

# Competition in Multi-Echelon Systems

*Awi Federgruen*

Graduate School of Business, Columbia University, New York, New York 10027,  
[af7@gsb.columbia.edu](mailto:af7@gsb.columbia.edu)

*Ming Hu*

Rotman School of Management, University of Toronto, Toronto, Ontario M5S 3E6, Canada,  
[ming.hu@rotman.utoronto.ca](mailto:ming.hu@rotman.utoronto.ca)

**Abstract** Around the start of this new millennium, scholars in the operations management/operations research field started to make important contributions to the study of price competition models. In this tutorial, we review these contributions and partition them into five broad areas. Most of this tutorial is devoted to the most recent category: price competition models for multi-echelon supply chains with an arbitrary number of competing firms and products at each echelon.

**Keywords** price competition models; Nash equilibria; globally stable equilibria; multi-echelon supply chains; operational cost structures; perfect coordination mechanisms

---

## 1. Introduction and Summary

Ever since the seminal paper by Bertrand [14], a plethora of papers has emerged addressing oligopolistic price competition models. The purpose of those models is to characterize and study various industries and interactions therein.

Until some 15–20 years ago, almost all of the contributions to this area appeared in the economics and marketing literature; see Vives [64] for a survey of this literature. Around the start of this new millennium, scholars in the operations management/operations research field started to make important contributions to the study of price competition models. The following provides a broad classification of the various types of contributions made:

- I. analyses of the equilibrium behavior of basic workhorse models for price competition under simple cost structures;
- II. the exploration of how more realistic operational cost structures impact the equilibrium behavior of basic price competition models;
- III. the impact, on price competition, of contractual arrangements among the different parties in a supply chain—in particular, when attempting to optimize the aggregate performance of a decentralized supply chain;
- IV. analyses of competition models in which firms compete by selecting operational standards, along with product prices; and
- V. price competition models for *multi-echelon* supply chains with an arbitrary number of competing firms and products at each echelon.

In this tutorial, we will give a few illustrations of contributions in each of these five areas. Each of the five areas is deserving of a full tutorial in its own right. However, the treatment of the first four areas will be brief; most of the tutorial is devoted to the last and most recent area (V).

## 2. Analyzing the Equilibrium Behavior of Basic Workhorse Models for Price Competition Under Simple Cost Structures

Every price competition model is based on a consumer choice model that predicts how consumers, or industrial buyers, select among the various products offered in the market. The consumer choice model predicts expected aggregate sales volumes in the market, for all of the products offered by the competing firms, as an implicit or explicit vector function of the prices selected for the various products. One of the most important classes of price competition models for differentiated products employs the so-called multinomial logit (MNL) discrete choice model. This model was proposed by Daniel McFadden [52], a contribution later awarded with the 2000 Nobel Prize in Economics. This model may be derived from the following well-known random utility model: Assume there are  $J$  firms in the market indexed  $j = 1, \dots, J$ , each offering a *single* product. Each of the products can be procured at a given cost rate and is sold at a given unit price to be selected by the corresponding firm. Let

- $x_j$  = an  $L$ -dimensional vector of observable *non-price* attributes for firm  $j$ ;
- $c_j$  = the variable cost rate for firm  $j$ ;
- $p_j$  = the price selected by firm  $j$ ;  $p_j \in [p_j^{\min}, p_j^{\max}]$ , with  $0 \leq p_j^{\min} \leq c_j \leq p_j^{\max}$ ;
- $S_j$  = the expected aggregate sales volume for firm  $j$ ;
- $H$  = the population size in the market.

The MNL model assumes that each consumer and buyer in the market selects at most one unit of one of the  $J$  products, by trading off price differentials to be determined by the competing firms and differences in product attributes that are exogenously given in the model. Each consumer makes her product choice by assigning a (random) utility value to each of the  $J$  products as well as the *no-purchase* option, referred to as product 0. The consumer then chooses the option with the highest utility value. Thus let

$$u_{ij} = u_j(x_j) + g_j(p_j) + \epsilon_{ij}, \quad j = 1, 2, \dots, J, \quad i = 1, 2, \dots, \quad (1)$$

denote the utility attributed to product  $j$  by consumer  $i$ . Similarly, the consumer assigns the following utility to the no-purchase option:

$$u_{i0} = u_0(x_1, \dots, x_J) + \epsilon_{i0}, \quad i = 1, 2, \dots \quad (2)$$

As mentioned above,  $x_j$  is a vector of *observable* product attributes for product  $j$ . By contrast,  $\epsilon_{ij}$  and  $\epsilon_{i0}$  denote a random *unobservable* utility component. The function  $g_j(\cdot)$  is decreasing in the price level  $p_j$ , with  $\lim_{p_j \rightarrow \infty} g_j(p_j) = -\infty$ . It is well known that when the random components  $\epsilon_{ij}$  are independent and identically distributed (i.i.d.) and follow a Type I extreme value or Gumbel distribution, a simple closed-form expression can be derived for the expected sales volumes  $\{S_j: j = 1, 2, \dots, J\}$  (Luce and Suppes [51] attribute this derivation to an unpublished manuscript by Holman and Marley):

$$S_j = H \frac{e^{u_j(x_j) + g_j(p_j)}}{e^{u_0(x_1, \dots, x_J)} + \sum_{m=1}^J e^{u_m(x_m) + g_m(p_m)}}, \quad j = 1, 2, \dots, J.$$

Almost all applications of the MNL models specify the  $g_j(\cdot)$  functions as affine. Under such an affine specification, Anderson et al. [5] establish the existence of a (unique) Nash equilibrium in the special case where all firms are symmetric (i.e., have identical characteristics). Bernstein and Federgruen [10] extend this result for the case of general asymmetric firms and a generalization of MNL models referred to as attraction models. The general attraction model can be retrieved by a special nonaffine choice of the  $g_j(\cdot)$  functions. For the same model, Gallego et al. [41] provide sufficient conditions for the existence of a unique

equilibrium, under cost structures that depend on the firm's sales volume according to an increasing convex function.

The MNL model satisfies the so-called independence of irrelevant alternatives (IIA) axiom, according to which the *ratio* of any pair of firms' market shares is independent of the set of other alternatives that are offered to the consumers. This axiom was first postulated by Luce [50], but Debreu [27] points out that the IIA property is highly restrictive, as illustrated by his famous red bus–blue bus example: the relative market share of an alternative is, in general, significantly affected if a close substitute to this alternative is added to the choice set.

Ben-Akiva [8] proposes an extension of the MNL model referred to as the *nested logit* (NL) model, which remedies this limitation. The model also allows for a consumer choice process in which consumers arrive at their ultimate product choice in two (or more) stages: the consumer first selects among broad classes of alternatives, referred to as nests. In a second stage, she chooses a specific variant among the selected class of alternatives. For example, when purchasing an automobile, the consumer may first decide on whether a subcompact, compact, midsize, SUV, or minivan best suits her needs and desires; after selecting one of these categories, she proceeds to select a specific make and model within the chosen category or nest. As a second example, in deciding on what travel option to purchase for a trip from New York to Washington, DC, the first-stage choice may be between air and train as two broad “nests”; in a second stage, the consumer selects a specific flight (train) if the former (latter) is chosen.

Assume, therefore, that the  $J$  products are partitioned into  $n$  nests, with nest  $l$  containing  $m_l$  products,  $l = 1, 2, \dots, n$ . Assume further that all price sensitivity functions  $g_j(\cdot)$  in (1) are linear. Using a double index to differentiate among the products, let

$$\alpha_{ls} = u_{ls}(x_{ls}) \quad \text{and} \quad g_{ls}(p_{ls}) = -\beta_{ls}p_{ls}, \quad l = 1, 2, \dots, n, \quad s = 1, 2, \dots, m_l.$$

Finally, let  $Q_l(p)$  denote the probability of a consumer selecting nest  $l$ , and  $q_{s|l}(p)$  the conditional probability of selecting product  $s$  in nest  $l$ , given nest  $l$  is selected in the first stage. In the nested logit models, there are positive constants  $\gamma_l$ ,  $l = 1, 2, \dots, n$  such that

$$Q_l(p) = \frac{(a_l(p))^{\gamma_l}}{1 + \sum_{r=1}^n (a_r(p))^{\gamma_r}},$$

$$q_{s|l}(p) = \frac{e^{\alpha_{ls} - \beta_{ls}p_{ls}}}{\sum_{t=1}^{m_l} e^{\alpha_{lt} - \beta_{lt}p_{lt}}},$$

where  $a_l(p) = \sum_{s=1}^{m_l} e^{\alpha_{ls} - \beta_{ls}p_{ls}}$  for each  $l$ , so that

$$S_{ls}(p) = H Q_l(p) q_{s|l}(p).$$

A second major restriction of the pure MNL model is that, except for the random  $\epsilon$  terms, all consumers make the same trade-offs among prices and all other observable attributes. However, in most applications, there is systematic customer heterogeneity based on, for example, demographic attributes, income level, geographic location, or past purchasing history.

To allow for such heterogeneity, and at the same time eliminate the above restriction (IIA), the so-called mixed multinomial logit (MMNL) model was proposed apparently first by Boyd and Mellman [16] and Cardell and Dunbar [20]; earlier papers in the 1970s (e.g., Westin [65]) had derived a similar model by treating, in a single-segment model, the attribute vector as random with a given distribution. McFadden and Train [53] show that, under mild conditions, any discrete choice model derived from random utility maximization generates choice probabilities that can be approximated, arbitrarily closely, by an MMNL model.

The MMNL model partitions the market into  $K$  segments, indexed by  $k = 1, 2, \dots, K$ , with  $h_k$  customers in segment  $k$  ( $\sum_{k=1}^K h_k = H$ ). A potentially *different* set of utility functions, of type (1) and (2), is assumed for each segment. Thus, let  $u_{ijk}$  denote the utility value attributed to product  $j$  by consumer  $i$  in segment  $k$ . Then,

$$\begin{aligned} u_{ijk} &= u_{jk}(x_j) + g_{jk}(p_j) + \epsilon_{ijk}, \\ u_{i0k} &= u_{0k}(x_1, \dots, x_J) + \epsilon_{i0k}, \end{aligned}$$

where the  $\epsilon$  terms are again i.i.d. random variables with a Gumbel distribution. Thus, it is easily shown that

$$S_j = \sum_{k=1}^K h_k \frac{e^{u_{jk}(x_j) + g_{jk}(p_j)}}{e^{u_{0k}(x_1, \dots, x_J)} + \sum_{m=1}^J e^{u_{mk}(x_m) + g_{mk}(p_m)}}.$$

In spite of the huge popularity of MMNL models in both the theoretical and empirical literatures, it is not known, in general, whether a Nash equilibrium of prices exists and whether the equilibria can be uniquely characterized as the solutions to the system of first-order-condition (FOC) equations. (This system of equations is obtained by specifying that all firms' marginal profit values equal 0.) Indeed, there are many elementary price competition models in which either *no* or a *multiplicity* of Nash equilibria exist. For example, the seminal paper by Berry et al. [13], which studies market shares in the U.S. automobile industry, introduced, at least in the empirical industrial organization literature, a new estimation methodology to circumvent the problem that prices, as explanatory variables of sales volumes, are typically endogenously determined. The paper postulates an MNL model with random coefficients for the industry. One of the empirical methods developed in the paper is based on estimating the model parameters as those under which the observed price vector satisfies the FOC equations. Berry et al. [13, Footnote 12] acknowledge that it is unclear whether their model possesses an equilibrium, let alone a unique equilibrium. Even if these questions can be answered in the affirmative, so that the observed price vector can be viewed as the unique price equilibrium, it is unclear whether it is necessarily identified by the FOC equations on which the estimation method relies.<sup>1</sup>

Gallego and Wang [40] recently characterized the equilibrium behavior in an important class of nested logit models where there are  $n$  firms in the market, each of which has one of the nests as its product assortment. The authors show that it is always optimal for each firm to choose a *uniform* "adjusted markup" for all of its products. In other words, each firm  $i$  chooses an adjusted markup  $\theta_i$  and sets its prices as follows:

$$p_{is} = c_{is} + \frac{1}{\beta_{is}} + \theta_i, \quad \text{for all } s = 1, 2, \dots, m_i.$$

Given this characterization, the price competition model reduces to a simpler game, in which each firm  $i$  selects a single "adjusted markup value"  $\theta_i$  as opposed to a complete price vector. Moreover, Gallego and Wang [40] show that the reduced game is log-supermodular.

**Theorem 1 (Gallego and Wang [40]).** *The reduced competition model, under the nested logit demand model, is strictly log-supermodular. Its equilibrium set is, therefore, a non-empty complete lattice and, in particular, has a component-wise largest and a component-wise smallest element, denoted by  $\theta^*$  and  $\underline{\theta}^*$ , respectively. Furthermore, the largest equilibrium is preferred by all the firms.*

<sup>1</sup> In Berry et al. [13, p. 853, Footnote 12] they write, "We assume that a Nash equilibrium to this pricing game exists, and that the equilibrium prices are in the interior of the firms' strategy sets (the positive orthant). . . . However, we are able to check numerically whether our final estimates are consistent with the existence of an equilibrium. Note that none of the properties of the estimates require uniqueness of equilibrium, although without uniqueness it is not clear how to use our estimates to examine the effects of policy and environmental changes."

Moreover, under a modest restriction of the model parameters, Gallego and Wang [40] show that the equilibrium is, in fact, *unique*.

