$, we can always increase α_0 and fix α_1 and α_2 to increase $-\alpha_3$. Hence, the dual objective value $-\alpha_3$ is maximized with $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)=0$. Note that $\alpha_0+\alpha_1\mu+\alpha_2(\mu^2+\sigma^2)=\alpha_0-\alpha_1^2/(4\alpha_2)+\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]=0$, and then $\alpha_0-\alpha_1^2/(4\alpha_2)=-\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$. Hence, $h(x_0)=\alpha_0-\alpha_1^2/(4\alpha_2)=-\alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]\ge-\alpha_3-(p-c)/(z-c)=\alpha_0+\alpha_1p+\alpha_2p^2-(p-c)/(z-c)=\alpha_0-\alpha_1^2/(4\alpha_2)+\alpha_2[\alpha_1/(2\alpha_2)+$

$p]^2 - (p-c)/(z-c)$, which implies $\alpha_2[\alpha_1/(2\alpha_2)+p]^2 \le (p-c)/(z-c)$. Note that $-\alpha_3 = \alpha_0 + \alpha_1 p + \alpha_2 p^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2)+p]^2 = \alpha_2[\alpha_1/(2\alpha_2)+p]^2 - \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$. If $[\alpha_1/(2\alpha_2)+p]^2 \le [(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$, $-\alpha_3 \le 0$, and otherwise, $-\alpha_3$ is increasing in α_3 by fixing $\alpha_1/(2\alpha_2)$. Consider $[\alpha_1/(2\alpha_2)+p]^2 > [(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$, and if $\alpha_2[\alpha_1/(2\alpha_2)+p]^2 < (p-c)/(z-c)$, we can always fix $\alpha_1/(2\alpha_2)$ and increase α_2 to increase $-\alpha_3$. Thus, the dual objective value is maximized with $\alpha_2[\alpha_1/(2\alpha_2)+p]^2 = (p-c)/(z-c)$, i.e., $\alpha_2 = (p-c)/[(z-c)[\alpha_1/(2\alpha_2)+p)]^2]$. We plug $\alpha_2 = (p-c)/[(z-c)[\alpha_1/(2\alpha_2)+p)]^2]$ into $-\alpha_3 = \alpha_2[\alpha_1/(2\alpha_2)+p]^2 - \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$, and then obtain $-\alpha_3 = (p-c)/(z-c) - (p-c)[(x_0-\mu)^2+\sigma^2]/[(z-c)(x_0-p)^2]$, which is equal to the last scenario.

We denote $H(x_0) = [(x_0-\mu)^2+\sigma^2]/(x_0-p)^2$ and take derivative of it with respect to x_0,

$$\frac{\partial H(x_0)}{\partial x_0} = \frac{2(x_0-p)[(\mu-p)(x_0-\mu)-\sigma^2]}{(x_0-p)^4},$$

which is less than zero for $x_0 \le \mu + \sigma^2/(\mu-p)$ and greater than zero for $x_0 \ge \mu+\sigma^2/(\mu-p)$ due to $x_0 > p$. Therefore, $-\alpha_3 = (p-c)/(z-c)(1-H(x_0))$ is maximized at $x_0 = \mu+\sigma^2/(\mu-p)$ and the corresponding value equals to $(p-c)(\mu-p)^2/[(z-c)((\mu-p)^2+\sigma^2)]$, which is decreasing in z.

In sum, if $p<\mu$, the optimal value of problem (S.5) is equal to $\frac{p-c}{z-c}\frac{(\mu-p)^2}{(\mu-p)^2+\sigma^2}$, and otherwise, it is equal to 0. Thus, the optimal value of problem (S.5) is finite in this case.

Recall $G^-(p) = \min_{z\le p}\min_{F\in\mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V\ge z)}$ and $\min_{F\in\mathscr{F}} \frac{\mathsf{E}_F[g(V,p)]}{(z-c)\mathsf{P}(V\ge z)}$ is the primal problem of (S.5). Now we will minimize the optimal value of problem (S.5) for $z\le p$ to obtain $G^-(p)$. From the analysis above, we obtain:

$$G^-(p) = \begin{cases} (\mu-p)^2/[(\mu-p)^2+\sigma^2], & \text{if} \quad p<\mu, \\ 0, & \text{if} \quad p\ge\mu. \end{cases} \tag{S.9}$$

We will prove that $\bar{p}$ maximizes $\rho(p) = \min\{G^+(p), G^-(p)\}$ by the following three steps.

Firstly, we show $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2]$ first increases and then decreases in p. Taking derivative of $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2]$ with respect to p,

$$\frac{\partial[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2]}{\partial p} = \frac{(\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2}{[(\mu-c)(\mu-p)+\sigma^2]^2}.$$

Note that $(\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2$ is decreasing in p, and it is greater than zero for $p=c$ and less than zero for $p=\mu$. Then there exists $c<p'<\mu$ satisfies $(\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2=0$, if $p\le p'$, $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2]$ is increasing in p, and otherwise, is decreasing in p.

Secondly, we show $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2] > (\mu-p)^2/[(\mu-p)^2+\sigma^2]$ for $p\ge p'$. $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2] > (\mu-p)^2/[(\mu-p)^2+\sigma^2]$ is equivalent to $(p-c)(\mu-p)^2+(p-c)\sigma^2 > (\mu-c)(\mu-p)^2+(\mu-p)\sigma^2$, i.e., $(p-c)(\mu-p)^2 > (\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2$. Noting that $p \geq p' > c$ and $(\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2 \leq 0$ for $p > p'$, we obtain $(p-c)(\mu-p)^2 > (\mu-c)(\mu-p)^2+(\mu+c-2p)\sigma^2$ for $p \geq p'$. Consequently, $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2] > (\mu-p)^2/[(\mu-p)^2+\sigma^2]$ for $p \geq p'$.

Thirdly, we will show $G^+(\bar{p}) = G^-(\bar{p})$; moreover, if $c < p \leq \bar{p}$, $G^+(p)$ is increasing in p and $G^+(p) \leq G^-(p)$, and if $p \geq \bar{p}$, $G^+(p) \geq G^-(p)$. Consequently, $\rho(p) = \min\{G^+(p), G^-(p)\}$ is increasing for $c < p \leq \bar{p}$ and decreasing for $p \geq \bar{p}$ since $G^-(p)$ is decreasing in p. Thus, $\bar{p}$ maximizes $\rho(p) = \min\{G^+(p), G^-(p)\}$. Let $[(p-c)(\mu-p)]/[(\mu-c)(\mu-p)+\sigma^2] = (\mu-p)^2/[(\mu-p)^2+\sigma^2]$, which is equivalent to $(2p-\mu-c)\sigma^2 = (\mu-p)^3$. Denote $p = \mu - k\sigma$, we have $(\mu-c-2k\sigma)\sigma^2 = k^3\sigma^3$, i.e., $k^3+2k = (\mu-c)/\sigma = \tau$. By Cardano's solution for a cubic function, the unique real root of it is $k = \sqrt[3]{\tau/2+\sqrt{(\tau/2)^2+(2/3)^3}} + \sqrt[3]{\tau/2-\sqrt{(\tau/2)^2+(2/3)^3}}$. It is easy to see that $G^-(p)$ is decreasing in p. Note that $G^+(p)$ is increasing in p for $p \leq p'$ due to the first step, and $G^+(p) \geq G^-(p)$ for $p \geq p'$ due to the second step. With $G^+(\bar{p}) = G^-(\bar{p})$, we have that if $c < p \leq \bar{p}$, $G^+(p)$ is increasing in p and $G^+(p) \leq G^-(p)$, and if $p \geq \bar{p}$, $G^+(p) \geq G^-(p)$.

Now we will show the worst relative regret for $\bar{p}$ is $1/(1+\bar{k}^2)$ and it is asymptotically achievable by $1 - \frac{\pi(\bar{p};F_\eta)}{\max_z\{\pi(z;F_\eta)\}}$ as $\eta \searrow 0$. Since the worst relative regret for $\bar{p}$ is $1-\rho(\bar{p})$, then it is equivalent to show $\rho(\bar{p}) = \bar{k}^2/(1+\bar{k}^2)$ and $\rho(\bar{p})$ is asymptotically achievable by $\frac{\pi(\bar{p};F_\eta)}{\max_z\{\pi(z;F_\eta)\}}$ as $\eta \searrow 0$. From the above analysis, we have $\rho(\bar{p}) = G^+(\bar{p}) = G^-(\bar{p})$. Recall $G^-(\bar{p}) = (\mu-\bar{p})^2/[(\mu-\bar{p})^2+\sigma^2] = (\bar{k}^2\sigma^2)/(\bar{k}^2\sigma^2+\sigma^2) = \bar{k}^2/(1+\bar{k}^2)$. On the one hand, note that $\pi(\mu-k_\eta\sigma;F_\eta) = \mu-k_\eta\sigma-c = \sigma(\tau-k_\eta)$ and $\pi(\mu+\frac{1}{k_\eta}\sigma;F_\eta) = (\mu+\frac{1}{k_\eta}\sigma-c)\frac{k_\eta^2}{1+k_\eta^2} = \sigma(\tau+\frac{1}{k_\eta})\frac{k_\eta^2}{1+k_\eta^2}$. Recall $k_\eta = \bar{k}+\eta$ and $\bar{k}^3+2\bar{k} = \tau$, then $k_\eta^3+2k_\eta > \tau$. Thus $\tau - k_\eta - (\tau+\frac{1}{k_\eta})\frac{k_\eta^2}{1+k_\eta^2} = \frac{\tau}{1+k_\eta^2} - \frac{k_\eta^3+2k_\eta}{1+k_\eta^2} < 0$, which implies $\pi(\mu-k_\eta\sigma;F_\eta) < \pi(\mu+\frac{1}{k_\eta}\sigma;F_\eta)$. Hence, $\max_z\{\pi(z;F_\eta)\} = \pi(\mu+\frac{1}{k_\eta}\sigma;F_\eta) = \sigma(\tau+\frac{1}{k_\eta})\frac{k_\eta^2}{1+k_\eta^2}$. On the other hand, $\pi(\bar{p};F_\eta) = (\mu-\bar{k}\sigma-c)\frac{k_\eta^2}{1+k_\eta^2} = \sigma(\tau-\bar{k})\frac{k_\eta^2}{1+k_\eta^2}$. Therefore, $\lim_{\eta\searrow 0}\frac{\pi(\bar{p};F_\eta)}{\max_z\{\pi(z;F_\eta)\}} = \lim_{\eta\searrow 0}\frac{\tau-\bar{k}}{\tau+\frac{1}{k_\eta}} = \frac{\tau-\bar{k}}{\tau+\frac{1}{\bar{k}}} = \frac{\bar{k}^3+2\bar{k}-\bar{k}}{\bar{k}^3+2\bar{k}+\frac{1}{\bar{k}}} = \frac{\bar{k}^2}{1+\bar{k}^2}$, where the second equality is due to $k_\eta = \bar{k}+\eta$. Thus, we obtain the announced results. □

Proof of Theorem 4. First, we show that the full characterization of $\tilde{p}$ is as follows:

(a) If $(\sigma^2+\mu^2)/(2\mu) \leq \mu$ and $c \leq (\sigma^2+\mu^2)/(2\mu) - (\sigma^2+\mu^2)^2/(8\mu\sigma^2)$, or $(\sigma^2+\mu^2)/(2\mu) \geq \mu$ and $c \leq \mu^3/(\sigma^2+\mu^2)$, then $\tilde{p}$ satisfies:

$$((\sigma^2+\mu^2)/\mu - p)/(1+\sigma^2/\mu^2) = \sigma^2(p-c)/(\sigma^2+(p-\mu)^2) \tag{S.10}$$

and $\tilde{p} \leq \min\{\mu, (\sigma^2+\mu^2)/(2\mu)\}$.

(b) If $(\sigma^2+\mu^2)/(2\mu) \geq \mu$ and $\mu^3/(\sigma^2+\mu^2) \leq c \leq \min\{(\sigma^2+\mu^2)/(4\mu), \mu\}$, then $\tilde{p}$ satisfies:

$$((\sigma^2+\mu^2)/\mu - p)/(1+\sigma^2/\mu^2) = (p-c)\mu/p \tag{S.11}$$

and $\tilde{p} \in [\mu, (\sigma^2+\mu^2)/(2\mu)]$.

(c) If $(\sigma^2+\mu^2)/(2\mu) \le \mu$ and $(\sigma^2+\mu^2)/(2\mu) - (\sigma^2+\mu^2)^2/(8\mu\sigma^2) \le c \le \mu - \sigma/2$, then $\tilde{p}$ satisfies:

$$[\sqrt{\sigma^2+(p-\mu)^2} - (p-\mu)]/2 = \sigma^2(p-c)/(\sigma^2+(p-\mu)^2) \tag{S.12}$$

and $\tilde{p} \in [(\sigma^2+\mu^2)/(2\mu), \mu]$.

(d) If $(\sigma^2+\mu^2)/(2\mu) \le \mu$ and $\mu - \sigma/2 \le c < \mu$, or $(\sigma^2+\mu^2)/(2\mu) \ge \mu$ and $\min\{(\sigma^2+\mu^2)/(4\mu), \mu\} \le c \le \mu$, then $\tilde{p}$ satisfies:

$$[\sqrt{\sigma^2+(p-\mu)^2} - (p-\mu)]/2 = (p-c)\mu/p \tag{S.13}$$

and $\tilde{p} \in [\max\{(\sigma^2+\mu^2)/(4\mu), \mu\}, (\sigma^2+\mu^2)/\mu)$.