As to the MMNL model, Aksoy-Pierson et al. [3] showed recently that the resulting competition model may fail to have *any* (pure) Nash equilibrium. At the same time, they identify a simple and very broadly satisfied condition under which a Nash equilibrium exists, and the set of Nash equilibria coincides with the solutions of the system of FOC equations—a property of essential importance to empirical studies. Their existence condition merely requires that any single product captures less than a *majority* of the *potential* customer population in any of the market segments; moreover, this restriction on the market shares only needs to hold when the product is priced at a level that under the condition is shown to be an *upper bound* for a rational price choice, irrespective of the prices charged for the competing products. To guarantee *uniqueness* of the Nash equilibrium, a second condition is needed, restricting any given product’s market share to be no more than a *third* of the potential market. No restrictions whatsoever are required with respect to the distribution of population sizes across the different market segments.

Many open questions remain for both the NL and MMNL models. For the former, the equilibrium behavior remains unknown, when a given firm’s product assortment intersects with multiple “nests.” However, this is commonplace. In the above automobile market example, with the subcompacts, compacts, etc., as the nests, all major automobile manufacturers offer products in all or several of the above nests. Similarly, in the MMNL model, the equilibrium behavior is unknown when firms offer more than a single product.

### 3. The Impact of Operational Cost Structures

Almost all price competition models assume simple cost structures—for example, costs that grow linearly or *convexly* with the sales volumes. However, more complex cost functions arise under realistic operational cost structures. Several papers in the recent operations management literature have explored what impact those cost structures have on the equilibrium behavior in the competition model. In this section, we discuss two examples.

#### 3.1. A Deterministic Model with Fixed Order Costs

Consider first a market with  $N$  independent and competing retailers, each of whom sells a single product that is procured from a *single* common supplier. Demands for each of the products occur at a constant deterministic rate that, in general, depends on the full vector of retail prices selected by the firms. Both the supplier and each of the retailers replenish their inventories in periodic batches, the size of which is driven by fixed procurement costs. In addition to these fixed procurement costs, each of the firms incurs holding costs that are proportional to the amount of inventory carried. In this deterministic setting, each retailer avoids any stockout.

In the case of a single retailer directly procuring from an outside uncapacitated supplier and selling at a *given* rate, the model reduces to the oldest, most basic, and still most frequently applied inventory model, usually referred to as the economic order quantity (EOQ) model; see Harris [44].

We use index  $i$  to differentiate the  $N$  retailers,  $i = 1, 2, \dots, N$ . The common supplier is referred to as firm 0. Let

$$p_i = \text{the retail price selected by firm } i, \quad i = 1, 2, \dots, N, \quad p \equiv (p_1, p_2, \dots, p_N); \quad \text{and}$$

$$d_i(p) = \text{the constant demand rate for product } i, \quad i = 1, 2, \dots, N.$$

We confine our analysis to the case where all demand functions are affine; i.e.,

$$d_i(p) = a_i - \beta_i p_i + \sum_{j \neq i} \beta_{ij} p_j, \quad i = 1, 2, \dots, N,$$

where  $a_i > 0$ ,  $\beta_i > 0$  and  $\beta_{ij} \geq 0$  are linear parameters. These affine demand functions can only be valid on the polyhedron

$$P = \{p \geq 0 \mid d(p) \geq 0\},$$

since, for any vector  $p \notin P$ , some of the products' sales volumes are negative. We therefore assume that each firm  $i$  selects its retail price within a closed interval  $[p_i^{\min}, p_i^{\max}]$  such that  $\times_{i=1}^N [p_i^{\min}, p_i^{\max}] \subseteq P$ . See, however, Section 6 for a tractable way to extend the specification of the demand function  $d(\cdot)$  beyond  $P$  onto the full positive orthant  $\mathbb{R}_+^N$ .

We now turn to a description of the cost structure. All deliveries to and from the supplier incur fixed and variable costs. In a decentralized setting, it is useful to decompose the fixed cost associated with a delivery to a retailer into a component incurred by this retailer and one incurred by the supplier (e.g., an order-processing cost). Inventory carrying costs are incurred for each location's inventory, and they are proportional to the prevailing inventory level. For  $i = 1, 2, \dots, N$ , define the stationary cost parameters:

- $K_0$  = fixed cost incurred for each delivery to the supplier;
- $K_i$  = fixed cost incurred for each delivery to retailer  $i$ ,  $i = 1, 2, \dots, N$ ;
- $K_i^s$  = the component of  $K_i$  incurred by the *supplier* in a decentralized setting;
- $K_i^r$  = the component of  $K_i$  incurred by retailer  $i$  in a decentralized setting;
- $K_i = K_i^s + K_i^r$ ;
- $h_0$  = annual holding cost per unit of inventory at the supplier;
- $\bar{h}_i$  = annual holding cost per unit of inventory at retailer  $i$ ;
- $h_i = \bar{h}_i - h_0$ , incremental or echelon holding cost at retailer  $i$ ;
- $c_0$  = cost per unit delivered to the supplier; and
- $c_i$  = transportation cost per unit shipped from the supplier to retailer  $i$ .

We assume that  $h_i \geq 0$  for all  $i$ , which means that the cost of carrying a unit at retailer  $i$  is at least as large as the cost of carrying it in the supplier's warehouse. We also assume, without loss of generality, that in a decentralized setting the variable transportation cost  $c_i$  is borne by the retailer.

We analyze the retailer competition model assuming the supplier applies a simple *linear* pricing scheme charging firm  $i$  a constant wholesale price  $w$ . However, in the next section, we revisit the equilibrium behavior under more sophisticated pricing schemes that apply a variety of discounts.

When all wholesale and retail prices are determined, each retailer faces a setting in which the EOQ model applies. This implies that firm  $i$ 's optimal long-run average cost value is given by the well-known EOQ formula:

$$\sqrt{2d_i(p)\bar{h}_i K_i^r}, \quad i = 1, 2, \dots, N,$$

and firm  $i$ 's long-run average profit function is therefore given by

$$\pi_i(p; w_i) = (p_i - c_i - w_i) \left( a_i - \beta_i p_i + \sum_{j \neq i} \beta_{ij} p_j \right) - \sqrt{2d_i(p)\bar{h}_i K_i^r}.$$

The competition model thus reduces to a price competition game in which each firm is faced with a *single* choice (i.e., its retail price) but with a cost function that is *concave* rather than *convex* in the sales volume; reflecting the economies of scale, embedded in fixed delivery costs. Under general parameter combinations, the profit functions, therefore, lack any of the known structural properties (e.g., quasiconcavity in the firm's own price variable, or supermodularity) under which the existence of a Nash equilibrium can be guaranteed.



Instead, Bernstein and Federgruen [9] derive a broad parameter condition under which the existence of a Nash equilibrium *can* be guaranteed. Let  $\epsilon_{ii} \equiv -(\partial d_i(p)/\partial p_i)(p_i/d_i(p))$  denote the absolute price elasticity of retailer  $i$ 's demand. In addition, let

$$\text{INV}_i = \sqrt{2d_i(p)\bar{h}_iK_i^r} \quad \begin{array}{l} \text{(the optimal total inventory and setup cost for retailer } i \\ \text{under the price vector } p), \end{array}$$

and

$$\text{REV}_i = p_i d_i(p) \quad \text{(the total gross revenue for retailer } i \text{ under the price vector } p).$$

**Assumption (C1).** For all  $i$ ,

$$\epsilon_{ii} \leq \frac{8 \times \text{REV}_i}{\text{INV}_i}.$$

Even if the annual inventory carrying cost is as large as 40% of the dollar value of the inventory (a comfortable upper bound, in practice), the ratio  $\text{REV}_i/\text{INV}_i$  is at least 2.5 times the (annual) sales-to-inventory ratio. Dun & Bradstreet [30] reports the sales-to-inventory ratios in 10 consumer product categories; see Table 2 therein. The average lower quartile of the retailers' sales-to-inventory ratios varies between 2.8 and 6.7. Thus, for retailers in any of these sectors with a sales-to-inventory value above the lower quartiles, the right-hand side of Assumption (C1) is bounded from below by 56 and in some product lines by 134. Compare these values with estimated elasticities of demand, which vary between 1.4 and 2.8. (These estimates are obtained from Tellis [61].) We conclude that Assumption (C1) is comfortably satisfied in most markets. Assumption (C1) is equivalent to the inequality  $d_i(p)^{3/2} \geq \frac{1}{8}\beta_i\sqrt{2\bar{h}_iK_i^r}$ , or  $d_i(p) \geq \alpha_i \equiv \frac{1}{4}(\beta_i\sqrt{2\bar{h}_iK_i^r})^{2/3}$ . Thus, in vector notation, Assumption (C1) holds on the compact polyhedron  $P' = \{p \geq 0 \mid d(p) = a - Bp \geq \alpha\}$ . This polyhedron has a largest element  $\bar{p}$  when the matrix  $B$  has a nonnegative inverse, where  $a = (a_i)$ ,  $B_{ii} = \beta_{ii} > 0$  for all  $i$ , and  $B_{ij} = -\beta_{ij} \leq 0$  for all  $j \neq i$  ( $B^{-1} \geq 0$  is guaranteed if  $B$  is a positive-definite matrix). We henceforth assume that the cube  $\times_{i=1}^N [p_i^{\min}, p_i^{\max}]$  is contained in  $P'$ .

Under Assumption (C1), we propose the following theorem.

**Theorem 2 (Bernstein and Federgruen [9]).** Suppose Assumption (C1) holds. Then the price competition model has a Nash equilibrium.

Moreover, a much more precise characterization of the equilibrium behavior can be achieved by strengthening Assumption (C1) as follows.

**Assumption (C2).** For all  $i$ ,

$$\epsilon_{ii} \leq \frac{4 \times \text{REV}_i}{\text{INV}_i}.$$

As demonstrated above, Assumption (C2) continues to be easily satisfied in the 10 product markets reviewed in Dun & Bradstreet [30]. In particular, under the stronger Assumption (C2), it is possible to show that the equilibrium is *unique* and *globally stable*. The latter means that the market converges to this equilibrium, irrespective of its starting point, if the firms, iteratively, adjust their price as a best response to the price selection of their competitors. More specifically, assuming Assumption (C2) applies throughout the feasible price space  $\times_{i=1}^N [p_i^{\min}, p_i^{\max}]$ , Bernstein and Federgruen [9] prove the following.

**Theorem 3 (Bernstein and Federgruen [9]).** Suppose Assumption (C2) holds.

- (a) The price competition model is supermodular and has a unique equilibrium  $p^*$ .
- (b) Equilibrium  $p^*$  is globally stable.
- (c) Equilibrium  $p^*$  is increasing in  $w$ .<sup>2</sup>

<sup>2</sup> The monotonicity in this tutorial is in its weaker sense unless explicitly stated otherwise.

### 3.2. A Stochastic Single-Season Model

The following is a second and final example of a price competition model in which the explicit consideration of operational costs and risks has a fundamental impact on the analysis of the equilibrium behavior and the ultimate price equilibria themselves. This model assumes a *single* sales season, and a single procurement opportunity at the start of the season, as in the most basic of stochastic inventory models: the famous “newsvendor model.” The demand faced by each firm  $i$  is random with a known but general cumulative distribution function (c.d.f.)  $\tilde{G}_i(\cdot | p_1, \dots, p_N)$ , which depends on the prices of all  $N$  products in the market. We restrict ourselves to the case where the random variables  $D_i(p)$ ,  $i = 1, 2, \dots, N$ , are of the multiplicative form  $D_i(p) = d_i(p)\epsilon_i$ , with  $\epsilon_i$  as a general random variable whose distribution is independent of the price vector  $p$ . Equivalently,

$$\tilde{G}_i(x | p) = G_i\left(\frac{x}{d_i(p)}\right),$$

with  $G_i(\cdot)$  as the c.d.f. of  $\epsilon_i$  and  $g_i(\cdot)$  its probability density function. (Most of the results carry over to the case of additive demand shocks (i.e.,  $D_i(p) = d_i(p) + \epsilon_i$ ), again with  $\epsilon_i$  as a random variable whose distribution is independent of the price vector  $p$ .) Without loss of generality, we normalize  $E\epsilon_i = 1$ ,  $i = 1, 2, \dots, N$  so that  $ED_i(p) = d_i(p)$ . Because the products offered by the retailers are substitutes, we make the standard assumption that the demand functions  $d_i(p)$  are differentiable with

$$\frac{\partial d_i(p)}{\partial p_i} \leq 0 \quad \text{and} \quad \frac{\partial d_i(p)}{\partial p_j} \geq 0, \quad j \neq i.$$

In addition, we assume the following.

**Assumption (A).** For each  $i = 1, 2, \dots, N$ , the function  $\log d_i(p)$  has increasing differences in  $(p_i, p_j)$  for all  $j \neq i$ .

Milgrom and Roberts [54] identified the linear, logit, Cobb–Douglas, and constant elasticity of substitution demand functions as satisfying Assumption (A). (It is easily verified that Assumption (A) is essentially *weaker* than the requirement that the expected demand functions  $d_i(p)$  have increasing differences in  $(p_i, p_j)$ ,  $j \neq i$  themselves.) Finally, we again assume that each retailer  $i$  chooses his price  $p_i$  from a closed interval  $[p_i^{\min}, p_i^{\max}]$ .

As in the previous deterministic model, we assume that all  $N$  retail firms share a common supplier. This supplier charges, again, a constant wholesale price  $w_i$  to retailer  $i$ . However, to mitigate and share the demand risk with the retailers, the supplier offers to buy back any unit left over at the end of the season at a rate  $b_i < w_i$ ,  $i = 1, 2, \dots, N$ . Such buy-back agreements are commonplace in many supply chains.

This competition model translates into a competition game in which each firm is required to select *two* quantities:

$$\begin{aligned} p_i &= \text{the retail price of firm } i, \text{ and} \\ y_i &= \text{firm } i\text{'s order quantity with the supplier.} \end{aligned}$$

However, it is easily shown that the two-dimensional competition game reduces to a *one-dimensional* price competition game. This reduction is possible because it is easily shown from the standard newsvendor analysis that, under a *given* price vector  $p$ , firm  $i$ 's expected profits are maximized by selecting a procurement quantity:

$$y_i(p) = d_i(p)G_i^{-1}\left(\frac{p_i - w_i}{p_i - b_i}\right), \quad i = 1, 2, \dots, N,$$

where  $G_i^{-1}(\cdot)$  denotes the inverse function of the c.d.f.  $G_i(\cdot)$ . Substituting these optimal procurement quantities into the well-known expected profit function in the newsvendor



model, one shows that the two-dimensional competition game reduces to a one-dimensional price competition game, with profit functions:

$$\tilde{\pi}_i(p) = d_i(p) \left\{ (p_i - w_i) G_i^{-1} \left( \frac{p_i - w_i}{p_i - b_i} \right) - (p_i - b_i) E \left[ G_i^{-1} \left( \frac{p_i - w_i}{p_i - b_i} \right) - \epsilon_i \right]^+ \right\}. \quad (3)$$

Under *deterministic* demands, there is no need for operational risk hedging, and the profit functions are given by  $\pi_i^{\text{det}}(p) = (p_i - w_i)d_i(p)$ ,  $i = 1, 2, \dots, N$ . Thus

$$\tilde{\pi}_i(p) = \pi_i^{\text{det}}(p) L_i(f_i(p_i)),$$

where

$$f_i(p_i) = (p_i - w_i)/(p_i - b_i) = \text{the critical fractile for retailer } i,$$

and

$$\begin{aligned} L_i(f) &= G_i^{-1}(f) - f^{-1} E[G_i^{-1}(f) - \epsilon_i]^+ \\ &= G_i^{-1}(f) - f^{-1} \int_{-\infty}^{G_i^{-1}(f)} G_i(u) du = \frac{\int_{-\infty}^{G_i^{-1}(f)} u g_i(u) du}{f} \end{aligned}$$

is the factor by which retailer  $i$ 's expected profits are discounted because of the prevailing demand uncertainty. (The second equality in the definition of  $L_i(f)$  follows by integration by parts.) In other words, under a given price vector  $p$  in the market, each retailer  $i$ 's expected profits equal the profit value in the corresponding *deterministic* system multiplied with a *loss factor* that depends only on the retailer's chosen critical fractile  $f_i$  and the shape of the c.d.f. of  $\epsilon_i$ . The profit functions,  $\tilde{\pi}_i(p)$ , under operational risk hedging, see (3), have a much more complex structure than their deterministic counterparts  $\pi_i^{\text{det}}(p)$ . Nevertheless, Bernstein and Federgruen [11] were able to characterize the equilibrium behavior, in this case under fully general parameter conditions, by establishing the reduced game as a log-supermodular game.

**Theorem 4 (Bernstein and Federgruen [11]).** *Suppose Assumption (A) holds. Fix the vectors  $b < w$ .*

(a) *There exists a Nash equilibrium  $p^*$  for the reduced retailer game that arises under the  $(w, b)$ -payment scheme, and  $(p^*, y(p^*))$  is a Nash equilibrium in the original game.*

(b) *If the reduced retailer game has multiple Nash equilibria, these equilibria constitute a sublattice of  $\mathbb{R}^N$ . In particular, there exists a component-wise smallest and largest equilibrium  $\underline{p}$  and  $\bar{p}$ , respectively.*

(c) *The equilibrium  $\bar{p}$  is preferred by all  $N$  retailers among all Nash equilibria.*

Bernstein and Federgruen [11] also show that the existence of a unique Nash equilibrium and its monotonicity with respect to wholesale prices and buy-back rates can be established under an additional restriction on the class of c.d.f.s  $\{G_i(\cdot)\}$  as well as the (deterministic) demand functions  $d(p)$ ; see Theorem 4 therein.

#### 4. Perfect Coordination Mechanisms

The operations management field has, from its onset, been interested in comparing the supply chain-wide aggregate performance in a decentralized setting with its *first-best* performance, achieved when the supply chain is fully centralized and all decisions are made by a *single* decision maker with the objective of maximizing aggregate profits in the supply chain; see Cachon [18], Lariviere [47], and Cachon and Lariviere [19] for early contributions in this area.

Of particular interest is the identification of so-called *perfect coordination mechanisms*. A coordination mechanism is defined as a complete set of commercial terms between the

suppliers and the retailers they service. Such terms include (i) a wholesale pricing scheme, (ii) return policies or revenue sharing agreements, (iii) restrictions or times at which orders may be placed, and (iv) reallocation mechanisms through annual fees, rebates, etc.

A coordination mechanism is *perfect* if, in the decentralized system, in equilibrium, the supply chain-wide profits *equal* the optimal aggregate profits in the centralized system. A mechanism is *acceptable* if and only if it permits channel members to achieve expected profits at least equal to their status quo.

In some settings, a perfect coordination mechanism can be obtained with a simple linear wholesale pricing scheme, where the supplier charges a constant per-unit wholesale price to each of the retailers, possibly differentiated by retailers. (We omit a discussion under which settings it is legal to adopt different wholesale prices for different retailers.) A linear wholesale pricing scheme that results in perfect coordination can be turned into an acceptable and perfect coordination mechanism by adding a fixed periodic payment from the retailers to the supplier, or from the latter to the retailers. Such schemes are referred to as *two-part tariff* schemes.

In other settings, no two-part tariff scheme can achieve perfect coordination but a more sophisticated mechanism does, often taken from mechanisms and commercial arrangements, frequently implemented in real-world supply chains. Such arrangements include buy-back arrangements, discount schemes, and revenue sharing schemes, among others. Below, we discuss a few examples.

We start with the stochastic single-season model, discussed in Section 3.2. Let  $c_i$  denote the cost rate per unit at which the supplier is able to procure item  $i$  and assume that, even in the absence of a buy-back agreement, each retailer  $i$  has the option to salvage any leftover inventory at a unit price  $v_i$ . There exists a pair of price and order quantity vectors  $(p^I, y^I)$  that maximizes aggregate supply chain profits in the centralized system.

Bernstein and Federgruen [11] prove that perfect coordination can be achieved with a (unique) set of linear wholesale price/buy-back arrangements  $\{w_i^*, b_i^* \mid i = 1, 2, \dots, N\}$  under which the supplier charges retailer  $i$  a constant wholesale price  $w_i^*$  but guarantees to buy back any leftover units at a rate  $b_i^*$ , where  $c_i \leq w_i^* < p_i^I$  and  $b_i^* = v_i$ ,  $i = 1, 2, \dots, N$ . The existence result is predicated on sufficient conditions that guarantee that, under an *arbitrary* choice of a wholesale price vector  $w$  and buy-back rates  $b$ , the competition game has a *unique* price equilibrium vector  $p^*$  that is component-wise increasing in the vector of wholesale price  $w$ ; see Bernstein and Federgruen [11, Theorem 6] for details. Finally, we have  $c_i < w_i^* < p_i^I$ ,  $i = 1, 2, \dots, N$ , implying a *strictly positive* margin for the supplier, as long as the deterministic demand functions  $\{d_i\}$  have the property  $\partial d_i / \partial p_j > 0$  for all  $i \neq j$ —in other words, as long as the demand for any product exhibits some sensitivity with respect to the prices charged for the competing items.

Thus, a linear wholesale pricing scheme, combined with a (linear) buy-back scheme, exists under which perfect coordination can be achieved; however, it exists only under certain restrictions for the deterministic demand functions  $d(p)$  and the shape of the c.d.f.s  $G_i(\cdot)$  of the random factors  $\{\epsilon_i \mid i = 1, 2, \dots, N\}$ .

In parallel, perfect coordination can be achieved, in a fully general setting, through the use of a so-called *price-discount* sharing scheme (PDS). As with buy-back arrangements, these schemes incentivize the retailers to purchase large quantities and to reduce their retail price to stimulate sales. The idea behind buy-back agreements is to mitigate the retailer's risk exposure. By contrast, the idea behind a PDS is to incentivize the retailer to reduce her retail price by offering to adjust the wholesale price downward when the retailer reduces her price from a given list price. The simplest PDS schemes are linear and anchored on a base wholesale price  $w^0$  and retail price  $p^0$ . Here, the wholesale price is a linear function of the retail price (i.e.,  $w = w(p)$ ) and is specified as

$$w = w^0 - \alpha \Delta p \text{ or } \Delta w = \alpha \Delta p, \quad \text{where } \Delta p = p^0 - p \text{ and } \Delta w = w^0 - w. \quad (4)$$

Here,  $w^0$  is a constant *base* or *gross* wholesale price,  $p^0$  is an arbitrary reference value (e.g., the “list price”), and  $\alpha > 0$  is a *constant*. The PDS scheme is to be combined with a traditional buy-back arrangement at a buy-back rate set at a given constant  $\delta$  below the (net) wholesale price; i.e.,

$$b = w - \delta. \quad (5)$$

Under the scheme (4) and (5), the supplier compensates the retailer for every sold unit at the rate of  $\alpha\%$  for every dollar the retailer discounts from the reference value  $p^0$ . Lal et al. [46] consider such a shared price discount scheme. However, they restrict the retailer to one possible discount size only; they also discuss a cooperative merchandising agreement in the consumer packaged goods industry, which embodies this type of PDS scheme. In an article in the *Sloan Management Review*, Ailawadi et al. [2] report on an increasing trend toward trade promotions<sup>3</sup> and explain that the most effective discount schemes tie the supplier’s price directly to the retail price according to a given PDS scheme.

To achieve perfect coordination in our model, a *nonlinear* PDS is required, where  $\Delta w_i = H_i(\Delta p_i)$ , with  $H_i(\cdot)$  being a specific nonlinear increasing adjustment function; see Bernstein and Federgruen [11, Theorem 5].

We now turn our attention to the first competition model covered in Section 3.1, where the various firms in the supply chain incur fixed delivery and order costs. Here, the design of a perfect coordination scheme is considerably more complex, for reasons explained below.

Even the assessment of the optimal solution strategy for the centralized system—the necessary benchmark for any perfect coordination scheme—is difficult, and this is the case even under a given choice of the retail price vector  $p^o$ ; hence a given vector of demand rates  $d^o = d(p^o)$ . To date, the optimal procurement strategy in the centralized system with fixed demand rates remains unknown but is undoubtedly of a prohibitively complex nature, precluding its practical implementability.

However, Roundy [58], in a landmark paper, shows that a simple so-called power-of-two replenishment strategy is guaranteed to come within 2% of optimality: under such a strategy, every facility  $i = 0, 1, \dots, N$ , brings in replenishment batches after a constant interval  $T_i$ , and all intervals  $\{T_i \mid i = 1, 2, \dots, N\}$  are *power-of-two* multiples of some base period  $T_b$ ; i.e.,  $T_i = 2^{m_i} T_b$ , where  $m_i$  is a positive or negative integer, or zero. The replenishment batches are brought into the system just in time when the inventory level drops to zero. The optimal power-of-two interval vector  $T$  can be computed with a simple  $O(N \log N)$  algorithm, and this algorithm can be embedded into a procedure that identifies the optimal price vector  $p^I$  for the centralized system.

To design a perfect coordination mechanism in the decentralized system, a *first* requirement is, therefore, that all facilities agree to commit themselves to placing their orders at replenishment epochs selected from the power-of-two grid  $\{\dots, \frac{1}{8}T_b, \frac{1}{4}T_b, \frac{1}{2}T_b, T_b, T_b, T_b, \dots\}$ . Even so, it has been shown that perfect coordination can *not* be achieved with a linear wholesale pricing scheme, or even with one that sets the wholesale price as a decreasing function of the retailer’s order size, in accordance with a general (nonlinear) discount scheme.

Bernstein and Federgruen [9] show that a perfect coordination scheme can be constructed, but it requires the utilization of no less than three types of discount schemes: (i) a traditional discount scheme under which the wholesale price is adjusted as a function of the order size, (ii) a scheme where the wholesale price is adjusted as a function of the chosen replenishment interval, and (iii) a scheme under which the wholesale price is adjusted as a function of the

<sup>3</sup> They note, “In 1996, the percentage of retail sales made ‘on deal’ across forty packaged goods categories included in the *Market Fact Book* averaged about 37 percent—up almost 5 percentage points from 1991” (Ailawadi et al. [2, p. 83]).

total annual sales volume. More specifically, if retailer  $i$  chooses a replenishment interval  $T_i$ —thus with an annual sales volume  $q_i = d_i(p)$ —the per-unit wholesale price  $w_i$  is set as follows:

$$w_i(q_i, T_i) = w_i^{(1)}(T_i q_i) + w_i^{(2)}(T_i) + w_i^{(3)}(q_i).$$

Quantity discounts based on the retailer’s replenishment frequency and annual sales volume, as reflected by components  $w_i^{(2)}$  and  $w_i^{(3)}$ , are prevalent in many industries. (See Brown and Medoff [17] and Munson and Rosenblatt [55]. See Chen et al. [22] for additional discussion on this issue.)

Many supply chains have adopted an alternative business model, referred to as vendor-managed inventories (VMIs). In its purest form, the retailers empower the supplier to determine when and how their inventories are to be replenished. All inventory and delivery costs are assigned to the supplier as well. Instead of having to respond passively to an arbitrary pattern of retailers’ orders, a VMI enables the supplier to determine an efficient coordinated procurement strategy to service *all* of the retailers. Indeed, Bernstein and Federgruen [9] show that, under such a VMI arrangement, perfect coordination *can* be achieved with a simple linear wholesale pricing scheme.