The minimax absolute regret criterion consists of minimizing $\rho(p) = \max_{F\in\mathscr{F}} \max_z \{\pi(z;F)\} - \pi(p;F)$, i.e.,

$$\rho^* = \min_p \rho(p) = \min_p \max_{F\in\mathscr{F}} \max_z \{\pi(z;F)\} - \pi(p;F). \tag{S.14}$$

We define

$$g(x,t) = \begin{cases} 0, & x < t, \\ t-c, & x \ge t. \end{cases}$$

Since $\pi(z;F) = (z-c)\mathsf{P}(V \ge z)$ and $\pi(p;F) = (p-c)\mathsf{P}(V \ge p)$, by inverting the order of maximization, we can change (S.14) into:

$$\min_p \rho(p) = \min_p \max_z \max_{F\in\mathscr{F}} \{\mathsf{E}_F[g(V,z) - g(V,p)]\}.$$

Since $\pi(p;F) < 0$ for $p < c$, and $\pi(z;F) < 0$ for $z < c$, then we just need to consider $p \ge c$ and $z \ge c$. When p and z are given, we consider the following optimization problem:

$$\begin{aligned} \max_F \quad & \mathsf{E}_F[g(V,z) - g(V,p)] \\ \text{s.t.} \quad & \mathsf{E}_F[1] = 1, \quad \mathsf{E}_F[V] = \mu, \quad \mathsf{E}_F[V^2] = \mu^2+\sigma^2, \quad V \ge 0, \end{aligned} \tag{S.15}$$

whose dual problem is:

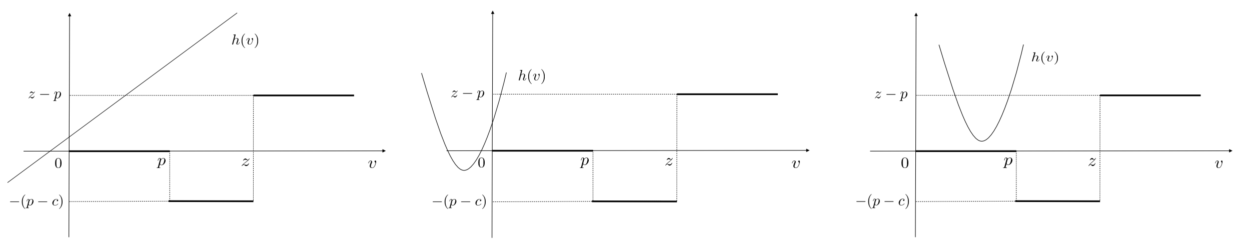
$$\begin{aligned} \min_{\alpha_0,\alpha_1,\alpha_2} \quad & \alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \\ \text{s.t.} \quad & \alpha_0 + \alpha_1 v + \alpha_2 v^2 \ge g(v,z) - g(v,p), \quad \forall v \ge 0. \end{aligned} \tag{S.16}$$

Theorem 1 of Isii (1962) shows that if the moment vector is an interior point of the set of feasible moment vectors, then strong duality holds in a moment problem. This is true here due to $\mu > 0$ and $\sigma > 0$ (a similar application of Isii 1962's result can be found in Natarajan et al. 2018). Thus, the optimal value of problem (S.15) is equal to that of problem (S.16).

We consider problem (S.16) with two cases: $z \ge p$ and $z \le p$. Let $G^+(p) = \max_{z\ge p} \max_{F\in\mathscr{F}} \mathsf{E}_F[g(V,z) - g(V,p)]$ and $G^-(p) = \max_{z\le p} \max_{F\in\mathscr{F}} \mathsf{E}_F[g(V,z) - g(V,p)]$.

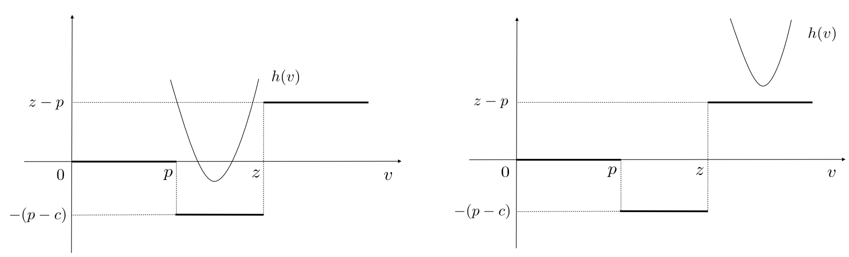
Case 1: $z \geq p$. The right hand side of the constraints of problem (S.16) is piecewise linear. A feasible function is any function $h(v) = \alpha_0 + \alpha_1 v + \alpha_2 v^2$ that lies above 0 for $0 \leq v < p$, above $-(p-c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$. We consider $h(v)$ is a straight line ($\alpha_2 = 0$) or quadratic function ($\alpha_2 > 0$ due to $h(v)$ needs to lie above 0 for $0 \leq v < p$, above $-(p-c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$. Denote $x_0 = -\alpha_1/(2\alpha_2)$ as the minimum of the quadratic function $h(v)$ for $\alpha_2 > 0$. There are five possible cases that we will analyze as follows.

Figure S.5 Graphical illustration of functions satisfying feasibility conditions of the dual problem(Case 1)



(a) Case 1.1 (b) Case 1.2 (c) Case 1.3

Figure S.6 Graphical illustration of functions satisfying feasibility conditions of the dual problem(Case 1 cont'd)



(a) Case 1.4 (b) Case 1.5

Case 1.1: $h(v)$ is a straight line. Since the line $h(v) = \alpha_0 + \alpha_1 v$ has to stay above 0 for $0 \leq v < p$, above $-(p-c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$, as illustrated in Figure S.5(a), all feasible straight lines should satisfy $h(0) = \alpha_0 \geq 0$, $\alpha_1 \geq 0$ and $h(z) = \alpha_0 + \alpha_1 z \geq z - p$.

If $\alpha_1 = 0$, i.e., $h(v)$ is a horizontal line, then the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 \geq z - p$ due to $h(v)$ needs to lie above $z - p$ for $v \geq z$. Hence, the optimal dual objective equals to $z - p$, which is achieved by $\alpha_0 = z - p$. If $\alpha_1 > 0$ and $z \leq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 + \alpha_1\mu \geq \alpha_0 + \alpha_1 z \geq z - p$. If $\alpha_1 > 0$ and $z \geq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 + \alpha_1 \geq \alpha_0 + (z - p - \alpha_0)\mu/z = (z-p)\mu/z + (1 - \mu/z)\alpha_0 \geq (z-p)\mu/z$, where the first inequality is due to $\alpha_0 + \alpha_1 z \geq z - p$ and $\alpha_1 > 0$, the second inequality is due to $z \geq \mu$ and $\alpha_0 \geq 0$.

In sum, if $z \leq \mu$, the straight line with α_0, α_1, α_2 that maximize the dual problem must be a horizontal line which corresponds to $\alpha_0 = \alpha_1 = \alpha_2 = 0$ with the associated dual objective value being equal to $z - p$; if $z \geq \mu$, the one that minimize the dual problem must pass through $(0, 0)$ and $(z, z - p)$, which corresponds to $\alpha_0 = \alpha_2 = 0$ and $\alpha_1 = (z - p)/z$ with the associated dual objective value being equal to $(z - p)\mu/z$.

Case 1.2: $x_0 = -\alpha_1/(2\alpha_2) \leq 0$. Since $h(v)$ has to stay above 0 for $0 \leq v < p$, above $-(p - c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$, as illustrated in Figure S.5(b), all feasible quadratic functions should satisfy $h(0) = \alpha_0 \geq 0$, and $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z - p$. Since $-\alpha_1/(2\alpha_2) \leq 0$, then $\alpha_1 \geq 0$ due to $\alpha_2 > 0$. The dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2)$ is increasing in α_0, α_1, α_2. If $\alpha_0 + \alpha_1 z + \alpha_2 z^2 > z - p$, we can always decrease α_1 or α_2 to decrease the dual objective value. Thus the dual objective value is minimized with $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z - p$. When $z \leq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) \geq \alpha_0 + \alpha_1 z + \alpha_2(z^2 + \sigma^2) > z - p$, where the last inequality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z - p$ and $\sigma > 0$. When $\mu \leq z \leq (\mu^2 + \sigma^2)/\mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) \geq \alpha_0 + \alpha_1\mu + \alpha_2\mu z = \alpha_0 + \mu(\alpha_1 + \alpha_2 z) = \alpha_0 + (z - p - \alpha_0)\mu/z = (z - p)\mu/z + (1 - \mu/z)\alpha_0 \geq (z - p)\mu/z$, where the second equality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z - p$, the last inequality is due to $\mu \leq z$ and $\alpha_0 \geq 0$. When $z \geq (\mu^2 + \sigma^2)/\mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 + \alpha_1\mu + (z - p - \alpha_0 - \alpha_1 z)(\mu^2 + \sigma^2)/z^2 = \alpha_0(1 - (\mu^2 + \sigma^2)/z^2) + \alpha_1(\mu - (\mu^2 + \sigma^2)/z) + (z - p)(\mu^2 + \sigma^2)/z^2$ is increasing in α_1 due to $z \geq (\mu^2 + \sigma^2)/\mu$, then the minimum is attained at $\alpha_1 = 0$, i.e., $x_0 = 0$, which can be included in the next case.

Case 1.3: $0 \leq x_0 = -\alpha_1/(2\alpha_2) \leq p$. Since $h(v)$ has to stay above 0 for $0 \leq v < p$, above $-(p - c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$, as illustrated in Figure S.5(c), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq 0$ and $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z - p$. Note that $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + z]^2$ is increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2 by fixing $\alpha_1/(2\alpha_2)$. The dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2]$, which is increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2. If $\alpha_0 + \alpha_1 z + \alpha_2 z^2 > z - p$, then we can always fix $\alpha_1/(2\alpha_2)$ and $\alpha_0 - \alpha_1^2/(4\alpha_2)$, and decrease α_2 to decrease the dual objective value. Thus the dual problem is maximized with $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z - p$, which implies $\alpha_0 - \alpha_1^2/(4\alpha_2) = z - p - \alpha_2(\alpha_1/(2\alpha_2) + z)^2$. Hence, $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[((\alpha_1/(2\alpha_2)) + \mu)^2 + \sigma^2] = z - p - \alpha_2(\alpha_1/(2\alpha_2) + z)^2 + \alpha_2[((\alpha_1/(2\alpha_2)) + \mu)^2 + \sigma^2] = z - p - \alpha_2[(\alpha_1/(2\alpha_2) + z)^2 - [(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2]]$, which is greater than $z - p$ when $[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] > (\alpha_1/(2\alpha_2) + z)^2$. Now consider $[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] \leq (\alpha_1/(2\alpha_2) + z)^2$, which implies the dual objective value is decreasing in α_2 by fixing $\alpha_1/(2\alpha_2)$. Note

$h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) = z - p - \alpha_2(\alpha_1/(2\alpha_2) + z)^2 \geq 0$, if $\alpha_0 - \alpha_1^2/(4\alpha_2) = z - p - \alpha_2(\alpha_1/(2\alpha_2) + z)^2 > 0$, we can fix $\alpha_1/(2\alpha_2)$ and increase α_2 to decrease the dual objective value. Hence, the dual objective value is minimized with $\alpha_0 - \alpha_1^2/(4\alpha_2) = 0$ and $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z - p$.

Thus the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] = (z-p)[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2]/[\alpha_1/(2\alpha_2) + z]^2 = (z-p)[(\mu - x_0)^2 + \sigma^2]/(z - x_0)^2$, where the second equality is due to $\alpha_0 - \alpha_1^2/(4\alpha_2) = 0$ and $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + z]^2 = z - p$.

We denote $L(x_0) = [(\mu - x_0)^2 + \sigma^2]/(z - x_0)^2$, consider the following FOC condition,

$$\frac{\partial L(x_0)}{\partial x_0} = \frac{2(z - x_0)[(\mu - x_0)(\mu - z) + \sigma^2]}{(z - x_0)^4} = 0.$$

Then we have $x_0^* = \mu + \frac{\sigma^2}{\mu - z}$. When $z \leq \mu$, $2(z - x_0)[(\mu - x_0)(\mu - z) + \sigma^2] > 0$ due to $x_0 \leq p \leq z \leq \mu$ and $\sigma > 0$, then $L(x_0)$ is increasing in $[0, p]$ and minimized at $x_0 = 0$. We next consider $z > \mu$, and have $\frac{\partial L(x_0)}{\partial x_0} \leq 0$ if $x_0 \leq x_0^*$, and $\frac{\partial L(x_0)}{\partial x_0} \geq 0$ if $x_0 \geq x_0^*$. When $\mu < z \leq (\sigma^2 + \mu^2)/\mu$, $x_0^* = \mu + \frac{\sigma^2}{\mu - z} \leq \mu + \frac{\sigma^2}{\mu - (\sigma^2 + \mu^2)/\mu} = 0$, then $L(x_0)$ is increasing in $[0, p]$ and minimized at $x_0 = 0$, and the corresponding dual objective value is equals to $(\mu^2 + \sigma^2)(z - p)/z^2$, which is greater than $(z - p)\mu/z$ due to $z \leq (\sigma^2 + \mu^2)/\mu$. When $z \geq (\sigma^2 + \mu^2)/\mu$ and $(z - \mu)(\mu - p) \geq \sigma^2$, $x_0^* = \mu + \frac{\sigma^2}{\mu - z} \geq \mu + \frac{(z-\mu)(\mu-p)}{\mu - z} = p$ due to $z \geq (\sigma^2 + \mu^2)/\mu > \mu$, then $L(x_0)$ is decreasing in $[0, p]$ and minimized at $x_0 = p$, and the corresponding dual objective value is equals to $((p - \mu)^2 + \sigma^2)/(z - p)$. When $z \geq (\sigma^2 + \mu^2)/\mu$ and $(z - \mu)(\mu - p) \leq \sigma^2$, we have $0 \leq x_0^* \leq p$. Then $L(x_0)$ is minimized at $x_0 = x_0^*$ and $L(x_0^*) = \sigma^2/((z - \mu)^2 + \sigma^2)$. Thus the corresponding dual objective value is equals to $(z - p)L(x_0^*) = (z - p)\sigma^2/((z - \mu)^2 + \sigma^2)$.

Case 1.4: $p \leq x_0 = -\alpha_1/(2\alpha_2) \leq z$. Since $h(v)$ has to stay above 0 for $0 \leq v < p$, above $-(p - c)$ for $p \leq v < z$, and above $z - p$ for $v \geq z$, as illustrated in Figure S.6(a), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq -(p - c)$, $h(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 \geq 0$ and $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z - p$. When $z \leq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\mu - x_0)^2 + \sigma^2] \geq \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(z - x_0)^2 + \sigma^2] = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + z)^2 + \sigma^2] = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_2\sigma^2 > z - p$, where the first inequality is due to $x_0 \leq z \leq \mu$, and the last inequality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z - p$ and $\alpha_2 > 0$. When $z \geq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] \geq z - p - \alpha_2[\alpha_1/(2\alpha_2) + z]^2 + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] = z - p + \alpha_2[(z - \mu)(-\alpha_1/\alpha_2 - \mu - z) + \sigma^2]$, where the inequality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + z]^2 \geq z - p$. Since $z - p + \alpha_2[(z - \mu)(-\alpha_1/\alpha_2 - \mu - z) + \sigma^2]$ is increasing in $-\alpha_1/(2\alpha_2)$ due to $z \geq \mu$, then the minimum of the dual problem is attained at $x_0 = -\alpha_1/(2\alpha_2) = p$, which can be included in the last case.

Case 1.5: $x_0 = -\alpha_1/(2\alpha_2) \geq z$. Since $h(v)$ has to stay above 0 for $0 \leq v < p$, above $-(p-c)$ for $p \leq v < z$, and above $z-p$ for $v \geq z$, as illustrated in Figure S.6(b), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq z-p$. Then the the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2 + \sigma^2] > z-p$ due to $\alpha_2 > 0$.

In sum, from the analysis of the five cases above, we have the optimal value of problem (S.16) is equal to: $z-p$ when $z \leq \mu$, $(z-p)\mu/z$ when $\mu \leq z \leq (\sigma^2+\mu^2)/\mu$, $\sigma^2(z-p)/((z-\mu)^2+\sigma^2)$ when $z \geq (\sigma^2+\mu^2)/\mu$ and $(z-\mu)(\mu-p) \leq \sigma^2$, $((p-\mu)^2+\sigma^2)/(z-p)$ when $z \geq (\sigma^2+\mu^2)/\mu$ and $(z-\mu)(\mu-p) \geq \sigma^2$. Recall $G^+(p) = \max_{z \geq p} \max_{F \in \mathscr{F}} \mathsf{E}_F[g(V,z) - g(V,p)]$ and $\max_{F \in \mathscr{F}} \mathsf{E}_F[g(V,z) - g(V,p)]$ is the primal problem of (S.16). Now we will maximize the optimal value of problem (S.16) for $z \geq p$ to obtain $G^+(p)$.

Note that $z-p$ and $(z-p)\mu/z$ are increasing in z, and $((p-\mu)^2+\sigma^2)/(z-p)$ is decreasing in z. We take the derivative of $\sigma^2(z-p)/((z-\mu)^2+\sigma^2)$ with respect to z:

$$\begin{aligned}\frac{\partial\sigma^2(z-p)/(\sigma^2+(z-\mu)^2)}{\partial z} &= \frac{\sigma^2(\sigma^2+\mu^2-z^2+2pz-2p\mu)}{(\sigma^2+(z-\mu)^2)^2} \\ &= \frac{-\sigma^2(z-p-\sqrt{\sigma^2+(p-\mu)^2})(z-p+\sqrt{\sigma^2+(p-\mu)^2})}{(\sigma^2+(z-\mu)^2)^2},\end{aligned}$$

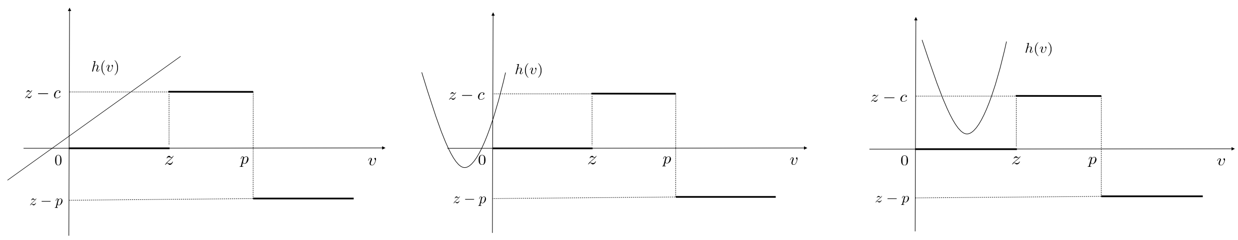
which is nonnegative if $p \leq z \leq p + \sqrt{\sigma^2+(p-\mu)^2}$ and nonpositive if $z \geq p + \sqrt{\sigma^2+(p-\mu)^2}$. It is easy to verify that $p + \sqrt{\sigma^2+(p-\mu)^2}$ is increasing in p. Let $p + \sqrt{\sigma^2+(p-\mu)^2} = (\sigma^2+\mu^2)/\mu$, we obtain $p = (\sigma^2+\mu^2)/(2\mu)$. If $p \leq (\sigma^2+\mu^2)/(2\mu)$, then $z \geq (\sigma^2+\mu^2)/\mu \geq p + \sqrt{\sigma^2+(p-\mu)^2}$ and $\sigma^2(z-p)/((z-\mu)^2+\sigma^2)$ is decreasing in z. If $p \geq (\sigma^2+\mu^2)/(2\mu)$, then $(\sigma^2+\mu^2)/\mu \leq p + \sqrt{\sigma^2+(p-\mu)^2}$, and $\sigma^2(z-p)/((z-\mu)^2+\sigma^2)$ is increasing in z for $(\sigma^2+\mu^2)/\mu \leq z \leq p + \sqrt{\sigma^2+(p-\mu)^2}$, $\sigma^2(z-p)/((z-\mu)^2+\sigma^2)$ is decreasing in z for $z \geq p + \sqrt{\sigma^2+(p-\mu)^2}$. Hence, if $p \leq (\sigma^2+\mu^2)/(2\mu)$, the optimal value of problem (S.16) is increasing in z for $z \leq (\sigma^2+\mu^2)/\mu$ and decreasing in z for $z \geq (\sigma^2+\mu^2)/\mu$; otherwise, the optimal value of problem (S.16) is increasing in z for $z \leq p + \sqrt{\sigma^2+(p-\mu)^2}$ and decreasing in z for $z \geq p + \sqrt{\sigma^2+(p-\mu)^2}$.

From the analysis above, we obtain:

$$G^+(p) = \begin{cases} ((\sigma^2+\mu^2)/\mu - p)/(1+\sigma^2/\mu^2), & \text{if} \quad p \leq (\sigma^2+\mu^2)/(2\mu), \\ [\sqrt{\sigma^2+(p-\mu)^2} - (p-\mu)]/2, & \text{if} \quad p \geq (\sigma^2+\mu^2)/(2\mu). \end{cases}$$

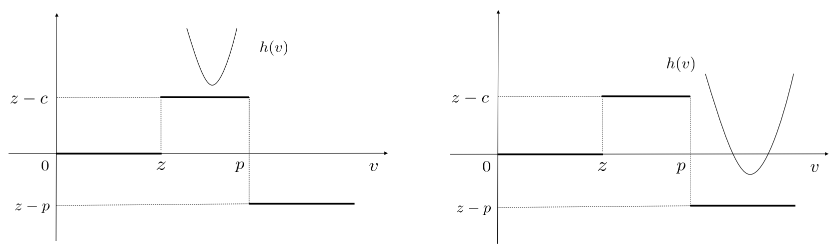
Case 2: $z \leq p$. The right hand side of the constraints of problem (S.16) is piecewise linear. A feasible function is quadratic function $h(v) = \alpha_0 + \alpha_1 v + \alpha_2 v^2$ that lies above above 0 for $0 \leq v < z$, above $z-c$ for $z \leq v < p$, and above $z-p$ for $v \geq p$. Similarly with Case 1, we consider $h(v)$ is a straight line ($\alpha_2 = 0$) or quadratic function ($\alpha_2 > 0$). There are three possible cases that we will analyze next.

Figure S.7 Graphical illustration of functions satisfying feasibility conditions of the dual problem(Case 2)



(a) Case 2.1 (b) Case 2.2 (c) Case 2.3

Figure S.8 Graphical illustration of functions satisfying feasibility conditions of the dual problem(Case 2 cont'd)



(a) Case 2.4 (b) Case 2.5

Case 2.1: $h(v)$ is a straight line. When $z \le \mu$ the optimal solution is a horizontal line. The associated dual objective value is equal to $z-c$. Since $h(v)$ has to stay above 0 for $0 \le v < z$, above $z-c$ for $z \le v < p$, and above $z-p$ for $v \ge p$, as illustrated in Figure S.7(a), all feasible straight lines should satisfy $h(0) = \alpha_0 \ge 0$, $\alpha_1 \ge 0$ and $h(z) = \alpha_0 + \alpha_1 z \ge z - c$.

If $\alpha_1 = 0$, i.e., $h(v)$ is a horizontal line, then the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 \ge z - c$ due to $h(v)$ needs to lie above $z-c$ for $v \ge z$. Hence, the optimal dual objective equals to $z-p$, which is achieved by $\alpha_0 = z - c$.

If $\alpha_1 > 0$ and $z \le \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 + \alpha_1\mu \ge \alpha_0 + \alpha_1 z \ge z - c$. If $\alpha_1 > 0$ and $z \ge \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 + \alpha_1 \ge \alpha_0 + (z-c-\alpha_0)\mu/z = (z-c)\mu/z + (1-\mu/z)\alpha_0 \ge (z-c)\mu/z$, where the first inequality is due to $\alpha_0 + \alpha_1 z \ge z - c$ and $\alpha_1 > 0$, the second inequality is due to $z \ge \mu$ and $\alpha_0 \ge 0$.

In sum, if $z \le \mu$, the straight line with α_0, α_1, α_2 that maximize the dual problem must be a horizontal line which corresponds to $\alpha_0 = \alpha_1 = \alpha_2 = 0$ with the associated dual objective value being equal to $z-c$; if $z \ge \mu$, the one that minimize the dual problem must pass through $(0,0)$ and $(z, z-c)$, which corresponds to $\alpha_0 = \alpha_2 = 0$ and $\alpha_1 = (z-c)/z$ with the associated dual objective value being equal to $(z-c)\mu/z$.

Case 2.2: $x_0 = -\alpha_1/(2\alpha_2) \leq 0$. Since $h(v)$ has to stay above 0 for $0 \leq v < z$, above $z-c$ for $z \leq v < p$, and above $z-p$ for $v \geq p$, as illustrated in Figure S.7(b), all feasible quadratic functions should satisfy $h(0) = \alpha_0 \geq 0$, and $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z-c$. Since $-\alpha_1/(2\alpha_2) \leq 0$, then $\alpha_1 \geq 0$ due to $\alpha_2 > 0$. The dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2)$ is increasing in α_0, α_1, α_2. If $\alpha_0 + \alpha_1 z + \alpha_2 z^2 > z-c$, we can always decrease α_1 or α_2 to decrease the dual objective value. Thus the dual objective value is minimized with $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z-c$. When $z \leq \mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \geq \alpha_0 + \alpha_1 z + \alpha_2(z^2+\sigma^2) > z-c$, where the last inequality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z-c$ and $\sigma > 0$. When $\mu \leq z \leq (\mu^2+\sigma^2)/\mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) \geq \alpha_0 + \alpha_1\mu + \alpha_2\mu z = \alpha_0 + \mu(\alpha_1 + \alpha_2 z) = \alpha_0 + (z-c-\alpha_0)\mu/z = (z-c)\mu/z + (1-\mu/z)\alpha_0 \geq (z-c)\mu/z$, where the second equality is due to $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z-c$, the last inequality is due to $\mu \leq z$ and $\alpha_0 \geq 0$. When $z \geq (\mu^2+\sigma^2)/\mu$, the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 + \alpha_1\mu + (z-c-\alpha_0-\alpha_1 z)(\mu^2+\sigma^2)/z^2 = \alpha_0(1-(\mu^2+\sigma^2)/z^2) + \alpha_1(\mu-(\mu^2+\sigma^2)/z) + (z-c)(\mu^2+\sigma^2)/z^2$ is increasing in α_1 due to $z \geq (\mu^2+\sigma^2)/\mu$, then the minimum is attained at $\alpha_1 = 0$, i.e., $x_0 = 0$, which can be included in the next case.

Case 2.3: $0 \leq x_0 = -\alpha_1/(2\alpha_2) \leq z$. Since $h(v)$ has to stay above 0 for $0 \leq v < z$, above $z-c$ for $z \leq v < p$, and above $z-p$ for $v \geq p$, as illustrated in Figure S.7(c), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq 0$ and $h(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 \geq z-c$. Note that $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2)+z]^2$, and the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]$, are increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2. With the similar analysis in Case 1.3, we can obtain that the dual objective value is minimized with $\alpha_0 - \alpha_1^2/(4\alpha_2) = 0$ and $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = z-c$. Thus the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2+\sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2] = (z-c)[(\alpha_1/(2\alpha_2)+\mu)^2+\sigma^2]/[\alpha_1/(2\alpha_2)+z]^2 = (z-c)[(\mu-x_0)^2+\sigma^2]/(z-x_0)^2$, where the second equality is due to $\alpha_0 - \alpha_1^2/(4\alpha_2) = 0$ and $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2)+z]^2 = \alpha_2[\alpha_1/(2\alpha_2)+z]^2 = z-c$.

Recall the analysis about $L(x_0) = [(\mu-x_0)^2+\sigma^2]/(z-x_0)^2$ in Case 1.3, we have that when $z \leq (\sigma^2+\mu^2)/\mu$, $L(x_0)$ is increasing in $[0,z]$ and minimized at $x_0 = 0$, and the corresponding dual objective value is equals to $(\mu^2+\sigma^2)(z-c)/z^2$, which is greater than $(z-c)\mu/z$ due to $z \leq (\sigma^2+\mu^2)/\mu$. When $z \geq (\sigma^2+\mu^2)/\mu$, $0 \leq x_0^* = \mu + \frac{\sigma^2}{\mu-z} < z$. Then $L(x_0)$ is minimized at $x_0 = x_0^*$ and $L(x_0^*) = \sigma^2/[(\mu-z)^2+\sigma^2]$. Thus the corresponding dual objective value is equals to $(z-c)L(x_0^*) = (z-c)\sigma^2/((\mu-z)^2+\sigma^2)$.

Case 2.4: $z \leq x_0 = -\alpha_1/(2\alpha_2) \leq p$. Since $h(v)$ has to stay above 0 for $0 \leq v < z$, above $z-c$ for $z \leq v < p$, and above $z-p$ for $v \geq p$, as illustrated in Figure S.8(a), we have the value of $h(v)$ at x_0

satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq z - c$. Then the the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] > z - c$ due to $\alpha_2 > 0$.

Case 2.5: $x_0 = -\alpha_1/(2\alpha_2) \geq p$. Since $h(v)$ has to stay above 0 for $0 \leq v < z$, above $z - c$ for $z \leq v < p$, and above $z - p$ for $v \geq p$, as illustrated in Figure S.8(b), we have the value of $h(v)$ at x_0 satisfies $h(x_0) = \alpha_0 - \alpha_1^2/(4\alpha_2) \geq z - p$, $h(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 \geq z - c$. Note that $\alpha_0 + \alpha_1 p + \alpha_2 p^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + p]^2$, and the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2]$, are increasing in $\alpha_0 - \alpha_1^2/(4\alpha_2)$ and α_2. With the similar analysis in Case 1.3, we can obtain that the dual objective value is minimized with $\alpha_0 - \alpha_1^2/(4\alpha_2) = z - p$ and $\alpha_0 + \alpha_1 p + \alpha_2 p^2 = z - c$. Thus the dual objective value $\alpha_0 + \alpha_1\mu + \alpha_2(\mu^2 + \sigma^2) = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2] = z - p + (p - c)[(\alpha_1/(2\alpha_2) + \mu)^2 + \sigma^2]/[\alpha_1/(2\alpha_2) + p]^2 = z - p + (p - c)[(x_0 - \mu)^2 + \sigma^2]/(x_0 - p)^2$, where the second equality is due to $\alpha_0 - \alpha_1^2/(4\alpha_2) = z - p$ and $\alpha_0 + \alpha_1 p + \alpha_2 p^2 = \alpha_0 - \alpha_1^2/(4\alpha_2) + \alpha_2[\alpha_1/(2\alpha_2) + p]^2 = z - p + \alpha_2[\alpha_1/(2\alpha_2) + p]^2 = z - c$. If $p \geq \mu$, $z - p + (p - c)[(x_0 - \mu)^2 + \sigma^2]/(x_0 - p)^2 > z - p + (p - c)[(x_0 - p)^2 + \sigma^2]/(x_0 - p)^2 > z - p + p - c = z - c$, where the first inequality is due to $x_0 \geq p \geq \mu$, and the second inequality is due to $\sigma > 0$ and $p > c$.