What, then, are the essential features of the supply chain cost structure that allow for perfect coordination with a linear wholesale pricing scheme? This question was addressed by Bernstein et al. [12]. Bernstein et al. [12] show that such a linear wholesale pricing scheme exists under the following general setting, referred to as echelon operational autonomy (EOA).

Consider a two-echelon supply chain with a supplier distributing a single (or closely substitutable) product to  $N$  retailers, who in turn sell the product to the consumer market. All demands must be satisfied in their entirety. For  $i = 1, 2, \dots, N$ , let  $p_i$  be the retail price charged by retailer  $i$ , and let  $q_i$  be its annual consumer demand. The two sets of variables are related to each other via general, continuously differentiable, inverse demand functions  $p_i = f_i(q_1, \dots, q_N)$ ,  $i = 1, 2, \dots, N$ . Let  $q_{\max}$  denote a nonrestrictively large upper bound for the retailers’ annual demand volumes. For substitutable products, each retailer’s price decreases when he or any of his competitors increases their targeted sales volume; i.e.,  $\partial f_i / \partial q_j \leq 0$ ,  $i, j = 1, 2, \dots, N$ .

Our main assumption reflects the supply chain members’ cost structures. Let  $\sigma_i$ ,  $i = 0, 1, \dots, N$ , denote the complete set of operational decisions that are controlled by firm  $i$ . For example,  $\sigma_i$  may denote a set of capacity decisions, possibly in combination with an infinite horizon replenishment policy. EOA requires that each retailer  $i$ ’s operational decisions  $\sigma_i$  impact his own cost only. However, the supplier’s operational decisions  $\sigma_0$  may impact the costs of all chain members. Under EOA, the chain members’ costs depend on the vector  $q$  and the operational decisions, and the members costs may be described by the following continuous functions:

$$\begin{aligned} \tilde{h}_0(q, \sigma_0) &= \text{cost incurred by the supplier,} \\ \tilde{h}_i(q, \sigma_0, \sigma_i) &= \text{cost incurred by retailer } i, \quad i = 1, 2, \dots, N. \end{aligned}$$

The supplier charges the retailers for their purchases according to a given pricing scheme and determines her operational decisions  $\sigma_0$  so as to induce perfect coordination. To do so,  $\sigma_0$  is chosen to minimize aggregate costs. Decisions are made in the following sequence, at the beginning of the planning horizon, and cannot be revoked thereafter: in the first stage, a wholesale pricing scheme is specified; in the second stage, the retailers simultaneously select their sales volumes; in the third stage, the supplier chooses her operational decisions  $\sigma_0$ ; and in the final stage, the retailers choose their operational decisions  $\sigma_i$ ,  $i = 1, 2, \dots, N$ .

Thus, for any given vector  $q$ , operational decisions are determined as follows: the supplier selects  $\sigma_0^*(q)$ , taking into account each retailer  $i$ ’s response  $\sigma_i^*(q, \sigma_0)$  to minimize its own cost  $\tilde{h}_i$ . This gives rise to reduced cost functions  $h_0(q)$  and  $h_i(q)$ ,  $i = 1, 2, \dots, N$ . Formally, let

$$\sigma_i^*(q, \sigma_0) = \arg \max_{\sigma_i} \tilde{h}_i(q, \sigma_0, \sigma_i)$$

and

$$\sigma_0^*(q) = \arg \min_{\sigma_0} \left\{ \tilde{h}_0(q, \sigma_0) + \sum_{i=1}^N \tilde{h}_i(q, \sigma_0, \sigma_i^*(q, \sigma_0)) \right\}.$$

Thus,  $h_0(q) = \tilde{h}_0(q, \sigma_0^*(q))$  and  $h_i(q) = \tilde{h}_i(q, \sigma_0^*(q), \sigma_i^*(q, \sigma_0^*(q)))$ .

It is easily verified that in the traditional business model, EOA is violated, hence requiring a complex, multitiered discount scheme, while EOA is regained under a VMI arrangement, thus enabling a linear wholesale pricing scheme to coordinate the system perfectly.

### 5. Multidimensional Competition

Firms rarely compete on the basis of prices and design attributes alone. Increasingly, various types of operational (for example, service) metrics play a large, perhaps dominant, role in determining the competitors' market shares and, hence, in the competitive dynamics. Such operational service metrics include waiting time standards in service systems (see, e.g., Allon and Federgruen [4]) and product availability measures when goods, rather than services, are sold. Modeling such multidimensional competition settings requires a consumer choice model in which the firms' expected sales volumes depend on the vector of selected operational (service) standards along with the firms' price vector. In this section, we review one such model, which may be viewed as a generalization of the model in Section 3.2.

Consider an industry with  $N$  independent retailers facing random demands. We analyze a periodic review infinite-horizon model in which each retailer  $i$  positions himself in the market by selecting a steady-state fill rate  $f_i$ , where the fill rate is defined as the fraction of customer demands satisfied from on-hand inventory as well as a retail price  $p_i$ . Without loss of practical generality, we restrict ourselves to service levels in the interval  $[0.5, 1)$ . While the fill-rate target levels  $f$  are stationary choices, the retail price may, in principle, be varied in each period. Each retailer may, at the beginning of each period, place an order with his supplier, assumed to have ample capacity to fill any size order completely and in time for the retailer to meet this period's demand. Stockouts are backlogged. Thus, each retailer  $i$  makes the following choices: (i) a one-time choice of  $f_i$  at the beginning of the planning horizon and (ii) at the beginning of each period  $t$  a retail price as well as the quantity to be ordered from the supplier.

One of the novel features of the model is that the demand faced by each retailer  $i$ , in any period  $t$ , has a distribution that may depend on the entire vector of retail prices  $p$  in that period as well as the entire vector of fill-rate target levels  $f$ . Thus, let  $D_{it}(p, f)$  be the random demand faced by retailer  $i$  in period  $t$  under the retail price vector  $p$  and the service-level vector  $f$ , where the c.d.f. of  $D_{it}$ , denoted by  $\tilde{G}_i(x | p, f)$ , depends on the entire vector  $p$  as well as the complete vector of service levels  $f$ . Thus, demands in any period depend on the target fill rates, not on the actual inventory levels. We assume that the demand variables are of the multiplicative form; i.e.,

$$D_{it} = d_i(p, f)\epsilon_{it},$$

with  $\epsilon_{it}$  as a general continuous random variable whose distribution is stationary and independent of the retail price vector  $p$  and the service-level vector  $f$ ; i.e., for all  $i = 1, 2, \dots, N$ , the sequence  $\{\epsilon_{it}\}$  has a common general c.d.f.  $G_i(\cdot)$  and density function  $g_i(\cdot)$  such that  $\tilde{G}_i(x | p, f) = G_i(x/d_i(p, f))$ .

As in Section 3.2, we normalize  $E(\epsilon_{it}) = 1$ ,  $i = 1, 2, \dots, N$  and  $t = 1, 2, \dots$ , so that  $ED_{it}(p, f) = d_{it}(p, f)$ . In other words, the functions  $\{d_i(p, f)\}$  may be viewed as representing the expected one-period sales volumes. As in virtually all inventory models, we assume that the sequence of random variables  $\{\epsilon_{it} | t = 1, 2, \dots\}$ , and hence the sequence  $\{D_{it} | t = 1, 2, \dots\}$ , is independent for all  $i = 1, \dots, N$ . At the same time, we allow for arbitrary correlations between the demands faced by the different retailers in any given period.

Information about the firms' service levels is not always as readily available as the unit price. In the business-to-business world, service-level guarantees are routinely provided by the vendors, often backed up with chargeback agreements for violations of these guarantees. Software systems allow retailers to monitor their vendors' compliance and provide them with comparative data regarding groups of vendors' service levels. In the business-to-consumer world, firms increasingly advertise service-level measures, as well as independent Internet services that rate online retailers in terms of their fill-rate performance (along with other service measures). Even when such information is not publicly available, consumers develop estimates on the basis of their own (repeat-purchase) experience as well as on the basis of word of mouth and other reputational information.

We assume the following basic monotonicity properties:

$$\frac{\partial d_i(p, f)}{\partial p_i} \leq 0, \quad \frac{\partial d_i(p, f)}{\partial f_i} \geq 0; \quad \frac{\partial d_i(p, f)}{\partial p_j} \geq 0, \quad \frac{\partial d_i(p, f)}{\partial f_j} \leq 0, \quad j \neq i.$$

In other words, if a retailer increases his retail price (fill rate), this results in a decrease (increase) of his own expected sales while those of his competitors increase (decrease).

No firm's sales are expected to increase under a uniform price increase.

**Assumption (D0).**

$$\sum_{j=1}^N \frac{\partial d_i}{\partial p_j} < 0, \quad \text{for all } i = 1, 2, \dots, N.$$

*Decisions are made in the following sequence: at the beginning of each period, all retailers simultaneously determine their prices and order quantities for that period; then, these orders are filled.*

Each retailer pays the supplier a constant per-unit wholesale price, inclusive of delivery costs, or he incurs production costs at a constant rate. Holding costs are proportional to end-of-period inventories. A retailer may incur direct, out-of-pocket, backlogging costs; if so, these are proportional to the backlog size. Thus, for each retailer  $i = 1, 2, \dots, N$ ,

- $w_i$  = the per-unit wholesale price paid by retailer  $i$ ,
- $h_i^+$  = the per-period holding cost for each unit carried in inventory, and
- $h_i^-$  = the per-period direct backlogging cost for each unit backlogged at retailer  $i$ .

Contrary to most standard inventory models, but more representative of actual cost/service trade-offs experienced in practice, our model does not require that direct backlogging costs exist (i.e., that  $h_i^- > 0$ ). Even if  $h_i^- = 0$ , every firm is incentivized to carry appropriate safety stocks because a large backorder frequency or, equivalently, a low fill rate reduces the retailer's average sales while increasing those of his competitors.

When characterizing the equilibrium behavior of the industry, it is necessary to distinguish among three possible scenarios.

1. *Price competition only:* Here, we assume that the firms' service levels are *exogenously* chosen but characterize how the price equilibrium and inventory strategy vary with the chosen service levels.

2. *Simultaneous price and service-level competition:* Here, each of the firms simultaneously chooses a service level and a combined price and inventory strategy.

3. *Two-stage competition:* The firms make their competitive choices sequentially. In the first stage, all firms simultaneously choose a service level; in the second stage, the firms simultaneously choose a combined pricing and inventory strategy with full knowledge of the service levels selected by all competitors.

Fudenberg and Tirole [38] refer to the equilibria arising under scenario 2 as *open-loop equilibria* and those under scenario 3 as *closed-loop equilibria*. See Gallego and Hu [39] for



the comparison of these equilibrium concepts in a context of an infinite-dimensional price competition over a finite horizon and under capacity constraints.

Bernstein and Federgruen [10] show that in each of the three scenarios the infinite-horizon periodic review game is equivalent to a single-season game, in which each retailer  $i$  competes with two instruments  $(p_i, f_i) \in [p_i^{\min}, p_i^{\max}] \times [0.5, 1)$  and a reduced profit function

$$\pi_i(p, f) = (p_i - w_i - k_i(f_i))d_i(p, f),$$

where

$$k_i(f_i) = h_i^+ E \left[ G_i^{-1} \left( \max \left\{ f_i, \frac{h_i^-}{h_i^- + h_i^+} \right\} \right) - \epsilon_i \right]^+ + h_i^- E \left[ \epsilon_i - G_i^{-1} \left( \max \left\{ f_i, \frac{h_i^-}{h_i^- + h_i^+} \right\} \right) \right]^+.$$

Note that  $k_i(f)$  is the expected (end-of-period) inventory cost per unit of sales, required to guarantee a given service level of at least  $f$ . It can be shown that the function  $k_i(\cdot)$  is convexly increasing and differentiable. The following represents the equivalency result in scenario 2 (i.e., under simultaneous price and service competition).

**Theorem 5 (Bernstein and Federgruen [10]).** *Consider scenario 2 (simultaneous price and service competition). Assume that  $(p^*, f^*)$  is a Nash equilibrium in the reduced single-stage game. The  $N$ -tuple of infinite-horizon stationary strategies under which retailer  $i$  adopts a stationary price  $p_i^*$ , a fill rate  $f_i^*$ , and a base-stock policy with (stationary) base-stock level*

$$y_i(p^*, f^*) = d_i(p^*, f^*) G_i^{-1} \left( \max \left\{ f_i^*, \frac{h_i^-}{h_i^- + h_i^+} \right\} \right)$$

*is a Nash equilibrium in the infinite-horizon game.*

It thus suffices to characterize the equilibrium behavior in the single-season reduced game. The latter depends on the structural form of the deterministic demand functions  $d(p, f)$ . Bernstein and Federgruen [10] provide the characterization of three classes of demand functions:

I. *Generalized MNL models:* This is an MNL model in which product  $j$ 's utility measure is specified as

$$u_{ij} = \tilde{a}_j(p_j, f_j) + \epsilon_{ij},$$

a variant of (1). (We omit the intercept term  $u_j(x_j)$ .) Within this model, special attention is given to the following structure:

$$u_{ij} = b_j(f_j) - \alpha_j p_j + \epsilon_{ij}; \tag{6}$$

i.e.,  $\tilde{a}_j(p_j, f_j) = b_j(f_j) - \alpha_j p_j$ , where  $b_j(\cdot)$  is a concave, increasing, and differentiable function.

II. *Affine demand functions:*

$$d_j(p, f) = a_j - b_j p_j + \sum_{l \neq j} c_{jl} p_l + \beta_j f_j - \sum_{l \neq j} \gamma_{jl} f_l,$$

with  $b_j, \beta_j$  positive constants and  $c_{jl}, \gamma_{jl}$  nonnegative constants.

III. *The log-separable model:*

$$d_j(p, f) = \psi_j(f) q_j(p).$$

As a mere illustration, we exhibit some of the equilibrium behavior in the special generalized MNL mode, specified by (6). See Bernstein and Federgruen [10] for a much more detailed treatment.

For each product  $j = 1, 2, \dots, N$ , define  $f_j^0$  as the unique root of the function

$$\delta_j(f) \equiv k'_j(f_j) - \frac{b'_j(f_j)}{\alpha_j}. \tag{7}$$

The first term on the right-hand side of (7) represents the incremental operational costs associated with a marginal increase in the service level, while the second term denotes the incremental retail price value (i.e., the price increase that this marginal increase permits without altering the expected utility of firm  $j$ ). Thus  $f_j^0$  represents the unique service level for which the incremental operational costs equal the incremental retail price value. We conclude the following.

**Theorem 6 (Bernstein and Federgruen [10]).** *Consider the generalized MNL model, and assume that Assumption (D0) holds. For every fixed service-level vector  $f$ , there exists a unique Nash equilibrium  $p^*(f)$  in the price game. The pair  $(p^*(f^0), f^0)$  is the unique Nash equilibrium in the simultaneous single-stage (reduced) retailer game.*

## 6. Price Competition Models for Multi-Echelon Supply Chains

Until recently, almost all of the literature on two-echelon settings was confined to very special settings—for example, a *single* supplier with a *single* product selling to several competing retailers, or several suppliers each with a *single* product competing for the business of a *single* buyer, or an assembly system with one or two final products. The models reviewed in Section 3, for example, assumed a single supplier selling a single product to a number of competing retailers.

In practice, however, oligopolistic competition prevails at all echelons of the market. Thus, it is highly desirable to build a general model in which, at each echelon of the supply process, an arbitrary number of firms compete. These firms offer one or multiple products to some or all of the firms at the next or possibly subsequent echelons, with firms in the most downstream echelon selling to the end consumer; at each echelon, the offered products are *differentiated*,<sup>4</sup> and the firms belonging to this echelon engage in price competition. In this section, we focus on building and analyzing such a general framework.

For simplicity of exposition, we consider a two-echelon system where a set  $\mathcal{J} \equiv \{1, 2, \dots, J\}$  of suppliers compete by selling any number of grossly substitutable products via a pool  $\mathcal{I} \equiv \{1, 2, \dots, I\}$  of competing retailers. We generalize this to settings with an arbitrary number of echelons in Section 6.5. As a concrete example, consider the market for television sets. Each of the mega brands (Samsung, RCA, Magnavox, Mitsubishi, etc.) sells a line of television types, differentiated by type (LCD or plasma), screen size (19", 27", 32", etc.), and screen resolution (720 versus 1080 pixels), among other features. Different brands offer different subsets of the collection of all possible combinations; each sells these to some or all of the consumer electronics chains and general department stores. Thus, we aim for a model in which the firms' product assortments themselves are *endogenously* determined, in contrast to the established literature.

We denote by  $\mathcal{N}$  the set of all products potentially offered in the market and let  $N \equiv |\mathcal{N}|$ . To differentiate among different products, we employ a triple of indices  $(i, j, k)$ :  $i$  denotes

<sup>4</sup> See Corbett and Karmarkar [25] for a framework of competition of multi-echelon systems in selling a *homogeneous* product, with firms at each echelon engaging in Cournot (i.e., quantity) competition. Under this framework, Cho [23] investigates the impact of horizontal mergers. Adida and DeMiguel [1] analyze a setting with  $M$  suppliers, each selling the same collection of  $P$  products to a set of  $N$  retailers. In the first stage, the suppliers engage in homogeneous Cournot competition for each of the  $P$  products. In the second stage, the retailers engage in quantity competition responding to announced wholesale prices, one for each of the  $P$  products. DeMiguel and Xu [28] analyze a sequential competition model involving two groups of suppliers ultimately delivering the same homogeneous good to the same consumer market. Bimpikis et al. [15] analyze a model with  $N$  suppliers and a number of separate markets; in each market, a homogeneous good is offered by a specific subset of the  $N$  suppliers who engage in Cournot competition.

the retailer through which the product is sold, and  $j$  denotes the supplier procuring the product. We allow any supplier to sell multiple products through any one of the retailers and use the index  $k$  to differentiate among the various products potentially sold by supplier  $j$  to retailer  $i$ . All suppliers apply a linear pricing scheme for all of their products and clients; variable distribution and handling costs are easily incorporated in these schemes.

Let  $\mathcal{K}(i, j)$  denote the set of products offered by supplier  $j$  to retailer  $i$ . For  $i \in \mathcal{I}, j \in \mathcal{J}$  and  $k \in \mathcal{K}(i, j)$ , let

- $c_{ijk}$  = the constant marginal supply cost of supplier  $j$  for product  $k$  sold at retailer  $i$ ,
- $p_{ijk}$  = the retail price charged by retailer  $i$  for product  $k$  provided by supplier  $j$ ,
- $w_{ijk}$  = the wholesale price charged by supplier  $j$  for product  $k$  sold at retailer  $i$ , and
- $d_{ijk}$  = the consumer demand for product  $k$  provided by supplier  $j$  at retailer  $i$ .

Let  $p$ ,  $w$ ,  $c$ , and  $d$  be the corresponding vectors. For any  $i \in \mathcal{I}$ , let  $\mathcal{N}(i)$  denote the set of products offered to retailer  $i$  by the various suppliers; i.e.,  $\mathcal{N}(i) = \{(i, j, k) \mid j \in \mathcal{J}, k \in \mathcal{K}(i, j)\}$ .

The sequence of events in such a two-echelon model is as follows. In the first stage, with the exogenously given supply costs  $c$ , the suppliers simultaneously decide on their product assortment and wholesale prices among all products they can potentially sell to the retailers. In the second stage, given wholesale prices selected in the first stage, all retailers simultaneously decide on their product assortment and retail prices to maximize their total profits among all products of all suppliers they choose to do business with.

### 6.1. The Demand Model

To start with, we need to have a well-defined demand system that has the potential to lead to tractable solutions for a multi-echelon sequential competition model. Along with variants of the MNL model, such as NL and MMNL models, the most frequently used demand model in economics, marketing, and operations management employs affine demand functions. The demand value of each product may depend on the prices of *all* products offered in the market. As in the majority of supply chain competition models, we assume that this dependence is *in principle* described by general *affine* functions (see, for example, the first model in Section 3). In matrix notation, this gives rise to a system of demand equations:

$$q(p) = a - Rp, \quad (8)$$

where  $a \in \mathbb{R}_+^N \equiv \{x \mid x \geq 0\}^N$  and  $R \in \mathbb{R}^{N \times N}$ . We call  $a$  the intercept vector, of which each entry measures the potential market size of a product, as it would be the demand vector if all products were offered for free. We call  $R$  the matrix of price sensitivity coefficients. Any of its diagonal elements,  $r_{ll}$ , measures the potential direct price sensitivity of product  $l$ 's demand with respect to its own price  $p_l$ , whereas any off-diagonal entry,  $r_{ll'}$ , with  $l' \neq l$ , measures the potential cross-product price sensitivity of product  $l$ 's demand with respect to price  $p_{l'}$  of product  $l'$ . In our base model, we assume that the various products are *substitutes*.

**Assumption (Z).** *The matrix  $R$  is a Z-matrix (i.e., has nonpositive off-diagonal entries).*

This means that any product's demand volume does *not* decrease when the price of an alternative product is increased (see, however, Section 6.7 for a generalized model allowing for certain types of complementarities).

As pointed out by many authors, starting with Shubik and Levitan [59] and reviewed in Section 3, the affine structure in Equation (8) cannot be used for arbitrary price vectors  $p \geq 0$ .<sup>5</sup> After all, unless the price vector  $p$  is chosen in the polyhedron

$$P \equiv \{p \geq 0 \mid q(p) = a - Rp \geq 0\},$$

<sup>5</sup> The inequalities in this tutorial are in the component-wise sense for a vector or matrix.

some of the components of the  $q$ -vector are negative. We call  $P$  the *effective retail price polyhedron*. (Note that  $P \neq \emptyset$ , since  $p = 0 \in P$ , as  $a \geq 0$ .) To solve this difficulty, some authors—for example, Allon and Federgruen [4]—have replaced the right-hand side of (8) by its positive part:  $d(p) = [a - Rp]^+ = \max\{a - Rp, 0\}$ , effectively truncating the demand functions at zero. This works well in *single-product* competition models. However, as pointed out by Soon et al. [60] and Farahat and Perakis [32], the truncation procedure fails when firms sell multiple products. Consider, for example, a monopoly with two products, products 1 and 2, with a demand function  $d_1(p_1, p_2) = \max\{a_1 - b_{11}p_1 + b_{12}p_2, 0\}$  and  $d_2(p_1, p_2) = \max\{a_2 - b_{22}p_2 + b_{21}p_1, 0\}$ , where  $a_i, b_{ii} > 0$ ,  $i = 1, 2$ , and  $b_{ij} > 0$  for  $i \neq j$ ; setting the price of product 1 above its marginal cost while increasing the price of product 2 to infinity leads to zero profits for product 2 but an infinite profit value for product 1.

Instead, Shubik and Levitan [59] suggest that the extension of the demand functions beyond  $P$  satisfy the following intuitive regularity condition.

**Definition 1 (Regularity).** A demand function  $D(p): \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  is said to be *regular* if for any price vector  $p$  and product  $l$ ,  $D_l(p) = 0 \Rightarrow D(p + \Delta e_l) = D(p)$  for any  $\Delta > 0$ , where  $e_l$  denotes the  $l$ th unit vector.

In other words, if a product is priced out of the market, under a given price vector, an increase in its price does not affect any of the demand volumes. While intuitive and seemingly innocuous, Soon et al. [60] show that there is one, and only one, *regular* extension  $d(p)$  of the affine demand functions  $q(p)$  in (8). Under this extension, the demand volumes generated under an arbitrary price vector  $p$  are obtained by applying the affine function  $q(\cdot)$  to the *projection*  $\Omega(p)$  of  $p$  onto the polyhedron  $P$ ; i.e.,

$$d(p) = q(\Omega(p)),$$

where we have the following definition.

**Definition 2 (Projection).** For any  $p \in \mathbb{R}_+^N$ , the projection  $\Omega(p)$  of  $p$  onto  $P$  is defined as the vector  $p' = p - t$ , with  $t$  as the unique solution to the following linear complementarity problem (LCP):  $0 \leq d(p) = a - R(p - t) \perp t \geq 0$ ; i.e.,

$$\begin{aligned} d(p) = a - R(p - t) &\geq 0, & t &\geq 0, & \text{and} \\ t^\top [a - R(p - t)] &= 0, \end{aligned} \tag{9}$$

where the superscript “ $\top$ ” is the transpose operator of a vector or matrix.

The above LCP has a unique solution so that the projection operator is well defined, as long as the matrix  $R$  is positive definite; see, e.g., Cottle et al. [26, Theorem 3.1.6]. We therefore assume the following.

**Assumption (P).** *The matrix  $R$  is positive definite.*

There are many standard numerical procedures to verify the positive definiteness property. The following dominant diagonality conditions are often assumed in the literature, as they are very intuitive and likely to hold in many applications:

- (D) (strict row-dominant diagonality)  $R_{l,l} > \sum_{l' \neq l} |R_{l,l'}|$ , for all  $l$ ,
- (D') (strict column-dominant diagonality)  $R_{l,l} > \sum_{l' \neq l} |R_{l',l}|$ , for all  $l$ .

Condition (D) is equivalent to the assumption that no product’s demand value increases as a result of a uniform price increase of *all* products by the same amount. Condition (D') is equivalent to the assumption that aggregate sales do not increase as a result of a price

increase of any one of the products. The two dominant diagonality conditions are sufficient to ensure that  $R$  is positive definite.

The following proposition provides equivalent characterizations of the demand function  $d(p)$ .

**Proposition 1 (Demand System).** *Under Assumptions (P) and (Z), the following specifications of the demand function  $d(p)$  are equivalent:*

- (a) (Regular extension)  $d(p) = a - Rp$  for  $p \in P$ ;  $d(p)$  is a regular function.
- (b) (LCP representation) *There exists a unique vector of price corrections,  $t$ , such that  $0 \leq d(p) = a - R(p - t) \perp t \geq 0$ .*
- (c) (Projection mapping)  $d(p) = q(\Omega(p))$ . *If  $p \in P$ ,  $\Omega(p) = p$ ; if  $p \notin P$ ,  $\Omega(p)$  is on the boundary of  $P$ , where  $\Omega(p)$  is computed by minimizing any linear objective  $\phi^\top t$  with  $\phi > 0$  over the polyhedron, described by (9).*
- (d) (Piecewise linear function) *For any  $p$ , there exists a unique product assortment  $A \subseteq \mathcal{N}$  such that  $d_A(p) = a^A - R^A p_A > 0$ ,  $d_{\bar{A}}(p) = 0$ , where  $a^A \equiv a_A - R_{A,\bar{A}} R_{\bar{A},\bar{A}}^{-1} a_{\bar{A}} \geq 0$ , and  $R^A \equiv R_{A,A} - R_{A,\bar{A}} R_{\bar{A},\bar{A}}^{-1} R_{\bar{A},A}$  is a positive definite  $Z$ -matrix.*

*In the special case where the matrix  $R$  is symmetric, the following specification is equivalent to all of the above:*

- (e) (Utility maximization)  $d(p)$  solves the quadratic utility maximization problem of a representative consumer:

$$\max_{d \geq 0} \{ (R^{-1}a - p)^\top d - \frac{1}{2} d^\top R^{-1} d \}.$$

The representative consumer utility maximization problem has been suggested as an alternative foundation for the extended affine functions  $d(\cdot)$ ; see, e.g., Farahat and Perakis [31, 32]. However, this formulation can only be used when  $R$  is symmetric. Indeed, the matrix  $R$  may not be symmetric, as is typically the case when it is estimated in empirical studies; see, e.g., Dubé and Manchanda [29] and Vilcassim et al. [63].