If $p < \mu$, we denote $H(x_0) = [(x_0 - \mu)^2 + \sigma^2]/(x_0 - p)^2$, consider the following FOC condition,

$$\frac{\partial H(x_0)}{\partial x_0} = \frac{2(p - x_0)[(\mu - x_0)(\mu - p) + \sigma^2]}{(x_0 - p)^4} = 0.$$

Then we have $x_0^* = \mu + \frac{\sigma^2}{\mu - p} > p$, and $H(x_0)$ is decreasing for $x_0 \leq x_0^*$ and increasing for $x_0 \geq x_0^*$. Then $H(x_0)$ minimized at $x_0 = x_0^*$ and $H(x_0^*) = \sigma^2/[(\mu - p)^2 + \sigma^2]$. Thus the corresponding dual objective value is equals to $z - p + (p - c)H(x_0^*) = z - p + (p - c)\sigma^2/(\sigma^2 + (p - \mu)^2)$.

In sum, from the analysis in the five cases, the optimal value of problem (S.16) is equal to: $z - p + (p - c)\sigma^2/(\sigma^2 + (p - \mu)^2)$ when $z \leq p < \mu$, $z - c$ when $z \leq \mu \leq p$, $(z - c)\mu/z$ when $p \geq \mu$ and $\mu \leq z \leq (\sigma^2 + \mu^2)/\mu$, $\sigma^2(z - c)/(\sigma^2 + (z - \mu)^2)$ when $p \geq \mu$ and $z \geq (\sigma^2 + \mu^2)/\mu$. Recall $G^-(p) = \max_{z \leq p} \max_{F \in \mathscr{F}} \mathsf{E}_F[g(V, z) - g(V, p)]$ and $\max_{F \in \mathscr{F}} \mathsf{E}_F[g(V, z) - g(V, p)]$ is the primal problem of problem (S.16). Now we will maximize the optimal value of problem (S.16) for $z \leq p$ to obtain $G^-(p)$.

Note that $z - c - (p - \mu)^2(p - c)/(\sigma^2 + (p - \mu)^2)$, $z - c$ and $(z - c)\mu/z$ are increasing in z, and $\sigma^2(z - c)/(\sigma^2 + (z - \mu)^2)(\leq (z - c)\mu/z)$ is decreasing in z when $z \geq (\sigma^2 + \mu^2)/\mu$. Thus,

$$G^-(p) = \begin{cases} \sigma^2(p - c)/(\sigma^2 + (p - \mu)^2), & \text{if} \quad p < \mu, \\ (p - c)\mu/p, & \text{if} \quad \mu \leq p \leq (\sigma^2 + \mu^2)/\mu, \\ \mu - c\mu^2/(\sigma^2 + \mu^2), & \text{if} \quad p \geq (\sigma^2 + \mu^2)/\mu. \end{cases}$$

Note that $G(p)^+$ is decreasing and $G^-(p)$ is increasing in p. Therefore, there exists some $\tilde{p}$ such that $G(p)^+ = G^-(p)$, and $\tilde{p}$ minimizes $\rho(p) = \max\{G(p)^+, G^-(p)\}$. Now we will verify the optimal price $\tilde{p}$ is as described. It is easy to see the parameter space can de divided into the following four cases. We will verify them one by one.

(a) First we consider $(\sigma^2+\mu^2)/(2\mu) \le \mu$. Then $G^+((\sigma^2+\mu^2)/(2\mu)) = \mu/2$ and $G^-((\sigma^2+\mu^2)/(2\mu)) = (2\sigma^2\mu)/(\sigma^2+\mu^2) - (4\sigma^2\mu^2 c)/(\sigma^2+\mu^2)^2$. It is easy to verify $G^+((\sigma^2+\mu^2)/(2\mu)) \le G^-((\sigma^2+\mu^2)/(2\mu))$ if and only if $c \le (\sigma^2+\mu^2)/(2\mu) - (\sigma^2+\mu^2)^2/(8\mu\sigma^2)$. Next consider $(\sigma^2+\mu^2)/(2\mu) \ge \mu$. Then $G^+(\mu) = (\sigma^2\mu)/(\sigma^2+\mu^2)$ and $G^-(\mu) = \mu - c$. It is easy to verify $G^+(\mu) \le G^-(\mu)$ if and only if $c \le \mu^3/(\sigma^2+\mu^2)$. Noting that $G^+(p)$ is decreasing and $G^-(p)$ is increasing, we thus get $\tilde{p} \le \min\{\mu, (\sigma^2+\mu^2)/(2\mu)\}$ and $\tilde{p}$ is given by Equation (S.10).

(b) If $(\sigma^2+\mu^2)/(2\mu) \ge \mu$, we know $G^+(\mu) \ge G^-(\mu)$ when $c \ge \mu^3/(\sigma^2+\mu^2)$ from the previous case (a). Note that $G^+((\sigma^2+\mu^2)/(2\mu)) = \mu/2$ and $G^-((\sigma^2+\mu^2)/(2\mu)) = \mu - (2\mu^2 c)/(\sigma^2+\mu^2)$. It is easy to verify $G^+((\sigma^2+\mu^2)/(2\mu)) \le G^-((\sigma^2+\mu^2)/(2\mu))$ if and only if $c \le (\sigma^2+\mu^2)/(4\mu)$. Recall $c \le \mu$ by Assumption (P), then $\mu \le \tilde{p} \le (\sigma^2+\mu^2)/(2\mu)$ and $\tilde{p}$ is given by Equation (S.11).

(c) If $(\sigma^2+\mu^2)/(2\mu) \le \mu$, then $G^+((\sigma^2+\mu^2)/(2\mu)) \ge G^-((\sigma^2+\mu^2)/(2\mu))$ when $c \ge (\sigma^2+\mu^2)/(2\mu) - (\sigma^2+\mu^2)^2/(8\mu\sigma^2)$ by the analysis in case (a). Moreover, $G^+(\mu) = \sigma/2 \le G^-(\mu) = \mu - c$ when $c \le \mu - \sigma/2$. Note that $G^+(p)$ is decreasing and $G^-(p)$ is increasing. Thus we obtain $(\sigma^2+\mu^2)/(2\mu) \le \tilde{p} \le \mu$ and $\tilde{p}$ is given by Equation (S.12).

(d) If $(\sigma^2+\mu^2)/(2\mu) \le \mu$, then $G^+(\mu) = \sigma/2 \ge G^-(\mu) = \mu - c$ when $c \ge \mu - \sigma/2$ by the previous case (c). If $(\sigma^2+\mu^2)/(2\mu) \ge \mu$, then $G^+((\sigma^2+\mu^2)/(2\mu)) \ge G^-((\sigma^2+\mu^2)/(2\mu))$ when $c \ge (\sigma^2+\mu^2)/(4\mu)$ by the analysis in case (b). We can verify $G^+(c) \le \mu - c\mu^2/(\sigma^2+\mu^2)$ and note that $G^+(p)$ is decreasing. Thus $G^+((\sigma^2+\mu^2)/\mu) < G^+(c) \le \mu - c\mu^2/(\sigma^2+\mu^2) = G^-((\sigma^2+\mu^2)/\mu)$. We thus get $\max\{(\sigma^2+\mu^2)/(4\mu), \mu\} \le \tilde{p} < (\sigma^2+\mu^2)/\mu$ and $\tilde{p}$ is given by Equation (S.13).

Lastly, we show $p^* < \bar{p} < \tilde{p}$ to complete this proof. $p^* < \bar{p}$ is equivalent to $k^* > \bar{k}$. Recalling that $k^{*3} + 3k^* = 2\tau$ and $\bar{k}^3 + 2\bar{k} = \tau$, we have $k^{*3} + 3k^* = 2\tau = 2\bar{k}^3 + 4\bar{k} > \bar{k}^3 + 3\bar{k}$, i.e., $k^* > \bar{k}$. We will prove $\tilde{p} > \bar{p}$. Recall $\bar{p} = \mu - \bar{k}\sigma < \mu$, and then it is true for cases (b) and (d). Note that $\bar{k}^3 + 2\bar{k} = \tau$, which is equivalent to $\bar{k} = (\tau - \bar{k})/(\bar{k}^2 + 1)$. Let $\tilde{p} = \mu - k'\sigma$, and then it is equivalent to verify $k' < \bar{k}$. We first consider case (a). Since $\tau = (\mu - c)/\sigma$, Equation (S.10) can be simplified into: $(k' + \sigma/\mu)/(1 + \sigma^2/\mu^2) = (\tau - k')/(k'^2 + 1)$, where the left hand side is increasing in k' and the right hand side is decreasing in k'. We can verify $(k' + \sigma/\mu)/(1 + \sigma^2/\mu^2) > k'$. Suppose $k' \ge \bar{k}$, then $(\tau - k')/(k'^2 + 1) \le (\tau - \bar{k})/(\bar{k}^2 + 1)$. On the other hand, $(k' + \sigma/\mu)/(1 + \sigma^2/\mu^2) > k' \ge \bar{k}$, which contradicts with $(k' + \sigma/\mu)/(1 + \sigma^2/\mu^2) = (\tau - k')/(k'^2 + 1) \le (\tau - \bar{k})/(\bar{k}^2 + 1) = \bar{k}$. We next consider case (c). Equation (S.12) can be simplified into: $(\sqrt{k'^2 + 1} + k')/2 = (\tau - k')/(k'^2 + 1)$. It is easy to see $(\sqrt{k'^2 + 1} + k')/2 \ge k'$, and then we can prove $k' < \bar{k}$ by contradiction with a similar argument as in case (a). $\square$

Proof of Proposition 9. We define r and r_b as follows:

$$r=\max_{\bar{p}_i,\forall i\in[n]}\left\{\min_{F_i\in\mathscr{F}_i,\forall i\in[n]}\frac{\sum_{i=1}^{n}(\bar{p}_i-c_i)\mathsf{P}(V_i\geq\bar{p}_i)}{\sum_{i=1}^{n}\max_{p_i}(p_i-c_i)\mathsf{P}(V_i\geq p_i)}\right\},$$

$$r_b=\max_{\bar{p}_b}\left\{\min_{F_i\in\mathscr{F}_i,\forall i\in[n]}\frac{(\bar{p}_b-\sum_{i=1}^{n}c_i)\mathsf{P}(\sum_{i=1}^{n}V_i\geq\bar{p}_b)}{\max_{p_b}(p_b-\sum_{i=1}^{n}c_i)\mathsf{P}(\sum_{i=1}^{n}V_i\geq p_b)}\right\}.$$

For separate sales, $\mathscr{M}=\mathcal{P}([n])$ and $\bar{p}_S=\sum_{i\in S}\bar{p}_i$ for any $S\in\mathscr{M}$ and the seller decides on each individual product's price $\bar{p}_i$, $i\in[n]$. For the pure bundle, $\mathscr{M}=\{[n]\}$ and the seller decides on the bundle price $\bar{p}_b$. Thus we can see that $1-r$ and $1-r_b$ are the optimal objective values for (4) under separate sales and pure bundling, respectively. Hence, the pure bundle is guaranteed to perform better than separate sales if $1-r\geq 1-r_b$, i.e., $r\leq r_b$. Similar to Proposition 5, the asymptotic optimality of the pure bundle is that, for any $\epsilon\in(0,1]$, there exists a threshold n^* such that $1-r_b<\epsilon$, i.e., $r_b>1-\epsilon$ for $n>n^*$. Furthermore, we define

$$r_i=\max_{\bar{p}_i}\left\{\min_{F_i\in\mathscr{F}_i}\frac{(\bar{p}_i-c_i)\mathsf{P}(V_i\geq\bar{p}_i)}{\max_{p_i}(p_i-c_i)\mathsf{P}(V_i\geq p_i)}\right\}.$$

We first show $r\leq\max_i\{r_i\}$. By the definitions of r_i and r, we have

$$\begin{aligned}r&=\max_{\bar{p}_i,\forall i\in[n]}\left\{\min_{F_i\in\mathscr{F}_i,\forall i\in[n]}\frac{\sum_{i=1}^{n}(\bar{p}_i-c_i)\mathsf{P}(V_i\geq\bar{p}_i)}{\sum_{i=1}^{n}\max_{p_i}(p_i-c_i)\mathsf{P}(V_i\geq p_i)}\right\}\\&\leq\max_{\bar{p}_i,\forall i\in[n]}\left\{\min_{F_i\in\mathscr{F}_i,\forall i\in[n]}\max_i\frac{(\bar{p}_i-c_i)\mathsf{P}(V_i\geq\bar{p}_i)}{\max_{p_i}(p_i-c_i)\mathsf{P}(V_i\geq p_i)}\right\}\\&=\max_i\max_{\bar{p}_i}\left\{\min_{F_i\in\mathscr{F}_i}\frac{(\bar{p}_i-c_i)\mathsf{P}(V_i\geq\bar{p}_i)}{\max_{p_i}(p_i-c_i)\mathsf{P}(V_i\geq p_i)}\right\}=\max_i\{r_i\},\end{aligned}$$

where the inequality is due to the fact that $\frac{\sum_{i=1}^{n}a_i}{\sum_{i=1}^{n}b_i}\leq\max_i\frac{a_i}{b_i}$ for $a_i\geq0$ and $b_i\geq0$, the second equality is due to that inverting the order of maximization or minimization does not change the optimal value.