### 6.2. The Retailer Competition Model

To characterize the equilibrium behavior in the two-echelon competition model, we need to understand how the retailers respond to a given vector of wholesale prices  $w \geq 0$  selected by the suppliers. Thus, let

$$\begin{aligned} \pi_{ijk}(p) &= \text{the profit earned by retailer } i \text{ from the sale of product } (j, k) \\ &= (p_{ijk} - w_{ijk})d_{ijk}(p), \quad \text{and} \\ \pi_i(p) &= \sum_{(j,k) \in \mathcal{N}(i)} \pi_{ijk}(p) = \text{the aggregate profit earned by retailer } i. \end{aligned}$$

In response to a given vector of wholesale prices  $w$ , the retailers engage in a price competition game, with profit functions  $\{\pi_i(p) : i \in \mathcal{I}\}$  and feasible price space  $p \in \mathbb{R}_+^N$ . Federgruen and Hu [37] show that these profit functions are *not* supermodular over the full feasible strategy space. Nevertheless, Federgruen and Hu [34] characterize the equilibrium behavior in this retailer competition game. To summarize the main results, we need to define the following vectors and matrices:

$$T^r(R) = \begin{pmatrix} R_{\mathcal{N}(1),\mathcal{N}(1)}^\top & 0 & \cdots & 0 \\ 0 & R_{\mathcal{N}(2),\mathcal{N}(2)}^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\mathcal{N}(I),\mathcal{N}(I)}^\top \end{pmatrix}, \tag{10}$$

$$\Psi^r(R) = T^r(R)[R + T^r(R)]^{-1}, \tag{11}$$

$$S \equiv \Psi^r(R)R = T^r(R)[R + T^r(R)]^{-1}R \quad \text{and} \quad b \equiv \Psi^r(R)a. \tag{12}$$

Federgruen and Hu [34] show that a pure Nash equilibrium always exists. Often, multiple, sometimes infinitely many, equilibria exist.<sup>6</sup> There may even be multiple equilibria within  $P$  itself, along with equilibria outside of  $P$ . For any equilibrium  $\tilde{p}$  outside of  $P$ ,  $\Omega(\tilde{p})$  ( $\in P$ , see Proposition 1(c)) is an equilibrium as well, which is *equivalent* to  $\tilde{p}$  in the sense of generating an identical product assortment with identical equilibrium sales volumes. The set of equilibria may therefore be partitioned into one or multiple equivalency classes, each with a component-wise smallest element that belongs to  $P$ .

Federgruen and Hu [37] recently showed, however, that *one* of the equilibria, which we refer to as  $(p^* | w)$ , has the unique property of being globally stable and this even when firms adjust their prices employing the so-called (unique) *robust* best response mapping  $RB(\cdot)$ . This mapping is defined as follows: For any price vector  $p \in \mathbb{R}_+^N$ ,

$$RB_i(p_{-\mathcal{N}(i)}) = \arg \max_{p_{\mathcal{N}(i)} \geq 0} \{ (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}), \quad (13)$$

$$\text{s.t. } a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)} \geq 0 \}.$$

This best response mapping is well defined since the optimal problem within the curly brackets in (13) is a strictly concave program. Global stability of this best response mapping means that, regardless of the market's starting point  $p^0$ , a dynamic adjustment process based on this best response converges to  $(p^* | w)$ , i.e.,

$$\lim_{n \rightarrow \infty} RB^{(n)}(p^0) = (p^* | w), \quad (14)$$

where  $RB^{(n)}(\cdot)$  denotes the  $n$ -fold application of the mapping  $RB(\cdot)$ . Moreover, convergence in (14) is geometrically fast since Federgruen and Hu [37] show that the robust best response mapping  $RB(\cdot)$  is a contraction mapping.

It should be noted that under the robust best response mapping  $RB(\cdot)$ , firms do not necessarily select the fully best response among all possible price vectors. Instead, it determines the best choice among the polyhedron  $P_i(p_{-\mathcal{N}(i)}) = \{p_{\mathcal{N}(i)} \geq 0 \mid a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)}p_{-\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)} \geq 0\}$ . The big advantage of *this* “best” response mapping is that it may be applied by each firm, with very *limited local* information only. That is, any firm only needs to know the (affine) raw demand functions  $q_{\mathcal{N}(i)}(\cdot)$  that pertain to its own products as well as its own products’ cost rates  $w_{\mathcal{N}(i)}$ . As discussed extensively in the literature, see, e.g., Vives [64] and Holt and Roth [45], this global stability property adds majorly to the predictability of the equilibrium. Finally, this special equilibrium  $(p^* | w)$  has a relatively simple analytical representation as follows:

$$(p^* | w) = w + [R + T^r(R)]^{-1}q(w) = [R + T^r(R)]^{-1}a + [R + T^r(R)]^{-1}T^r(R)w,$$

if  $w \in W \equiv \{w \geq 0 \mid Q(w) = b - Sw \geq 0\}$ , where  $b = \Psi^r(R)a$  and  $S = \Psi^r(R)R$ . In other words, as long as the wholesale price vector  $w \in W$ , our special equilibrium  $(p^* | w)$  is given by an affine vector function of  $w$ , with the same structural property as the function  $q(\cdot)$ , i.e.,  $b \geq 0$  and  $S$  is a  $ZP$ -matrix.

When  $w \notin W$ , we have

$$(p^* | w) = w' + [R + T^r(R)]^{-1}q(w') = [R + T^r(R)]^{-1}a + [R + T^r(R)]^{-1}T^r(R)w',$$

where  $w' = \Theta(w)$  is the projection of  $w$  onto the *effective wholesale price polyhedron*  $W$ , defined as in Definition 2. That is,  $w' \equiv w - t$ , with  $t$  the unique vector to the following LCP:  $t \geq 0, b - S(w - t) \geq 0$  and  $t^\top [b - S(w - t)] = 0$ . (The uniqueness of the solution to

<sup>6</sup> The characterizations below represent a refinement of the results in Federgruen and Hu [34]. The latter contains an erroneous result; in particular, contrary to the statement in Theorem 1 there,  $P$  may contain more than one (non-equivalent) equilibrium.



the above LCP is guaranteed by the positive definiteness of matrix  $S$ , which can be proven under Assumption (P.)

These characterizations of the equilibrium behavior in the retailer competition model are all obtained under the minimal conditions (P) and (Z) (i.e., a positive definite matrix  $R$  with nonpositive off-diagonal elements), except that an additional condition is needed to ensure that the projection  $\Theta(\cdot)$  onto the effective wholesale price polyhedron  $W$  indeed projects any  $w \in \mathbb{R}_+^N$  onto a nonnegative price vector  $w' = \Theta(w) \in \mathbb{R}_+^N$ . For the projection operator  $\Omega(\cdot)$  on the effective retail price polyhedron  $P$ , we mentioned that  $\Omega(p) \in \mathbb{R}_+^N$  whenever  $p \in \mathbb{R}_+^N$ ; see Federgruen and Hu [34, Lemma 1 and Proposition 1(a)]. However, the proof of that result relies on the fact that  $a \geq 0$  and on Assumptions (P) and (Z). The analogous properties almost invariably apply to the intercept vector  $b$  and the matrix of price sensitivities  $S$  in the supplier competition game as well, but they occasionally fail, as a counterexample in Federgruen and Hu [34, Example C.1] reveals. To have the projection onto  $W$  well defined, we need to have that  $b \geq 0$  and that  $S$  is a (positive definite)  $Z$ -matrix. This condition is easily verified from the primitives of the model (i.e., the vector  $a$  and the matrix  $R$ ), with a single matrix inversion and a few matrix multiplications; see (10)–(12). It can be shown that this condition is satisfied when  $R$  is symmetric. See, however, Federgruen and Hu [34, Proposition 3] for more general sufficient conditions.

One such sufficient condition is that the matrix  $T^r(R)$  is symmetric. Symmetry of the matrix  $T^r(R)$  means that the cross-price sensitivity coefficients need to be identical only for pairs of products sold by the same retailer. This symmetry assumption is considerably weaker than the global symmetry assumption of the full matrix  $R$ , typically assumed in models with affine demand functions; see, e.g., Farahat and Perakis [32]. (When demand functions  $d(p)$  are derived from a representative consumer maximizing a quadratic utility function, the resulting matrix  $R$  of price sensitivity coefficients is always symmetric.)

We are now ready to summarize the equilibrium behavior in the retailer competition model.

**Theorem 7 (Retailer Competition Model; Federgruen and Hu [34]).** Fix  $w \geq 0$ . Suppose  $R$  is symmetric and Assumptions (P) and (Z) hold.

- (a) The retailer competition game has a pure Nash equilibrium.
- (b) Multiple pure Nash equilibria may exist; however, there exists a unique equilibrium  $p^*$  which is globally stable under the robust best response mapping, and  $p^* \in P$ .
- (c) If  $w \in W$ ,  $p^* = w + [R + T^r(R)]^{-1}q(w) = [R + T^r(R)]^{-1}a + [R + T^r(R)]^{-1}T^r(R)w$  is an affine function of  $w$ .
- (d) If  $w \notin W$ ,  $p^* = w' + [R + T^r(R)]^{-1}q(w') = [R + T^r(R)]^{-1}a + [R + T^r(R)]^{-1}T^r(R)w'$ , where  $w' = \Theta(w)$  is the projection of  $w$  onto  $W$ .
- (e) The equilibrium  $p^*$  is increasing in  $w$ .

Symmetry of  $R$  is only required when  $w \notin W$ . Even when  $w \in W$ , many of the results in Theorem 7 can be guaranteed simply on the basis of Assumptions (P) and (Z) alone; in particular, there exists at most one robust and globally stable equilibrium  $p^*$  in  $P$ , and if  $\tilde{p} \in P$  is an equilibrium, then its projection  $\Omega(\tilde{p})$  is an equilibrium as well. Thus, if an equilibrium exists, there is a component-wise smallest equivalent equilibrium. As mentioned above, the symmetry assumption regarding the matrix  $R$  may be replaced by one of several more complex but weaker conditions—in particular, assumption (NPW) in Federgruen and Hu [34]; see Proposition 3 therein.

### 6.3. The Two-Echelon Competition Model

To simplify the exposition, we henceforth confine ourselves to symmetric  $R$  matrices, assuming the following.

**Assumption (S).** The matrix  $R$  is symmetric.

By Theorem 7, any wholesale price vector  $w \geq 0$  induces a retailer competition game with a unique equilibrium that is globally stable (under the robust best response mapping). We, henceforth, assume that the retailers adopt this equilibrium ( $p^* | w$ ) with its *unique* characteristics. If  $w \in W$ , the equilibrium sales volumes are given by

$$\begin{aligned} Q(w) &= d(p^*) = q(p^*) = a - R w - R[R + T^r(R)]^{-1}q(w) \\ &= \{I - R[R + T^r(R)]^{-1}\}q(w) = \{[R + T^r(R)][R + T^r(R)]^{-1} - R[R + T^r(R)]^{-1}\}q(w) \\ &= T^r(R)[R + T^r(R)]^{-1}q(w) = \Psi^r(R)q(w) = b - S w, \end{aligned}$$

where  $I$  is an identity matrix, the second identity follows from  $p^* \in P$ , and the third identity from Theorem 7(c). Similarly, if  $w \notin W$ , we have by Theorem 7(d) that the equilibrium sales volumes are given by

$$Q(w') = \Psi^r(R)q(w') = b - S w'.$$

Thus, the induced demand functions encountered by the suppliers are the (unique) regular extension of the affine functions  $Q(w) = b - S w$ , as long as the matrix  $S$  is positive definite and has nonpositive off-diagonal elements in the same way the original matrix of price sensitivity coefficients  $R$  does, in view of Assumptions (P) and (Z). Fortunately, both properties can be shown to apply under Assumptions (P), (Z), and (S). This implies that the supplier competition model is structurally analogous to the retailer competition model.

To draw an analogy between the retailer and supplier competition models, we reorder the products so that their supplier index in the triple of indices  $(i, j, k)$  comes *first*. Let  $\mathcal{M}(1), \mathcal{M}(2), \dots, \mathcal{M}(J)$  denote the sets of products supplied by supplier  $1, 2, \dots, J$ , respectively. Define the matrix

$$T^s(S) \equiv \begin{pmatrix} S_{\mathcal{M}(1), \mathcal{M}(1)}^\top & 0 & \cdots & 0 \\ 0 & S_{\mathcal{M}(2), \mathcal{M}(2)}^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\mathcal{M}(J), \mathcal{M}(J)}^\top \end{pmatrix},$$

and  $\Psi^s(S) \equiv T^s(S)[S + T^s(S)]^{-1}$ . Analogous to the definition of the effective wholesale price polyhedron  $W$ , define the *effective supply cost polyhedron*  $C$  as follows:  $C = \{c \geq 0 | \Psi^s(S)Q(c) \geq 0\}$ . Let  $\Gamma(\cdot)$  denote the projection operator onto  $C$ .