Second, we will show that $r_b\geq\max_i\{r_i\}$ under any condition in Propositions 4, 6, 7, Corollaries 2, 3. We consider the independent case, and denote $\tau_i=\frac{\mu_i-c_i}{\sigma_i}=\frac{(1-\gamma)\mu_i}{\sigma_i}=\frac{1-\gamma}{\delta_i}$ for all i, and $\tau_{\bar{n}}=\frac{(1-\gamma)\sum_i\mu_i}{\sqrt{\sum_i\sigma_i^2}}$. Moreover, let $\bar{k}_i$ and $\bar{k}_{\bar{n}}$ be the unique real root to

$$\bar{k}_i^3+2\bar{k}_i=\tau_i=\frac{1}{\sigma_i}\mu_i(1-\gamma),\quad \bar{k}_{\bar{n}}^3+2\bar{k}_{\bar{n}}=\tau_{\bar{n}}=\frac{\sum_i\mu_i}{\sqrt{\sum_i\sigma_i^2}}(1-\gamma),$$

respectively. By Theorem 3, we have $1-r_i=\frac{1}{1+\bar{k}_i^2}$ and $1-r_b=\frac{1}{1+\bar{k}_{\bar{n}}^2}$. Then $r_b\geq\max_i\{r_i\}$ is equivalent to $\bar{k}_{\bar{n}}\geq\max_i\bar{k}_i$. Since k^3+2k is increasing in k, then $\bar{k}_{\bar{n}}\geq\max_i\bar{k}_i$ implies $\tau_{\bar{n}}\geq\max_i\tau_i$, i.e., $\frac{\sqrt{\sum_i\sigma_i^2}}{\sum_i\mu_i}\leq\min_i\delta_i$, which is indeed the condition in Proposition 4. Moreover, we have shown that Corollaries 2 and 3 are special cases of Proposition 4. Hence, we have verified $r_b\geq\max_i\{r_i\}$ under any condition in Proposition 4, Corollaries 2, 3. Following the same logic in proofs of Propositions 6 and 7, we can verify $r_b\geq\max_i\{r_i\}$ under their conditions.

Lastly, we show the asymptotic optimality. Consider n products have the same mean μ, standard deviation σ, and marginal cost c. Then,

$$\bar{k}_{\bar{n}}^3+2\bar{k}_{\bar{n}}=\tau_{\bar{n}}=\frac{\sum_i\mu_i}{\sqrt{\sum_i\sigma_i^2}}(1-\gamma)=\sqrt{n}\frac{1-\gamma}{\delta}.$$

Let n^* be the unique root of $\frac{\bar{k}_{\bar{n}}^2}{1+\bar{k}_{\bar{n}}^2}=1-\epsilon$. It is easy to see n^* is decreasing in ϵ. Therefore, if $n>n^*$, we have $r_b=1-\frac{1}{1+\bar{k}_{\bar{n}}^2}=\frac{\bar{k}_{\bar{n}}^2}{1+\bar{k}_{\bar{n}}^2}>1-\epsilon$. $\square$

Proof of Proposition OA.1. We will show the price heuristic p_s^* achieves the following performance guarantee:

$$\max\left\{\rho_s\equiv\frac{2(1-\frac{k_s^*}{\tau})^2}{2-\frac{k_s^*}{\tau}}/(1+\frac{\gamma}{2(1-\gamma)\beta}),\frac{1}{2+\frac{\gamma}{(1-\gamma)\beta}}\right\}(\geq\rho),$$

where $\beta\equiv\tau^2 1_{\{\tau>1\}}+1_{\{\tau\leq1\}}\geq1$ and $1_{\{.\}}$ is the indicator function.

By the two-sided Chebyshev's inequality, $\mathsf{P}(|V-\mu|\geq k\sigma)\leq\frac{1}{k^2}$. If V has a symmetric distribution, we have

$$\mathsf{P}(V-\mu\leq-k\sigma)\leq\frac{1}{2k^2}. \tag{S.17}$$

The bound in (S.17) is not very meaningful if $k\leq1$. We can establish a distribution-free upper bound by following the same approach as the proof of Theorem 1(a) (see (OA.1) and (OA.2)) to provide an upper bound on $cF(c)$. Since $cF(c)=c\mathsf{P}(V\leq c)=c\mathsf{P}(\frac{V-\mu}{\sigma}\leq-\frac{\mu-c}{\sigma})\leq\frac{c}{2(\frac{\mu-c}{\sigma})^2}=\frac{c}{2\tau^2}$, where the inequality is due to (S.17). Moreover, since the valuation distribution is symmetric, $F(c)\leq\frac{1}{2}$. Thus $cF(c)\leq\frac{1}{2\beta}c$, where $\beta\equiv\tau^2 1_{\{\tau>1\}}+1_{\{\tau\leq1\}}$. Then we can obtain an upper bound: for all $F\in\mathscr{F}_s$, $\max_p\pi(p;F)\leq\mu-c+\frac{c}{2\beta}\equiv U_s$ in view of (OA.2).

Now we establish a lower bound. Write $p=\mu-k\sigma$, then

$$\pi(p;F)=(p-c)\bar{F}(p^-)=(\mu-k\sigma-c)\bar{F}((\mu-k\sigma)^-)\geq(\mu-c)(1-\frac{k}{\tau})(1-\frac{1}{2k^2}),$$

where the inequality is due to (S.17). Denote $\varphi_s(k)=(1-\frac{k}{\tau})(1-\frac{1}{2k^2})$. Then $\varphi_s'(k)=-\frac{1}{\tau}-\frac{1}{2k^2\tau}+\frac{1}{k^3}=\frac{-2k^3-k+2\tau}{2k^3\tau}$. The roots of $\varphi_s'(k)=0$ are characterized by:

$$2k^3+k=2\tau. \tag{S.18}$$

By Cardano's solution for a cubic function, the unique real root of (S.18), denoted by k_s^*, is in the form as in Proposition OA.1. If $k \le k_s^*$, $\varphi_s'(k) \ge 0$ and if $k \ge k_s^*$, $\varphi_s'(k) \le 0$, thus k_s^* is the unique maximizer of $\varphi_s(k)$. By (S.18),

$$\varphi_s(k_s^*) = \frac{(2(k_s^*)^2 - 1)^2}{2(k_s^*)^2(2(k_s^*)^2+1)} = \frac{2(1-\frac{k_s^*}{\tau})^2}{2-\frac{k_s^*}{\tau}}.$$

Thus a distribution-free lower bound is $L_s = (\mu - c)\varphi_s(k_s^*)$. Therefore, we can obtain a distribution-free performance guarantee: $\rho_s = \frac{L_s}{U_s} = (\frac{2(1-\frac{k_s^*}{\tau})^2}{2-\frac{k_s^*}{\tau}})/(1+\frac{\gamma}{2(1-\gamma)\beta})$.

We may improve the performance bound ρ_s with another price. Because inequality (S.17) is loose when $k \le 1$, we can improve the lower bound L_s and hence the performance guarantee. Note that when we set $p = \mu$, the profit is $\frac{1}{2}(\mu - c)$. This is because due to the valuation distribution symmetry, with one half of chances, customers will make a purchase. Hence, if $\varphi_s(k_s^*) \le \frac{1}{2}$, the lower bound can be improved to $\frac{1}{2}(\mu - c)$, otherwise, we use the lower bound L_s. Since $\varphi_s(k)$ is increasing in k, then we have that when $k_s^* \le \sqrt{\frac{5+\sqrt{17}}{4}} \approx 1.5102$ (where the latter constant is solved from $\varphi_s(k) = \frac{1}{2}$), $\varphi_s(k_s^*) \le \frac{1}{2}$. Combining the two heuristics $p = \mu - k_s^*\sigma$ and $p = \mu$, we can obtain the following improved performance guarantee:

$$\rho_s' = \begin{cases} \rho_s & \text{if } \tau \ge \frac{2(\sqrt{\frac{5+\sqrt{17}}{4}})^3 + \sqrt{\frac{5+\sqrt{17}}{4}}}{2} \approx 4.1996, \\ \frac{1}{2+\frac{\gamma}{(1-\gamma)\beta}} & \text{otherwise.} \end{cases}$$

Lastly, we verify that with more information that the valuation distribution is symmetric, our heuristic is guaranteed to perform better than without this additional information (as a by-product, we show $p^* \le p_s^*$). It is sufficient to show that $\rho_s > \rho$, in particular, $U_s \le U$ and $L_s > L$. Compare upper bounds. If $\tau > 1$, $U_s = \mu - c + \frac{c}{2\tau^2} < \mu - \frac{\tau^2}{\tau^2+1}c = U$; if $\tau \le 1$, $U_s = \mu - \frac{1}{2}c \le \mu - \frac{\tau^2}{\tau^2+1}c = U$. Thus $U_s \le U$. Compare price heuristics. If $\tau \ge \frac{2(\sqrt{\frac{5+\sqrt{17}}{4}})^3 + \sqrt{\frac{5+\sqrt{17}}{4}}}{2}$, then $k_s^* \ge \sqrt{\frac{5+\sqrt{17}}{4}}$, hence, $(k_s^*)^2 > 2$. Since $(2k^3 + k) - (k^3 + 3k) = k^3 - 2k > 0$ holds if $k^2 > 2$, thus by (OA.4) and (S.18), $2\tau = (k^*)^3 + 3(k^*) = 2(k_s^*)^3 + (k_s^*) > (k_s^*)^3 + 3(k_s^*)$. Since $k^3 + 3k$ is increasing in k, we have $k^* > k_s^*$. As a result, $p^* \le p_s^*$. Recall the definition of L_s and L, $L_s = (\mu - c)\frac{(2(k_s^*)^2-1)^2}{2(k_s^*)^2(2(k_s^*)^2+1)} = (\mu - c)\frac{(2(k_s^*)^2-1)^2}{4k_s^*\tau} = (\mu - c)(\frac{(k_s^*)^3}{\tau} - \frac{k_s^*}{\tau} + \frac{1}{4\tau k_s^*}) = (\mu - c)(\frac{\tau - \frac{1}{2}k_s^*}{\tau} - \frac{k_s^*}{\tau} + \frac{1}{4\tau k_s^*}) = (\mu - c)(1 - \frac{3k_s^*}{2\tau} + \frac{1}{4\tau k_s^*})$, and $L = (\mu - c)(1 - \frac{3k^*}{2\tau}) < (\mu - c)(1 - \frac{3k_s^*}{2\tau}) < L_s$, where the first inequality is due to $k^* > k_s^*$. □

Proof of Proposition OA.2. By Proposition OA.1, $p^* \le p_s^*$, then $\bar{F}(p^{*-}) \ge \bar{F}(p_s^{*-})$, and we have

$$\frac{\pi(p^*; F)}{\pi(p_s^*; F)} = \frac{(p^* - c)\bar{F}(p^{*-})}{(p_s^* - c)\bar{F}(p_s^{*-})} = \frac{(\mu - c)(1 - \frac{k^*(\tau)}{\tau})\bar{F}(p^{*-})}{(\mu - c)(1 - \frac{k_s^*(\tau)}{\tau})\bar{F}(p_s^{*-})} \ge \frac{1 - \frac{k^*(\tau)}{\tau}}{1 - \frac{k_s^*(\tau)}{\tau}}.$$

By (OA.4) and (S.18), then $1 - \frac{k^*(\tau)}{\tau} = 1 - \frac{2}{(k^*(\tau))^2+3}$ and $1 - \frac{k_s^*(\tau)}{\tau} = 1 - \frac{2}{2(k_s^*(\tau))^2+1}$, which are increasing in $k^*(\tau)$ and $k_s^*(\tau)$, respectively. Because $k^*(\tau)$ and $k_s^*(\tau)$ are increasing in τ, thus

$1-\frac{k^*(\tau)}{\tau}$ and $1-\frac{k_s^*(\tau)}{\tau}$ are increasing in τ. When $\tau=130$, we compute $1-\frac{k^*(\tau)}{\tau}=0.9521$. Then if $\tau\geq 130$,

$$\frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{k_s^*(\tau)}{\tau}}\geq 1-\frac{k^*(\tau)}{\tau}\geq 1-\frac{k^*(130)}{130}>95\%,$$

where the first inequality is due to $1-\frac{k_s^*(\tau)}{\tau}\leq 1$ and the second inequality is due to that $1-\frac{k^*(\tau)}{\tau}$ is increasing in τ.

Now divide $[4.2,130]$ into 19 segments as follows: $[4.2,4.7]$, $[4.7,5.2]$, $[5.2,5.7]$, $[5.7,6.2]$, $[6.2,6.7]$, $[6.7,7.2]$, $[7.2,7.7]$, $[7.7,8.3]$, $[8.3,9.1]$, $[9.1,10.0]$, $[10.0,11.1]$, $[11.1,12.5]$, $[12.5,14.3]$, $[14.3,16.8]$, $[16.8,20.6]$, $[20.6,27.0]$, $[27.0,39.9]$, $[39.9,76.9]$, $[76.9,130]$. We evaluate $1-\frac{k^*(\tau)}{\tau}$ at every endpoint of the segments. The values are 0.6303, 0.6497, 0.6668, 0.6819, 0.6954, 0.7076, 0.7186, 0.7286, 0.7395, 0.7524, 0.7651, 0.7787, 0.7933, 0.8089, 0.8263, 0.8464, 0.8698, 0.8979, 0.9328, 0.9521, respectively. Similarly, we evaluate $1-\frac{k_s^*(\tau)}{\tau}$ at those endpoints. The values are 0.6404, 0.6647, 0.6853, 0.7030, 0.7183, 0.7318, 0.7438, 0.7545, 0.7660, 0.7793, 0.7923, 0.8058, 0.8201, 0.8351, 0.8514, 0.8699, 0.8909, 0.9156, 0.9452, 0.9613, respectively. We want to verify that $\frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{k_s^*(\tau)}{\tau}}\geq 95\%$ for each segment. The idea is: for a segment $[a,b]$, if $\tau\in[a,b]$, since $1-\frac{k^*(\tau)}{\tau}$ and $1-\frac{k_s^*(\tau)}{\tau}$ are increasing in τ, $\frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{k_s^*(\tau)}{\tau}}\geq \frac{1-\frac{k^*(a)}{a}}{1-\frac{k_s^*(b)}{b}}$. With the values of endpoints computed, we can easily verify for each segment $[a,b]$, it is indeed that $\frac{1-\frac{k^*(a)}{a}}{1-\frac{k_s^*(b)}{b}}\geq 95\%$. $\square$

Proof of Proposition OA.3. With the additional information of unimodality, we can resort to the Vysochanskij-Petunin inequality

$$\mathsf{P}(|X-\mu|\geq k\sigma)\leq\begin{cases}\dfrac{4}{9k^2} & \text{if } k\geq\sqrt{\dfrac{8}{3}}\approx 1.633,\\ \dfrac{4}{3k^2}-\dfrac{1}{3} & \text{otherwise,}\end{cases}$$

to derive an improved performance guarantee. In particular, we show that the price heuristic

$$p_u^*=\begin{cases}\mu-k_{u_1}^*\sigma & \text{if } \tau\geq\dfrac{7\sqrt{6}}{3}\approx 5.72,\\ \mu-\sqrt{\frac{8}{3}}\sigma & \text{if } \dfrac{11}{6}\sqrt{\dfrac{8}{3}}\leq\tau\leq\dfrac{7\sqrt{6}}{3},\\ \mu-k_{u_2}^*\sigma & \text{if } 2.65\leq\tau\leq\dfrac{11}{6}\sqrt{\dfrac{8}{3}}\approx 2.99,\\ \mu-k^*\sigma=p^* & \text{if } \tau\leq 2.65,\end{cases}$$

where

$$\begin{aligned}k_{u_1}^* &= \sqrt[3]{\tfrac{4}{9}\tau+\sqrt{(\tfrac{4}{9})^2\tau^2+(\tfrac{4}{27})^3}}+\sqrt[3]{\tfrac{4}{9}\tau-\sqrt{(\tfrac{4}{9})^2\tau^2+(\tfrac{4}{27})^3}},\\ k_{u_2}^* &= \sqrt[3]{\tau+\sqrt{\tau^2+(\tfrac{1}{3})^3}}+\sqrt[3]{\tau-\sqrt{\tau^2+(\tfrac{1}{3})^3}},\end{aligned}\qquad\text{(S.19)}$$

achieves a sharper performance guarantee as follows:

$$(\rho \leq)\rho_u = \frac{L_u}{U_u} = \begin{cases} \frac{(1-\frac{k^*_{u1}}{\tau})(1-\frac{4}{9k^{*2}_{u1}})}{1+\frac{4}{9\tau^2}\frac{\gamma}{1-\gamma}} & \text{if } \tau \geq \frac{7\sqrt{6}}{3}, \\ \frac{(1-\sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6}}{1+\frac{4}{9\tau^2}\frac{\gamma}{1-\gamma}} & \text{if } \frac{11}{6}\sqrt{\frac{8}{3}} \leq \tau \leq \frac{7\sqrt{6}}{3}, \\ \frac{(1-\frac{k^*_{u2}}{\tau})(\frac{4}{3}-\frac{4}{3k^{*2}_{u2}})}{1+\frac{4}{9\tau^2}\frac{\gamma}{1-\gamma}} & \text{if } 2.65 \leq \tau \leq \frac{11}{6}\sqrt{\frac{8}{3}}, \\ \frac{L}{\mu-c+\frac{4}{9\tau^2}c} & \text{if } \sqrt{\frac{8}{3}} \leq \tau \leq 2.65, \\ \frac{L}{\mu-c+(\frac{4}{3\tau^2}-\frac{1}{3})c} & \text{if } \sqrt{2} \leq \tau \leq \sqrt{\frac{8}{3}}, \\ \rho & \text{if } \tau \leq \sqrt{2}, \end{cases}$$

where L is specified in Theorem 1(b) and ρ is specified in Theorem 2.

First consider the upper bound. If $\tau \geq \sqrt{\frac{8}{3}}$, $cF(c) \leq \frac{4}{9\tau^2}c$, if $\sqrt{2} \leq \tau \leq \sqrt{\frac{8}{3}}$, $cF(c) \leq c(\frac{4}{3\tau^2} - \frac{1}{3})$ by the Vysochanskij-Petunin inequality, otherwise, $cF(c) \leq c\frac{1}{\tau^2+1}$ by the one-sided Chebyshev's inequality. This is because when $\tau \leq \sqrt{2}$, $c\frac{1}{\tau^2+1} \leq c(\frac{4}{3\tau^2} - \frac{1}{3})$, one-sided Chebyshev's inequality gives a smaller upper bound. In view of (OA.1) and (OA.2), we can obtain a sharper upper bound:

$$U_u = \begin{cases} \mu - c + \frac{4}{9\tau^2}c & \text{if } \tau \geq \sqrt{\frac{8}{3}}, \\ \mu - c + (\frac{4}{3\tau^2} - \frac{1}{3})c & \text{if } \sqrt{2} \leq \tau \leq \sqrt{\frac{8}{3}}, \\ U & \text{otherwise.} \end{cases}$$

Now consider the lower bound. Denote $\varphi_{u_1}(k) = (1 - \frac{k}{\tau})(1 - \frac{4}{9k^2})$ and $\varphi_{u_2}(k) = (1 - \frac{k}{\tau})(\frac{4}{3} - \frac{4}{3k^2})$, then $\varphi'_{u_1}(k) = -\frac{1}{\tau} - \frac{4}{9}\frac{1}{k^2\tau} + \frac{8}{9}\frac{1}{k^3} = \frac{-9k^3-4k+8\tau}{9k^3\tau}$, $\varphi'_{u_2}(k) = \frac{4}{3}\frac{-k^3-k+2\tau}{\tau k^3}$. The roots of $\varphi'_{u_1}(k) = 0$ and $\varphi'_{u_2}(k) = 0$ are characterized by

$$k^3 + \frac{4}{9}k = \frac{8}{9}\tau, \tag{S.20}$$

and

$$k^3 + k = 2\tau, \tag{S.21}$$

respectively. By Cardano's solution for a cubic function, the unique real root to (S.20) is in the form of S.19. If $k \leq k^*_{u_1}$, $\varphi'_{u_1}(k) \geq 0$ and if $k \geq k^*_{u_1}$, $\varphi'_{u_1}(k) \leq 0$, thus $k^*_{u_1}$ is the unique maximizer of $\varphi_{u_1}(k)$. Similarity, if $k \leq k^*_{u_2}$, $\varphi'_{u_2}(k) \geq 0$ and if $k \geq k^*_{u_2}$, $\varphi'_{u_2}(k) \leq 0$, thus $k^*_{u_2}$ is the unique maximizer of $\varphi_{u_2}(k)$.

If $\tau = \frac{7\sqrt{6}}{3}$, $k^*_{u_1}(\tau) = \sqrt{\frac{8}{3}}$; $\tau = \frac{11}{6}\sqrt{\frac{8}{3}}$, $k^*_{u_2}(\tau) = \sqrt{\frac{8}{3}}$; $\tau = \frac{3}{2}\sqrt{2}$, $k^*_{u_2}(\tau) = \sqrt{2}$. Because $k^3 + \frac{4}{9}k$ and $k^3 + k$ is increasing in k, if $\tau \geq \frac{7\sqrt{6}}{3}$, then $k^*_{u_1}(\tau) \geq \sqrt{\frac{8}{3}}$, in which case we can apply the Vysochanskij-Petunin inequality for $k \geq \sqrt{\frac{8}{3}}$. If $\frac{11}{6}\sqrt{\frac{8}{3}} \leq \tau \leq \frac{7\sqrt{6}}{3}$, $k^*_{u_1}(\tau) \leq \sqrt{\frac{8}{3}}$ and $k^*_{u_2}(\tau) \geq \sqrt{\frac{8}{3}}$, no matter using

Vysochanskij-Petunin inequality for $k \geq \sqrt{\frac{8}{3}}$ or $k \leq \sqrt{\frac{8}{3}}$, the optimal choice is letting $k^*_{u_1}(\tau) = \sqrt{\frac{8}{3}}$ or $k^*_{u_2}(\tau) = \sqrt{\frac{8}{3}}$. If $\tau \leq \frac{11}{6}\sqrt{\frac{8}{3}}$, $k^*_{u_2}(\tau) \leq \sqrt{\frac{8}{3}}$, in which case we can apply the Vysochanskij-Petunin inequality for $k \leq \sqrt{\frac{8}{3}}$. But if $\tau \leq \frac{3}{2}\sqrt{2} \approx 2.12$, $k^*_{u_2}(\tau) \leq \sqrt{2}$, $(1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3}-\frac{4}{3(k^*_{u_2})^2}) \leq (1-\frac{k^*_{u_2}(\tau)}{\tau})\frac{(k^*_{u_2})^2}{(k^*_{u_2})^2+1} \leq (1-\frac{k^*(\tau)}{\tau})\frac{(k^*)^2}{(k^*)^2+1}$, where the first inequality is due to $\frac{4}{3}-\frac{4}{3(k^*_{u_2})^2} \leq \frac{(k^*_{u_2})^2}{(k^*_{u_2})^2+1}$, and the second inequality is due to k^* is the maximum point of $\varphi(k)$, in which case we can resort to the one-sided Chebyshev's inequality and obtain a better lower bound L (see Theorem 1(b)). Now consider $2.12 \leq \tau \leq \frac{11}{6}\sqrt{\frac{8}{3}}$. Since $k^*(2.65) = 1.20$, $k^*_{u_2}(2.65) = 1.55$, then we can verify, $\varphi(k^*(2.65)) = (1-\frac{k^*(2.65)}{2.65})(\frac{(k^*(2.65))^2}{(k^*(2.65))^2+1}) = (1-\frac{k^*_{u_2}(2.65)}{2.65})(\frac{4}{3}-\frac{4}{3(k^*_{u_2}(2.65))^2}) = \varphi_{u_2}(k^*_{u_2}(2.65))$, which shows when $\tau = 2.65$, the lower bound given by Vysochanskij-Petunin inequality and one-sided Chebyshev's inequality are the same. We will show if $2.12 \approx \frac{3}{2}\sqrt{2} \leq \tau \leq 2.65$, $\varphi(k^*(\tau)) \geq \varphi_{u_2}(k^*_{u_2}(\tau))$, if $2.65 \leq \tau \leq \frac{11}{6}\sqrt{\frac{8}{3}} \approx 2.99$, $\varphi(k^*(\tau)) \leq \varphi_{u_2}(k^*_{u_2}(\tau))$. Thus it is better to use one-sided Chebyshev's inequality if $\tau \leq 2.65$. By (OA.4) and (S.21), $\varphi(k^*(\tau)) = (1-\frac{k^*(\tau)}{\tau})(\frac{(k^*(\tau))^2}{(k^*(\tau))^2+1}) = \frac{(k^*(\tau))^2}{(k^*(\tau))^2+3}$, $\varphi_{u_2}(k^*_{u_2}(\tau)) = (1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3}-\frac{4}{3(k^*_{u_2}(\tau))^2}) = \frac{4}{3}\frac{((k^*_{u_2}(\tau))^2-1)^2}{(k^*_{u_2}(\tau))^2((k^*_{u_2}(\tau))^2+1)}$, because $k^*(\tau)$ and $k^*_{u_2}(\tau)$ are increasing in τ, then $\varphi(k^*(\tau))$ and $\varphi_{u_2}(k^*_{u_2}(\tau))$ are increasing in τ. Dividing $[2.12, 2.99]$ into 8 segments: $[2.12, 2.33]$, $[2.33, 2.49]$, $[2.29, 2.60]$, $[2.60, 2.65]$, $[2.65, 2.67]$, $[2.67, 2.77]$, $[2.77, 2.87]$, $[2.87, 2.99]$, by computing, we obtain the values of every endpoints for $\varphi(k^*(\tau))$ are $0.26, 0.29, 0.31, 0.32, 0.32, 0.32, 0.33, 0.34, 0.35$, and $0.22, 0.26, 0.29, 0.31, 0.32, 0.33, 0.34, 0.36, 0.38$ for $\varphi_{u_2}(k^*_{u_2}(\tau))$. We want to verify that $\frac{\varphi(k^*(\tau))}{\varphi_{u_2}(k^*_{u_2}(\tau))} \geq 1$ for the first 4 segments and $\frac{\varphi(k^*(\tau))}{\varphi_{u_2}(k^*_{u_2}(\tau))} \leq 1$ for the last 4 segments. The idea of proof is: for a segment $[a, b]$, if $\tau \in [a, b]$, since $\varphi(k^*(\tau))$ and $\varphi_{u_2}(k^*_{u_2}(\tau))$ are increasing in τ, thus $\frac{\varphi(k^*(b))}{\varphi_{u_2}(k^*_{u_2}(a))} \geq \frac{\varphi(k^*(\tau))}{\varphi_{u_2}(k^*_{u_2}(\tau))} \geq \frac{\varphi(k^*(a))}{\varphi_{u_2}(k^*_{u_2}(b))}$. With the values of endpoints, we can easily verify it.

From the above analysis, we obtain a lower bound with 4 segments:

$$L_u = \begin{cases} (\mu-c)\varphi_{u_1}(k^*_{u_1}) & \text{if } \tau \geq \frac{7\sqrt{6}}{3}, \\ (\mu-c)(1-\sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6} & \text{if } \frac{11}{6}\sqrt{\frac{8}{3}} \leq \tau \leq \frac{7\sqrt{6}}{3}, \\ (\mu-c)\varphi_{u_2}(k^*_{u_2}) & \text{if } 2.65 \leq \tau \leq \frac{11}{6}\sqrt{\frac{8}{3}}, \\ L & \text{if } \tau \leq 2.65. \end{cases}$$

Then we can obtain the desired distribution-free performance guarantee by taking the ratio of L_u and U_u.