**Theorem 8 (Supplier Competition in the First Stage of the Two-Stage Game; Federgruen and Hu [35]).** Fix  $c \geq 0$ . Suppose Assumptions (P), (Z), and (S) hold.

- (a) The supplier competition game has a pure Nash equilibrium.
- (b) Multiple pure Nash equilibria may exist; however, there exists a unique equilibrium  $w^*$ , which is globally stable under the robust best response mapping.
- (c)  $C \neq \emptyset$ , since  $0 \leq c^0 = S^{-1}b = R^{-1}a \in C$ .
- (d) If  $c \in C$ ,  $w^* = c + [S + T^s(S)]^{-1}Q(c) = [S + T^s(S)]^{-1}b + [S + T^s(S)]^{-1}T^s(S)c$  is an affine function of  $w$ .
- (e) If  $c \notin C$ ,  $w^* = c' + [S + T^s(S)]^{-1}Q(c') = [S + T^s(S)]^{-1}b + [S + T^s(S)]^{-1}T^s(S)c'$ , where  $c' = \Gamma(c)$  is the projection of  $c$  onto  $C$ .

Under Assumptions (P), (Z), and (S), we have  $\Psi^s(S)b \geq 0$ , and  $\Psi^s(S)S$  is a  $Z$ -matrix, which ensures that the projection onto the polyhedron  $C$  in the space of cost rate vectors is well defined, in the sense that any vector  $c \in \mathbb{R}_+^N$  is projected onto a *nonnegative* vector  $c' \in C$ .

### 6.4. Comparative Statics

In this subsection, we characterize the impact of the model parameters on equilibrium wholesale and retail prices, equilibrium sales volumes, and product assortments under Assumptions (P), (Z), and (S). Among the model primitives, we focus on the suppliers' cost rate vector  $c$  and the intercept vector  $a$  of the demand functions  $q(\cdot)$ , as both relate to important managerial questions.

**6.4.1. Comparative Statics with Respect to Cost Rates.** In the marketing literature, much attention has been devoted to understanding the “pass-through” rates of exogenous cost changes: when a supplier changes the wholesale price for a given product, how will the different retailers respond to this price change for the same product (“direct pass-through rate”) as well as for other products in the same product category (“cross-brand pass-through rates”)? Here, we characterize the impact of changes in the suppliers' cost rates on equilibrium prices, sales volumes, and the product assortment. All effects are computable with little effort, requiring *at most* a few matrix multiplications and inversions and the solution of a single linear program with  $N$  variables and constraints.

**Theorem 9 (Comparative Statics with Respect to  $c$ ; Federgruen and Hu [35]).**

Fix a cost rate vector  $c^0$  and a product  $l = (i, j, k)$ , and consider the impact of an increase of  $c_{ijk}$  from  $c_{ijk}^0$  to  $c'_{ijk} = c_{ijk}^0 + \delta$ .

(a) (Demand volumes) *There exists a minimal threshold  $\Delta^+ \geq 0$  such that an increase of  $\delta$  beyond  $\Delta^+$  has no impact on any of the equilibrium demand volumes; when  $\delta \leq \Delta^+$ , product  $l$ 's demand volume decreases and the demand volume of all other products increases.*

(b) (Assortment) *An increase of  $\delta$  beyond  $\Delta^+$  has no impact on the equilibrium assortment; when  $\delta \leq \Delta^+$ , the equilibrium assortment remains the same or expands. There exists a second threshold  $\Delta \leq \Delta^+$  such that, for  $\delta \leq \Delta$ , the equilibrium assortment does not change while product  $l$ 's demand volume decreases and that of all other products  $l' \neq l$  increases proportionally with  $\delta$ .*

(c) (Wholesale and retail prices) *The unique equilibrium retail and wholesale price vectors  $p^*$  and  $w^*$ , which are globally stable under the robust best response mapping, increase concavely with  $\delta$ .*

Beyond the various monotonicity properties in Theorem 9, our model allows for simple expressions of the marginal pass-through rates of cost changes. We derive simpler lower and upper bounds that provide insights into the pass-through rates. For example, the lower bound shows that at least 50% of a reduction in the wholesale price of a product and at least 25% of a reduction in the supply cost rate are passed on to the consumer.

**Proposition 2. (Bounds for the Cost Pass-Through Rates; Federgruen and Hu [35]).**

(a) *Consider the retailer competition model under a given wholesale price vector  $w$ . Let  $\mathcal{A}$  denote the equilibrium assortment:*

$$\frac{1}{2} \leq \left( \frac{\partial p_{\mathcal{A}}^*}{\partial w_{\mathcal{A}}} \right)^- = [R^{\mathcal{A}} + T(R^{\mathcal{A}})]^{-1} T(R^{\mathcal{A}}) \leq \frac{(R^{\mathcal{A}})^{-1} T(R^{\mathcal{A}})}{2},$$

where  $(\partial p^*/\partial w)^-$  denotes the matrix of left-hand derivatives, which is shown to always exist.

(b) *Fix  $c \in \mathbb{R}_+^N$ . Let  $\mathcal{A}$  denote the unique assortment associated with the price equilibrium  $p^*$  in the two-stage game:*

$$\begin{aligned} \frac{1}{4} \leq \left( \frac{\partial p_{\mathcal{A}}^*}{\partial c_{\mathcal{A}}} \right)^- &= [R^{\mathcal{A}} + T(R^{\mathcal{A}})]^{-1} T(R^{\mathcal{A}}) [S^{\mathcal{A}} + T(S^{\mathcal{A}})]^{-1} T(S^{\mathcal{A}}) \\ &\leq \frac{(R^{\mathcal{A}})^{-1} T(R^{\mathcal{A}}) (S^{\mathcal{A}})^{-1} T(S^{\mathcal{A}})}{4}. \end{aligned}$$

**6.4.2. Comparative Statics with Respect to the Intercept Vector.** Goldfarb et al. [42] have argued that a firm’s brand value should be measured in an equilibrium framework. More specifically, consumer demand functions should be modeled as a function of the suppliers’ and/or retailers’ brands, represented by brand indicator variables. The brand value of a firm is then defined as the *difference* between its profit value when the brand indicator variable equals 1 (i.e., in the presence of the brand effect) and a counterfactual equilibrium value when it is set equal to 0 (i.e., in the absence of the brand effect). More specifically, Goldfarb et al. [42] assume that demands for the various products are specified by a mixed MNL model in which the *intercept* of the utility measure of each product is specified as an affine function of suppliers’ brand indicator variables. Following the same approach, in our demand model, we specify the intercepts as follows:

$$a_{ijk} = \alpha^\top x_{ijk} + \sum_{j'=1}^J \beta_{j'} z_{jj'}, \quad i = 1, 2, \dots, I, j = 1, 2, \dots, J, k = 1, 2, \dots, |\mathcal{K}(i, j)|,$$

where  $z_{jj'} = 1$  if  $j = j'$  and  $z_{jj'} = 0$  if  $j \neq j'$ ;  $x_{ijk}$  represents a vector of observable attribute values for product  $(i, j, k)$ . The same methodology may be used to measure brand values associated with the different retailers or with different sub-brands (i.e.,  $(j, k)$ -combinations). All of these brand value estimations amount to conducting comparative statics analyses with respect to the intercept vector  $a$ .

Here, we summarize those comparative statics results. We show, in particular, that all equilibrium retail and wholesale prices increase and that the equilibrium product assortment expands when the intercept vector  $a$  increases. In addition, an increase of one of or more of the intercept values causes all of the suppliers’ and retailers’ profit margins to grow, as well as their aggregate profit values. One implication is that all brand values, as defined in Goldfarb et al. [42], are *positive*.

**Theorem 10 (Comparative Statics with Respect to  $a$ ; Federgruen and Hu [35]).**

Fix  $c \geq 0$  and  $0 \leq a^1 \leq a^2$ . An increase in  $a$  elicits an increase in the equilibrium wholesale and retail prices, assortment, demand volumes, and the retailers’ and suppliers’ profit margins for all products. It also increases each firm’s profit level and expands the product assortment. In other words,

- (a) (Wholesale prices)  $w^*(\Gamma(c, a^1), a^1) \leq w^*(\Gamma(c, a^2), a^2)$ .
- (b) (Retail prices)  $p^*(w^*(\Gamma(c, a^1), a^1), a^1) \leq p^*(w^*(\Gamma(c, a^2), a^2), a^2)$ .
- (c) (Demand volumes)  $d(p^*(w^*(\Gamma(c, a^1), a^1), a^1)) \leq d(p^*(w^*(\Gamma(c, a^2), a^2), a^2))$ .
- (d) (Assortment)  $\mathcal{A}(a^1) \subseteq \mathcal{A}(a^2)$ .
- (e) (Retail profit margins)  $p^*(w^*(\Gamma(c, a^1), a^1), a^1) - w^*(\Gamma(c, a^1), a^1) \leq p^*(w^*(\Gamma(c, a^2), a^2), a^2) - w^*(\Gamma(c, a^2), a^2)$ .
- (f) (Wholesale profit margins)  $w^*(\Gamma(c, a^1), a^1) - \Gamma(c, a^1) \leq w^*(\Gamma(c, a^2), a^2) - \Gamma(c, a^2)$ .
- (g) (Profit levels) The profit earned by each firm increases with the intercept vector  $a$ .

**6.5. The Multi-Echelon Competition Model**

In this subsection, we discuss the generalization of the two-echelon model to one in which products (potentially) travel through an arbitrary number of distribution/production stages before reaching the end consumer. In the chain of oligopolies, there are  $m$  echelons,  $E_1, \dots, E_m$ , each with an arbitrary number of competing distributors. We number the echelons sequentially, starting with the most downstream echelon of retailers until reaching the most upstream echelon  $m$ . We assume that firms in a given echelon only sell to firms in the next more downstream echelon; i.e., firms in echelon  $e$  only sell to those in echelon  $e - 1$ , while the retailers in echelon 1 sell to the consumer. (Settings where for some of the products one or more echelons are skipped can be handled as well, analogous to the treatment of the “direct sales” channel; see Federgruen and Hu [36, Section 5].)

Products are partially differentiated by the route  $r$  traveled in the above multipartite network. For any such path  $r \in \mathcal{R}$ , the set of all possible paths, there may be up to  $K$  distinct products. We thus label each distinct product with a pair of indices  $(r, k)$ : product  $(r, k)$  is the  $k$ th product distributed along the route  $r$ ,  $r \in \mathcal{R}$ , and  $k = 1, \dots, K$ .

Our starting point is, again, a set of retailer demand functions  $d^{(1)}(p^{(1)})$ , with  $p^{(1)}$  the vector of retail prices, specified as follows on  $\mathbb{R}_+^N$ :

$$d^{(1)}(p^{(1)}) \equiv \begin{cases} q^{(1)}(p^{(1)}) = a^{(1)} - R^{(1)}p^{(1)} & \text{if } p^{(1)} \in P^{(1)} \equiv \{p^{(1)} \geq 0 \mid q^{(1)}(p^{(1)}) \geq 0\}, \\ q^{(1)}(\Omega^{(1)}(p^{(1)})) & \text{if } p^{(1)} \notin P^{(1)}. \end{cases} \quad (15)$$

Here,  $a^{(1)}$  and  $R^{(1)}$  are exogenously given, and  $\Omega^{(1)}(p^{(1)})$  is the projection of the vector  $p^{(1)}$  onto  $P^{(1)}$ , defined by Definition 2, with  $a$  and  $R$  replaced by  $a^{(1)}$  and  $R^{(1)}$ . Following the analysis of Section 6.3, one verifies that *each* of the remaining echelons  $e = 2, \dots, m$  experiences equilibrium demand functions of a similar structure. Define, recursively,

$$\begin{aligned} a^{(e)} &\equiv \Psi^{(e)}(R^{(e-1)})a^{(e-1)}, \\ R^{(e)} &\equiv \Psi^{(e)}(R^{(e-1)})R^{(e-1)}, \end{aligned} \quad (16)$$

where  $\Psi^{(e)}(R^{(e-1)}) \equiv T^{(e-1)}(R^{(e-1)})[R^{(e-1)} + T^{(e-1)}(R^{(e-1)})]^{-1}$ ,  $e = 2, \dots, m$ . The matrix  $T^{(e)}(R^{(e)})$  is obtained from the matrix  $R^{(e)}$  by replacing by zero any entry that corresponds to a pair of products that is distributed via different distributors in echelon  $e$ ; for any pair of products  $(l, l')$  that is distributed via the *same* distributor in echelon  $e$ ,  $T^{(e)}(R^{(e)})_{l,l'} = R^{(e)}_{l,l'}$ . As shown in Section 6.3, after appropriate sequencing of the products, the matrix  $T^{(e)}(R^{(e)})$  is block diagonal, with each block corresponding to a specific distributor in echelon  $e$ . The indirect demand functions for the firms in echelon  $e$  are, again, of the form given by (15):

$$d^{(e)}(p^{(e)}) \equiv \begin{cases} q^{(e)}(p^{(e)}) = a^{(e)} - R^{(e)}p^{(e)} & \text{if } p^{(e)} \in P^{(e)} \equiv \{p^{(e)} \geq 0 \mid q^{(e)}(p^{(e)}) \geq 0\}, \\ q^{(e)}(\Omega^{(e)}(p^{(e)})) & \text{if } p^{(e)} \notin P^{(e)}, \end{cases}$$

where  $\Omega^{(e)}(p^{(e)})$  is the projection of the price vector  $p^{(e)}$  onto  $P^{(e)}$ ,  $e = 2, \dots, m$ . As before  $P^{(e)} \neq \emptyset$ , since  $0 \leq (R^{(e)})^{-1}a^{(e)} = (R^{(e-1)})^{-1}a^{(e-1)} = \dots = R^{-1}a \in P^{(e)}$ ,  $e = 2, \dots, m$ .