Lastly, we show that with more information that the valuation distribution is unimodal, we can verify that our heuristic is guaranteed to perform better than without this additional information. It is sufficient to show that $\rho_u > \rho$, in particular, $U_u \leq U$ and $L_u > L$. Compare upper bounds. If $\tau \geq \sqrt{\frac{8}{3}}$, $U_u = \mu - c + \frac{4}{9\tau^2}c \leq \mu - \frac{\tau^2}{\tau^2+1}c = \mu - c + \frac{c}{\tau^2+1} = U$, which is due to $\frac{4}{9\tau^2} \leq \frac{1}{\tau^2+1}$; if $\sqrt{2} \leq \tau \leq \sqrt{\frac{8}{3}}$,

$U_u = \mu - c + (\frac{4}{3\tau^2} - \frac{1}{3})c \le \mu - \frac{\tau^2}{\tau^2+1}c = \mu - c + \frac{c}{\tau^2+1} = U$, which is due to $\frac{4}{3\tau^2} - \frac{1}{3} \le \frac{1}{\tau^2+1}$; otherwise, $U_u = U$. Thus $U_u \le U$. Compare price heuristics and their lower bounds. If $\tau \ge \frac{7\sqrt{6}}{3}$, $k^*_{u_1} \ge \sqrt{\frac{8}{3}}$, hence, $(k^*_{u_1})^2 \ge \frac{8}{5}$. Since $(\frac{9}{4}k^3 + k) - (k^3 + 3k) = \frac{5}{4}k^3 - 2k > 0$ holds if $k^2 > \frac{8}{5}$, thus by (OA.4) and (S.20), $2\tau = (k^*)^3 + 3k^* = \frac{9}{4}(k^*_{u_1})^3 + k^*_{u_1} > (k^*_{u_1})^3 + 3k^*_{u_1}$. Since $k^3 + 3k$ is increasing in k, we have $k^* > k^*_{u_1}$. As a result, $p^* \le p^*_u$. $L_u = (\mu - c)\varphi_{u_1}(k^*_{u_1}) = (\mu - c)\frac{(9(k^*_{u_1})^2-4)^2}{9(k^*_{u_1})^2(9(k^*_{u_1})^2+4)} = (\mu - c)\frac{(9(k^*_{u_1})^2-4)^2}{(9k^*_{u_1})(8\tau)} = (\mu - c)(\frac{9(k^*_{u_1})^3}{8\tau} - \frac{k^*_{u_1}}{\tau} + \frac{2}{9\tau k^*_{u_1}}) = (\mu - c)(\frac{8\tau - 4k^*_{u_1}}{8\tau} - \frac{k^*_{u_1}}{\tau} + \frac{2}{9\tau k^*_{u_1}}) = (\mu - c)(1 - \frac{3k^*_{u_1}}{2\tau} + \frac{2}{9\tau k^*_{u_1}})$. Then $L = (\mu - c)(1 - \frac{3k^*}{2\tau}) < (\mu - c)(1 - \frac{3k^*_2}{2\tau} + \frac{2}{9\tau}) = L_u$, which is due to $k^* > k^*_{u_1}$. If $\frac{11}{6}\sqrt{\frac{8}{3}} \le \tau \le \frac{7\sqrt{6}}{3}$, $L_u = (\mu - c)(1 - \sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6}$, $p^*_u = \mu - \sqrt{\frac{8}{3}}\sigma$. Since $k^* = \sqrt{\frac{8}{3}}$, $\tau = \frac{17\sqrt{2}}{3\sqrt{3}} \approx 4.63 > \frac{11}{6}\sqrt{\frac{8}{3}} \approx 2.99$, hence, when $\frac{11}{6}\sqrt{\frac{8}{3}} \le \tau \le \frac{17\sqrt{2}}{3\sqrt{3}}$, $k^* \le \sqrt{\frac{8}{3}}$, $p^* = \mu - k^*\sigma \ge p^*_u$; when $\frac{17\sqrt{2}}{3\sqrt{3}} \le \tau \le \frac{7\sqrt{6}}{3}$, $k^* \ge \sqrt{\frac{8}{3}}$, $p^* = \mu - k^*\sigma \le p^*_u$. Now we compare L_u and L. If $k^* \le \sqrt{\frac{8}{3}}$, $L = (\mu - c)(1 - \frac{k^*}{\tau})\frac{k^*}{(k^*)^2+1} < (\mu - c)\varphi_{u_2}(k^*) = (\mu - c)(1 - \frac{k^*}{\tau})(\frac{4}{3} - \frac{4}{3(k^*)^2}) \le (\mu - c)\varphi_{u_2}(\sqrt{\frac{8}{3}}) = L_u$, which the first inequality is due to $\frac{k^*}{(k^*)^2+1} \le \frac{4}{3} - \frac{4}{3(k^*)^2}$ when $k^* \ge \sqrt{2}$, and the last inequality is due to $k^*_{u_2} \ge \sqrt{\frac{8}{3}}$, thus $\varphi_{u_2}(k)$ is increasing when $k \le \sqrt{\frac{8}{3}}$. If $k^* \ge \sqrt{\frac{8}{3}}$, $L = (\mu - c)(1 - \frac{k^*}{\tau})\frac{k^*}{(k^*)^2+1} < (\mu - c)\varphi_{u_1}(k^*) = (\mu - c)(1 - \frac{k^*}{\tau})(1 - \frac{4}{9(k^*)^2}) \le \varphi_{u_1}(\sqrt{\frac{8}{3}}) = L_u$, which the first inequality is due to $\frac{k^*}{(k^*)^2+1} \le 1 - \frac{4}{9(k^*)^2}$ when $k^* \ge \sqrt{\frac{4}{5}}$, and the last inequality is due to $k^*_{u_1} \le \sqrt{\frac{8}{3}}$, thus $\varphi_{u_1}(k)$ is decreasing when $k \ge \sqrt{\frac{8}{3}}$. If $2.65 \le \tau \le \frac{11}{6}\sqrt{\frac{8}{3}}$, by (OA.4) and (S.21), we have $2\tau = (k^*_{u_2})^3 + k^*_{u_2} = (k^*)^3 + 3k^* > (k^*)^3 + k^*$, since $k^3 + k$ is increasing in k, then $k^*_{u_2} > k^*$. As a result, $p^* = \mu - k^*\sigma > p^*_u = \mu - k^*_{u_2}\sigma$. $L_u = \varphi_{u_2}(k^*_{u_2}(\tau)) \ge L = \varphi(k^*(\tau))$. If $\tau \le 2.65$, $L_u = L$ and $p^*_u = p^*$. By summary, $L_u > L$, if $\tau \ge \frac{17\sqrt{2}}{3\sqrt{3}}$, $p^*_u \ge p^*$, otherwise, $p^*_u \le p^*$. □

Proof of Proposition OA.4. If $\tau \ge \frac{17\sqrt{2}}{3\sqrt{3}} \approx 4.63$, first consider $\tau \ge \frac{7\sqrt{6}}{3} \approx 5.72$. By Proposition OA.3, $p^* \le p^*_u$, then $\bar{F}(p^{*-}) \ge \bar{F}(p_u^{*-})$, and we have

$$\frac{\pi(p^*; F)}{\pi(p^*_u; F)} = \frac{(p^* - c)\bar{F}(p^{*-})}{(p^*_u - c)\bar{F}(p_u^{*-})} = \frac{(\mu - c)(1 - \frac{k^*(\tau)}{\tau})\bar{F}(p^{*-})}{(\mu - c)(1 - \frac{k^*_{u_1}(\tau)}{\tau})\bar{F}(p_u^{*-})} \ge \frac{1 - \frac{k^*(\tau)}{\tau}}{1 - \frac{k^*_{u_1}(\tau)}{\tau}}.$$

By (OA.4) and (S.19), note that $k^*_{u_1}$ is the unique root of $k^3 + \frac{4}{9}k = \frac{8}{9}\tau$, then $1 - \frac{k^*(\tau)}{\tau} = 1 - \frac{2}{(k^*(\tau))^2+3}$ and $1 - \frac{k^*_{u_1}(\tau)}{\tau} = 1 - \frac{2}{2(k^*_{u_1}(\tau))^2+1}$, which are increasing in $k^*(\tau)$ and $k^*_{u_1}(\tau)$, respectively. Because $k^*(\tau)$ and $k^*_{u_1}(\tau)$ are increasing in τ, $1 - \frac{k^*(\tau)}{\tau}$ and $1 - \frac{k^*_{u_1}(\tau)}{\tau}$ are increasing in τ. When $\tau = 130$, we compute $1 - \frac{k^*(\tau)}{\tau} = 0.9521$. If $\tau \ge 130$, $\frac{1 - \frac{k^*(\tau)}{\tau}}{1 - \frac{k^*_{u_1}(\tau)}{\tau}} \ge 1 - \frac{k^*(\tau)}{\tau} \ge 95.21\% > 95\%$, where the first inequality is due to $1 - \frac{k^*_{u_1}(\tau)}{\tau} \le 1$ and the second inequality is due to that $1 - \frac{k^*(\tau)}{\tau}$ is increasing in $\tau(\ge 130)$.

Now divide $[5.72, 130]$ into 43 segments as follows: $[5.72, 5.82]$, $[5.82, 5.92]$, $[5.92, 6.02]$, $[6.02, 6.12]$, $[6.12, 6.22]$, $[6.22, 6.32]$, $[6.32, 6.42]$, $[6.42, 6.52]$, $[6.52, 6.62]$, $[6.62, 6.72]$, $[6.72, 6.82]$, $[6.82, 6.92]$, $[6.92, 7.02]$, $[7.02, 7.12]$, $[7.12, 7.22]$, $[7.22, 7.32]$, $[7.32, 7.42]$, $[7.42, 7.52]$, $[7.52, 7.62]$, $[7.62, 7.72]$, $[7.72, 7.82]$, $[7.82, 7.92]$, $[7.92, 8.12]$, $[8.12, 8.32]$, $[8.32, 8.52]$, $[8.52, 8.82]$, $[8.82, 9.12]$, $[9.12, 9.42]$, [9.42, 9.72], [9.72, 10.12], [10.12, 10.52], [10.52, 11.02], [11.02, 11.52], [11.52, 12.22], [12.22, 13.12], [13.12, 14.32], [14.32, 15.92], [15.92, 18.12], [18.12, 21.42], [21.42, 26.82], [26.82, 37.22], [37.22, 63.82], [63.82, 130]. We evaluate $1-\frac{k^*(\tau)}{\tau}$ at every endpoint of the segments. The values are: 0.6825, 0.6853, 0.6880, 0.6907, 0.6933, 0.6959, 0.6984, 0.7009, 0.7033, 0.7057, 0.7080, 0.7103, 0.7125, 0.7147, 0.7169, 0.7190, 0.7210, 0.7231, 0.7251, 0.7270, 0.7289, 0.7308, 0.7327, 0.7363, 0.7398, 0.7432, 0.7481, 0.7527, 0.7571, 0.7614, 0.7676, 0.7718, 0.7777, 0.7833, 0.7906, 0.7990, 0.8091, 0.8207, 0.8340, 0.8500, 0.8693, 0.8934, 0.9242, 0.9521, respectively. Similarly, we evaluate $1-\frac{k^*_{u_1}(\tau)}{\tau}$ at those endpoints. The values are: 0.7144, 0.7175, 0.7206, 0.7235, 0.7271, 0.7292, 0.7325, 0.7346, 0.7371, 0.7397, 0.7421, 0.7445, 0.7649, 0.7492, 0.7514, 0.7536, 0.7558, 0.7519, 0.7600, 0.7620, 0.7639, 0.7659, 0.7678, 0.7714, 0.7750, 0.7784, 0.7878, 0.7922, 0.7963, 0.8015, 0.8064, 0.8121, 0.8174, 0.8242, 0.8321, 0.8414, 0.8519, 0.8639, 0.8779, 0.8946, 0.9150, 0.9404, 0.9628, respectively. We want to verify that $\frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{k^*_{u_1}(\tau)}{\tau}} \geq 95\%$ for each segment. The idea is: for a segment $[a,b]$, if $\tau\in[a,b]$, since $1-\frac{k^*(\tau)}{\tau}$ and $1-\frac{k^*_{u_1}(\tau)}{\tau}$ are increasing in τ, thus $\frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{k^*_{u_1}(\tau)}{\tau}} \geq \frac{1-\frac{k^*(a)}{a}}{1-\frac{k^*_{u_1}(b)}{b}}$. With the values of endpoints computed, we can easily verify for each segment $[a,b]$, it is indeed that $\frac{1-\frac{k^*(a)}{a}}{1-\frac{k^*_{u_1}(b)}{b}} \geq 95\%$.

Now consider $4.63 \approx \frac{17\sqrt{2}}{3\sqrt{3}} \leq \tau \leq \frac{7\sqrt{6}}{3} \approx 5.72$. By Proposition OA.3,

$$\frac{\pi(p^*;F)}{\pi(p_u^*;F)} = \frac{(p^*-c)\bar{F}(p^{*-})}{(p_u^*-c)\bar{F}(p_u^{*-})} = \frac{(\mu-c)(1-\frac{k^*(\tau)}{\tau})\bar{F}(p^{*-})}{(\mu-c)(1-\frac{\sqrt{\frac{8}{3}}}{\tau})\bar{F}(p_u^{*-})} \geq \frac{1-\frac{k^*(\tau)}{\tau}}{1-\frac{\sqrt{\frac{8}{3}}}{\tau}},$$

where the inequality is due to that $p^* \leq p_u^*$ and $\bar{F}(p^{*-}) \geq \bar{F}(p_u^{*-})$.

Divide [4.63, 5.72] into 4 segments as follows: [4.63, 5.12], [5.12, 5.43], [5.43, 5.62], [5.62, 5.72]. We evaluate $1-\frac{k^*(\tau)}{\tau}$ at every endpoint of the segments. The values are: 0.6472, 0.6642, 0.6740, 0.6796, 0.6825, respectively. Similarly, we evaluate $1-\frac{\sqrt{\frac{8}{3}}}{\tau}$ at those endpoints. The values are: 0.6473, 0.6813, 0.6993, 0.7094, 0.7145, respectively. Using the same idea as before and noting that $1-\frac{\sqrt{\frac{8}{3}}}{\tau}$ is increasing in τ, it is sufficient to verify that for each of the above 4 segments $[a,b]$, $\frac{1-\frac{k^*(a)}{a}}{1-\frac{\sqrt{\frac{8}{3}}}{b}} \geq 95\%$, which is indeed the case.

If $2.65 \leq \tau \leq \frac{17\sqrt{2}}{3\sqrt{3}} \approx 4.63$, first consider $2.99 \leq \tau \leq 4.63$. By Proposition OA.3, $L_u(\tau) = (\mu - \sqrt{\frac{8}{3}}\sigma - c)\frac{5}{6} = (\mu-c)(1-\sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6}$, obviously $(1-\sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6}$ is increasing in τ, by Theorem 1(b), $L(\tau) = (\mu-c)(1-\frac{3}{2\tau}k^*(\tau))$, by (OA.4), $k^*(\tau)$ is increasing by increasing τ, then $\frac{k^*(\tau)}{\tau} = \frac{2}{k^{*2}(\tau)+3}$ is decreasing by increasing τ, thus $1-\frac{3}{2\tau}k^*(\tau)$ is increasing in τ, $\frac{L(\tau)}{L_u(\tau)} = \frac{(1-\frac{3}{2\tau}k^*(\tau))}{(1-\sqrt{\frac{8}{3}}\frac{1}{\tau})\frac{5}{6}}$.

Now divide [2.99,4.63] into 29 segments as follows: [2.99, 3.19], [3.19, 3.35], [3.35, 3.48], [3.48, 3.59], [3.59, 3.69], [3.69, 3.77], [3.77, 3.84], [3.84, 3.90], [3.90, 3.96], [3.96, 4.01], [4.01, 4.06], [4.06, 4.10], [4.10, 4.14], [4.14, 4.18], [4.18, 4.22], [4.22, 4.25], [4.25, 4.28], [4.28, 4.31], [4.31, 4.34], [4.34, 4.37], [4.37, 4.40], [4.40, 4.43], [4.43, 4.46], [4.46, 4.49], [4.49, 4.52], [4.52, 4.55], [4.55, 4.58], [4.58, 4.61], [4.61, 4.63]. We evaluate $1-\frac{3}{2\tau}k^*(\tau)$ at every endpoint of the segments. The values are: 0.3543, 0.3725, 0.3856, 0.3958, 0.4041, 0.4114, 0.4171, 0.4219, 0.4260, 0.4301, 0.4334, 0.4366, 0.4392, 0.4418, 0.4443, 0.4468, 0.4486, 0.4504, 0.4523, 0.4541, 0.4558, 0.4576, 0.4594, 0.4611, 0.4629, 0.4646, 0.4663, 0.4680, 0.4697, respectively. Similarly, we evaluate $(1-\sqrt{\frac{8}{3}\frac{1}{\tau}})\frac{5}{6}$ at those endpoints. The values are: 0.4067, 0.4271, 0.4423, 0.4543, 0.4645, 0.4724, 0.4790, 0.4844, 0.4897, 0.4940, 0.4982, 0.5014, 0.5046, 0.5078, 0.5109, 0.5131,0.5154, 0.5176, 0.5198, 0.5219, 0.5241, 0.5261, 0.5282, 0.5303, 0.5323, 0.5343, 0.5362, 0.5381,0.5394, respectively. Using the same idea of the proof of the situation $\tau \geq 4.63$, it is sufficient to verify that for each of the above 29 segments $[a,b]$, $\frac{1-\frac{3}{2a}k^*(a)}{(1-\sqrt{\frac{8}{3}\frac{1}{b}})\frac{5}{6}} \geq 87\%$, which is indeed the case.

Now consider $2.65 \leq \tau \leq 2.99$, By Proposition OA.3, $L_u(\tau) = (\mu - k^*_{u_2}(\tau)\sigma - c)(\frac{4}{3} - \frac{4}{3}k^{*2}_{u_2}(\tau)) = (\mu - c)(1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3} - \frac{4}{3}k^{*2}_{u_2}(\tau))$, then $\frac{L(\tau)}{L_u(\tau)} = \frac{(1-\frac{3}{2\tau}k^*(\tau))}{(1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3}-\frac{4}{3}k^{*2}_{u_2}(\tau))}$. By (S.19), note that $k^*_{u_2}(\tau)$ is the unique root of $k^3+k=2\tau$, $k^*_{u_2}(\tau)$ is increasing in τ, then $1-\frac{k^*_{u_2}(\tau)}{\tau} = 1-\frac{2}{k^{*2}_{u_2}(\tau)+1}$ and $\frac{4}{3}-\frac{4}{3}k^{*2}_{u_2}(\tau)$ are increasing in τ, thus $(1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3}-\frac{4}{3}k^{*2}_{u_2}(\tau))$ is increasing in τ.

Divide [2.65,2.99] into two segments: [2.65, 2.94], [2.94, 2.99], we evaluate $1-\frac{3}{2\tau}k^*(\tau)$ at every endpoint of the segments. The values are: 0.3229 and 0.3506. Similarly, we evaluate $(1-\frac{k^*_{u_2}(\tau)}{\tau})(\frac{4}{3} - \frac{4}{3}k^{*2}_{u_2}(\tau))$ at those endpoints. The values are: 0.3705 and 0.3782. It is sufficient to verify that for each of the above 2 segments $[a,b]$, $\frac{1-\frac{3}{2a}k^*(a)}{(1-\frac{k^*_{u_2}(b)}{b})(\frac{4}{3}-\frac{4}{3}k^{*2}_{u_2}(b))} \geq 87\%$, which is indeed the case. ☐

B. Ancillary Results

Proposition S.1. *(a) For a truncated distribution on* $[0,\infty)$ *of the normal distribution with mean* $\overline{\mu}$ *and standard deviation* $\overline{\sigma}$*, the performance of the optimal robust price is given by:* $\frac{(\frac{\mu-k^*\sigma}{\sigma}-\beta)\bar{\Phi}(\frac{\mu-k^*\sigma}{\sigma}-\alpha)}{\max_p(p-(\beta-\alpha))\bar{\Phi}(p)}$*, where* $\Phi(x)$ *is the c.d.f. of standard normal distribution,* $k^* = \sqrt[3]{\tau+\sqrt{\tau^2+1}} + \sqrt[3]{\tau-\sqrt{\tau^2+1}}$*,* μ *and* σ *are the mean and standard deviation of the truncated normal distribution on* $[0,\infty)$*,* $\alpha = \frac{\overline{\mu}}{\overline{\sigma}}$ *and* $\beta = \frac{c}{\overline{\sigma}}$*. Moreover, the performance only depends on* α *and* β*.*

(b) For a truncated distribution on $[0,\infty)$ *of the logit distribution with mean* $\overline{\mu}$ *and scale parameter* $\overline{\sigma}$*, the performance of the optimal robust price is given by:* $\frac{(\frac{\mu-k^*\sigma}{\sigma}-\beta)\bar{L}(\frac{\mu-k^*\sigma}{\sigma}-\alpha)}{\max_p(p-(\beta-\alpha))\bar{L}(p)}$*, where* $L(x) = \frac{1}{1+e^{-x}}$ *is the c.d.f. of logit distribution with mean 0 and scale parameter 1,* $k^* = \sqrt[3]{\tau+\sqrt{\tau^2+1}} + \sqrt[3]{\tau-\sqrt{\tau^2+1}}$*,* μ *and* σ *are the mean and standard deviation of the truncated logit distribution on* $[0,\infty)$*,* $\alpha = \frac{\overline{\mu}}{\overline{\sigma}}$ *and* $\beta = \frac{c}{\overline{\sigma}}$*. Moreover, the performance only depends on* α *and* β*.*

(c) For a lognormal distribution $f(x) = \frac{1}{\sqrt{2\pi}\overline{\sigma}x}e^{-\frac{(\ln x-\overline{\mu})^2}{2\overline{\sigma}^2}}$*, the performance of the optimal robust price is given by:*

$$\frac{(1-k^*\delta-\gamma)\bar{\Phi}(\frac{\frac{\ln(\delta^2+1)}{2}+\ln(1-k^*\delta)}{\sqrt{\ln(\delta^2+1)}})}{\max_p(p-\gamma)\bar{\Phi}(\frac{\ln p+\frac{\ln(\delta^2+1)}{2}}{\sqrt{\ln(\delta^2+1)}})},$$

where $\Phi(x)$ *is the c.d.f. of standard normal distribution,* $k^* = \sqrt[3]{\tau+\sqrt{\tau^2+1}} + \sqrt[3]{\tau-\sqrt{\tau^2+1}}$*, and* $\delta = \sqrt{e^{\overline{\sigma}^2}-1}$*.*

Proof of Proposition S.1. (a) For a given normal distribution $x \sim N(\overline{\mu}, \overline{\sigma})$, we first compute the mean μ and standard deviation σ of its truncated version on $[0,\infty)$, which are:

$$\mu = \frac{\int_0^\infty x \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x}{\int_0^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x} = \frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty (y\overline{\sigma}+\overline{\mu}) \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y}{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y} = \overline{\sigma} \frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y}{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y} + \overline{\mu},$$

$$\sigma = \sqrt{\frac{\int_0^\infty x^2 \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x}{\int_0^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x} - \mu^2} = \overline{\sigma} \sqrt{\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y^2 \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y}{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y} - \left(\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y}{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{y^2}{2}} \mathrm{d}y}\right)^2}.$$

Given the truncated normal distribution F, the profit function is

$$\pi(p;F) = \frac{(p-c)\bar{F}(p)}{\int_0^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x} = \frac{(p-c)\bar{\Phi}(\frac{p-\overline{\mu}}{\overline{\sigma}})}{\bar{\Phi}(-\frac{\overline{\mu}}{\overline{\sigma}})}.$$

Denote $p_1 = \frac{p-\overline{\mu}}{\overline{\sigma}}$ and $c_1 = \frac{c-\overline{\mu}}{\overline{\sigma}} = \beta - \alpha$. Then the profit function can be written as $\Pi(p;F) = \frac{(p-c)\bar{\Phi}(\frac{p^*-\overline{\mu}}{\overline{\sigma}})}{\bar{\Phi}(-\frac{\overline{\mu}}{\overline{\sigma}})} = \frac{(p_1-c_1)\bar{\Phi}(p_1)}{\bar{\Phi}(-\frac{\overline{\mu}}{\overline{\sigma}})} = \frac{\overline{\sigma}(p_1-(\beta-\alpha))\bar{\Phi}(p_1)}{\bar{\Phi}(-\alpha)}$.

The optimal robust price is $\mu - k^*\sigma$. Thus the profit given by the optimal robust price is:

$$\frac{(\mu-k^*\sigma-c)\bar{F}(\mu-k^*\sigma)}{\int_0^\infty \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x} = \frac{(\mu-k^*\sigma-c)\bar{\Phi}(\frac{\mu-k^*\sigma-\overline{\mu}}{\overline{\sigma}})}{\bar{\Phi}(-\frac{\overline{\mu}}{\overline{\sigma}})} = \overline{\sigma}\frac{(\frac{\mu-k^*\sigma}{\overline{\sigma}}-\beta)\bar{\Phi}(\frac{\mu-k^*\sigma}{\overline{\sigma}}-\alpha)}{\bar{\Phi}(-\alpha)}.$$

Thus the performance is $\frac{(\frac{\mu-k^*\sigma}{\overline{\sigma}}-\beta)\bar{\Phi}(\frac{\mu-k^*\sigma}{\overline{\sigma}}-\alpha)}{\max_p(p-(\beta-\alpha))\bar{\Phi}(p)}$, With the definition of μ and σ, we know $\frac{\mu}{\overline{\sigma}}$ and $\frac{\sigma}{\overline{\sigma}}$ only depend on α. Moreover, k^* only depends on τ and hence only depends on α and β. Therefore, $\frac{\mu-k^*\sigma}{\overline{\sigma}}$ and the performance only depend on α and β.

(b) For a given logit distribution $F(x) = \frac{1}{1+e^{-\frac{x-\overline{\mu}}{\overline{\sigma}}}}$, we first compute the mean μ and standard deviation σ of its truncated version on $[0,\infty)$, which are:

$$\mu = \frac{\int_0^\infty x \mathrm{d}\frac{1}{1+e^{-\frac{x-\overline{\mu}}{\overline{\sigma}}}}}{\int_0^\infty \mathrm{d}\frac{1}{1+e^{-\frac{x-\overline{\mu}}{\overline{\sigma}}}}} = \frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty (\overline{\sigma}y+\overline{\mu})\mathrm{d}\frac{1}{1+e^{-y}}}{\frac{1}{1+e^{\frac{\overline{\mu}}{\overline{\sigma}}}}} = \overline{\sigma}\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y\mathrm{d}\frac{1}{1+e^{-y}}}{\frac{1}{1+e^{\frac{\overline{\mu}}{\overline{\sigma}}}}} + \overline{\mu},$$

$$\sigma = \sqrt{\frac{\int_0^\infty x^2 \mathrm{d}\frac{1}{1+e^{-\frac{x-\overline{\mu}}{\overline{\sigma}}}}}{\int_0^\infty \mathrm{d}\frac{1}{1+e^{-\frac{x-\overline{\mu}}{\overline{\sigma}}}}} - \mu^2} = \sqrt{\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty (\overline{\sigma}y+\overline{\mu})^2\mathrm{d}\frac{1}{1+e^{-y}}}{\frac{1}{1+e^{\frac{\overline{\mu}}{\overline{\sigma}}}}} - \mu^2} = \overline{\sigma}\sqrt{\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y^2\mathrm{d}\frac{1}{1+e^{-y}}}{\frac{1}{1+e^{\frac{\overline{\mu}}{\overline{\sigma}}}}} - \left(\frac{\int_{-\frac{\overline{\mu}}{\overline{\sigma}}}^\infty y\mathrm{d}\frac{1}{1+e^{-y}}}{1+e^{\frac{\overline{\mu}}{\overline{\sigma}}}}\right)^2}.$$

The remaining analysis is similar to part (a), with $\Phi(x)$ replaced by $L(x)$.

(c) For a lognormal distribution with pdf $f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(lnx-\overline{\mu})^2}{2\overline{\sigma}^2}}$, the mean and standard deviation are $\mu = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}$ and $\sigma = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}\sqrt{e^{\overline{\sigma}^2}-1}$, then $\delta = \sqrt{e^{\overline{\sigma}^2}-1}$, k^* depends on γ and σ. The optimal robust price is $\mu - k^*\sigma = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(1 - k^*\sqrt{e^{\overline{\sigma}^2}-1})$, thus the profit is $e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(1 - k^*\sqrt{e^{\overline{\sigma}^2}-1} - \gamma)\int_{e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(1-k^*\sqrt{e^{\overline{\sigma}^2}-1})}^{\infty} \frac{1}{\sqrt{2\pi}\overline{\sigma}x} e^{-\frac{(lnx-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(1 - k^*\sqrt{e^{\overline{\sigma}^2}-1} - \gamma)\int_{\frac{\frac{\overline{\sigma}^2}{2}+ln(1-k^*\sqrt{e^{\overline{\sigma}^2}-1})}{\overline{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathrm{d}y = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(1 - k^*\sqrt{e^{\overline{\sigma}^2}-1} - \gamma)\Phi(\frac{\frac{\overline{\sigma}^2}{2}+ln(1-k^*\sqrt{e^{\overline{\sigma}^2}-1})}{\overline{\sigma}})$. The profit function $\Pi(p;F) = (p - c)\int_p^{\infty} \frac{1}{\sqrt{2\pi}\overline{\sigma}x} e^{-\frac{(lnx-\overline{\mu})^2}{2\overline{\sigma}^2}} \mathrm{d}x = (p - c)\int_{\frac{lnp-\overline{\mu}}{\overline{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathrm{d}p$, let $p = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}} p_1 = \mu p_1$, then $\Pi(p;F) = e^{\overline{\mu}+\frac{\overline{\sigma}^2}{2}}(p_1 - \gamma)\bar{\Phi}(\frac{lnp+\frac{\overline{\sigma}^2}{2}}{\overline{\sigma}})$. Thus we have the performance is $\frac{(1-k^*\sqrt{e^{\overline{\sigma}^2}-1}-\gamma)\bar{\Phi}(\frac{\frac{\overline{\sigma}^2}{2}+ln(1-k^*\sqrt{e^{\overline{\sigma}^2}-1})}{\sigma})}{\max_p (p-\gamma)\bar{\Phi}(\frac{lnp+\frac{\overline{\sigma}^2}{2}}{\overline{\sigma}})}$. With $\delta = \sqrt{e^{\overline{\sigma}^2}-1}$, it is translated to $\frac{(1-k^*\delta-\gamma)\bar{\Phi}(\frac{\frac{ln(\delta^2+1)}{2}+ln(1-k^*\delta)}{\sqrt{ln(\delta^2+1)}})}{\max_p (p-\gamma)\bar{\Phi}(\frac{lnp+\frac{ln(\delta^2+1)}{2}}{\sqrt{ln(\delta^2+1)}})}$. $\square$