One recursively verifies that the matrix  $R^{(e)}$ ,  $e = 2, \dots, m$ , is positive definite. By Theorem 7(a), this, by itself, guarantees the existence of equilibria at any stage of the sequential competition game.

However, to ensure that the indirect equilibrium demand functions for echelon  $e$  are well defined, we, of course, need to establish that an essentially unique sales volume equilibrium exists in a certain sense at any downstream echelon  $l = 1, 2, \dots, e - 1$ —i.e., the equivalency of equilibria in a robust sense of generating the identical sales volumes and profit levels throughout all downstream echelons (if multiple equilibria exist, they all project onto the *same* price vector in  $P^{(l)}$  that is globally stable under the robust best response mapping for any downstream echelon  $l = 1, 2, \dots, e - 1$ ). By Theorem 7 (or Theorem 8), this is guaranteed as long as each of the matrices  $R^{(1)}, R^{(2)}, \dots, R^{(e)}$ , is a  $Z$ -matrix (i.e., has nonpositive off-diagonal elements), and  $a^{(1)}, a^{(2)}, \dots, a^{(e)} \geq 0$ . This condition can easily be checked numerically. In addition, the condition can be guaranteed, inductively, when  $R^{(1)} = R$  is a symmetric matrix. (If  $R^{(e)}$  is symmetric, then  $R^{(e+1)} = \Psi^{(e+1)}(R^{(e)})R^{(e)} = [(R^{(e)})^{-1} + T^{(e)}(R^{(e)})^{-1}]^{-1}$ ; thus,  $R^{(e+1)\top} = [(R^{(e)\top})^{-1} + (T^{(e)}(R^{(e)}))^\top]^{-1} = [(R^{(e)})^{-1} + (T^{(e)}(R^{(e)}))^{-1}]^{-1} = R^{(e+1)}$  is symmetric as well.)

We conclude that as long as we can guarantee that each of the matrices  $R^{(1)}, R^{(2)}, \dots, R^{(m+1)}$  is a  $Z$ -matrix and  $a^{(1)}, a^{(2)}, \dots, a^{(m+1)} \geq 0$  (where  $R^{(m+1)}$  and  $a^{(m+1)}$  follow the same type of definition as  $R^{(e)}$  and  $a^{(e)}$ ,  $e = 2, \dots, m$ ), one essentially unique equilibrium, which is globally stable under the robust best response mapping, exists at *each* stage of the sequential competition game; the resulting chainwide equilibrium is a subgame perfect Nash

equilibrium. Moreover, for any echelon  $e$ , the values of the unique, robustly and globally stable, price equilibrium in  $P^{(e)}$  among all equilibria, are computed as follows: Let  $c$  denote the vector of marginal cost rates incurred by the firms in the most upstream echelon:

$$p^{*(m)} = \begin{cases} c + [R^{(m)} + T^{(m)}(R^{(m)})]^{-1}q^{(m)}(c) & \text{if } c \in C \equiv P^{(m+1)}, \\ \Omega^{(m+1)}(c) + [R^{(m)} + T^{(m)}(R^{(m)})]^{-1}q^{(m)}(\Omega^{(m+1)}(c)) & \text{if } c \notin C, \end{cases} \quad (17)$$

where  $P^{(m+1)}(\ni R^{-1}a)$  follows the same type of definition as  $P^{(e)}$ ,  $e = 2, \dots, m$ , and

$$p^{*(e)} = p^{*(e+1)} + [R^{(e)} + T^{(e)}(R^{(e)})]^{-1}q^{(e)}(p^{*(e+1)}), \quad e = m - 1, \dots, 1. \quad (18)$$

To verify (17) and (18), invoke Theorem 7. Moreover, since  $p^{*(m)} \in P^{(m)}$ , it follows from Theorem 7 that  $p^{*(m-1)}$  is given by (18) and  $p^{*(m-1)} \in P^{(m-1)}$ . One thus verifies by induction that  $p^{*(e)} \in P^{(e)}$  for all echelons  $e = 1, \dots, m$ , so that (18) applies to all echelons  $e = 1, \dots, m - 1$ .

The computation of all echelons' equilibrium price vector  $\{p^{*(e)} \mid e = 1, \dots, m\}$  is thus confined to the following: one first recursively computes the matrices  $R^{(e)}$  and intercept vectors  $a^{(e)}$  for  $e = 1, \dots, m + 1$ , via (16). Determination of  $p^{*(m)}$  may involve the computation of the unique solution of an LCP—but only if  $c \notin C$ —which can be achieved by solving a single linear program with  $N$  variables and  $N$  constraints. The remaining computations involve only multiplications and inversions of matrices related to the price sensitivity matrix  $R$ .

Under a symmetric matrix  $R$ , as we move upstream, the sequence of effective price polyhedra expands in the supply chain network. Moreover, as  $m$ , the number of echelons, grows, the sequence of effective price polyhedra  $\{P^{(e)}\}$  converges to a limiting polyhedron.

**Proposition 3 (Multi-Echelon Competition; Federgruen and Hu [35]).** *Suppose Assumptions (P), (Z), and (S) hold.*

(a) *For any  $m \in \mathbb{N}$ ,  $P^{(e)} = \{p \geq 0 \mid a^{(e)} - R^{(e)}p \geq 0\} \subseteq P^{(e+1)}$  for all  $e = 1, 2, \dots, m$ .*

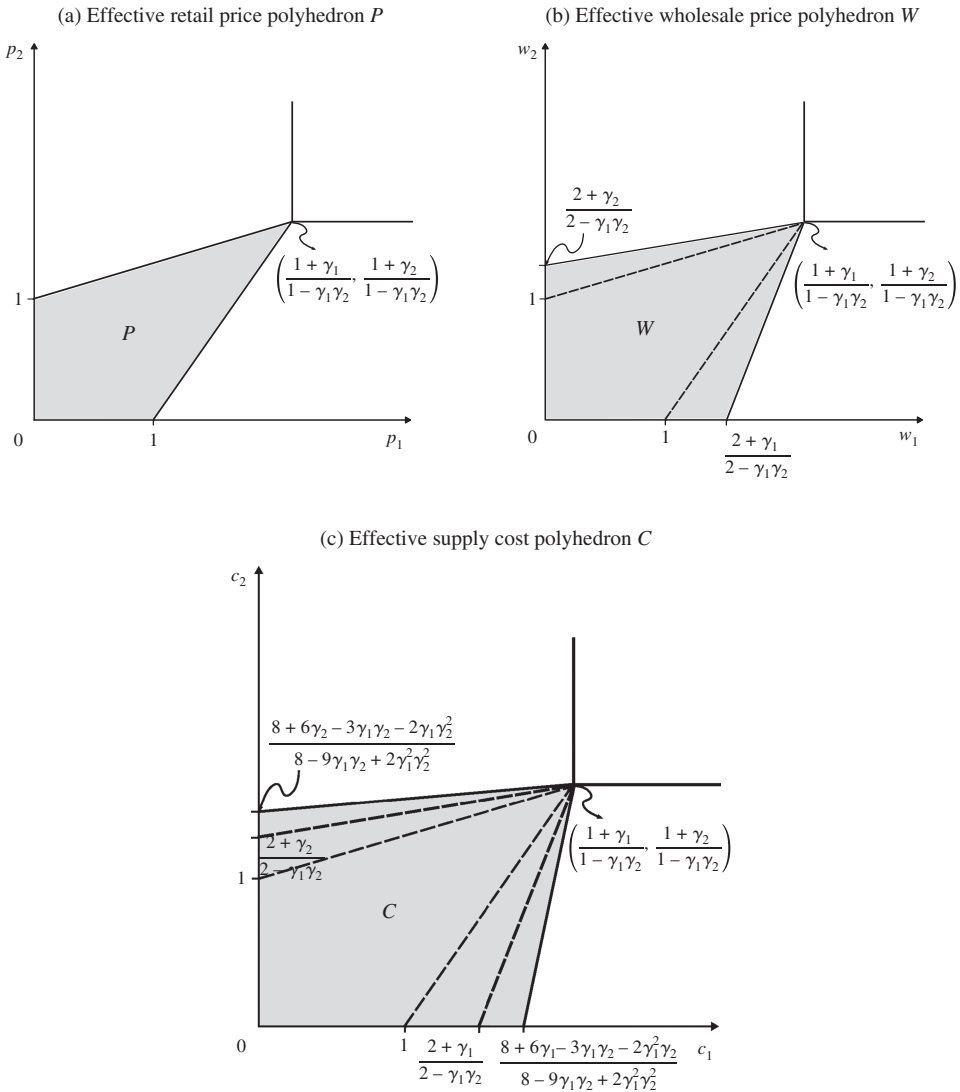
(b) *For any  $m \in \mathbb{N}$ ,  $P^{(e)} \subseteq H \equiv \{p \mid 0 \leq p \leq R^{-1}a\}$  for all  $e = 1, 2, \dots, m + 1$ .*

(c) *The sequence  $\{P^{(e)}, e = 1, 2, \dots, m + 1\}$  converges to a limiting polyhedron  $P^*$  as  $m$  increases.*

Proposition 3 has an implication for the impact of disintermediation: That is, what happens when retailers can buy products directly from their manufacturers as opposed to procuring these via a wholesaler? Consider a market with  $J$  manufacturers each selling a group of products to a manufacturer-associated chain of independently owned retailers. Initially, each manufacturer  $j$  sells its products via a dedicated wholesaler at a given price vector  $c^j$ . By Proposition 3, the effective price polyhedra are nested:  $P \subseteq W \subseteq C$ ; see Figure 1 for an example with two products for which the symmetry of matrix  $R$  is *not* required for the price polyhedra to be nested. Assume  $c = (c^1, \dots, c^J)^\top$  is in the interior of  $C \setminus W$ , and consider how the equilibrium is affected when the middlemen are cut out (i.e., the intermediate echelon of wholesalers is eliminated). On the one hand, it is possible to show that retail prices will come down. More surprisingly, however, product variety will *decrease*, all cost efficiencies notwithstanding: in the presence of the intermediary wholesalers, we get  $w^*(c) \in W^\circ$ ; hence,  $p^*(w^*(c)) \in P^\circ$  (i.e., all products are sold in the market). Without the intermediaries, the vector  $c$  becomes the new vector of “wholesale” purchase prices for the retailers. Since  $c \notin W$ ,  $p^*(c)$  is on the boundary of  $P$ , implying that some products are no longer part of the retailer assortment. Proposition 3 implies that disintermediation by eliminating a whole echelon of intermediaries would lead to lower prices in the market, therefore generating larger consumer surplus, but this can be at the cost of reducing the product variety. From the opposite angle, Proposition 3 also implies that adding intermediaries can reduce the intensity of price competition, but the marginal benefit of doing so is diminishing in view of part (c). Disintermediation involves the elimination of one echelon of firms. In the next subsection, we study the impact of vertical integration, which may only involve one or a few firms at a given echelon.



FIGURE 1. Nested effective price polyhedra.



Notes. Consider two parallel supply chains in which supplier  $i$ ,  $i = 1, 2$ , sells product  $i$  exclusively through retailer  $i$ . Let  $a = (1, 1)^T$  and  $R = \begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix}$ , with  $0 \leq \gamma_1, \gamma_2 \leq 1$ .

### 6.6. The Impact of Changes in the Network Structure: Vertical Mergers

Vertical integration occurs when a supplier acquires or merges with one or more of its client firms. Alternatively, vertical integration may result from *internal growth*: a firm in an intermediate echelon of a supply chain may integrate *forward* by creating its own direct distribution channel to final consumers or buyers at an echelon further downstream from that in the very next downstream echelon. Similarly, a firm may integrate *backward* by deciding to manufacture the products it used to buy from an upstream supplier, or to purchase them directly from firms further upstream than their traditional supplier(s). Riordan [56] surveys the economics literature on the competitive effects of vertical integration. This survey paper also reviews antitrust regulation and legal challenges of proposed vertical integrations by the U.S. Department of Justice (DOJ) and the Federal Communications Commission.

Downloaded from informs.org by [142.1.10.4] on 03 October 2017, at 20:28 . For personal use only, all rights reserved.

Till now, almost all of the literature has focused on special, relatively simple market structures in a two-echelon setting, usually with up to two firms at each echelon and at most *one* product being transacted between any pair of upstream and downstream firms, hereafter referred to as suppliers and retailers. It is, however, important to establish *robust* price effects that pertain to markets with an arbitrary number of general, nonidentical suppliers and retailers and any one of the retailers procuring any number of differentiated products from any given subset of the suppliers. Beyond characterizing whether vertical mergers result in price increases or decreases, it is important to assess the actual resulting magnitude in price and consumer welfare changes. Last but not least, it is important to understand what impact vertical mergers have on the *product variety* offered in the market. For example, even if prices decline, consumers may be disenfranchised if the vertical merger results in reduced product variety.

Product variety is one of the principal “non-price effects” used in antitrust and merger investigations in the United States (along with “quality,” “service,” and “innovation”). For example, the 2010, most recent merger guidelines by the DOJ and the Federal Trade Commission (FTC) state that, in addition to price effects, “[e]nhanced market power can also be manifested in non-price terms and conditions that adversely affect customers, including reduced product quality, reduced *product variety*, reduced service, or diminished innovation” (see U.S. Department of Justice and Federal Trade Commission [62, p. 2]). While the 2010 guidelines are focused on horizontal mergers, the concerns are no less prevalent for vertical mergers—and for good reasons, as shown in Federgruen and Hu [36]. Moreover, legal scholars such as Averitt and Lande [6, 7], economists such as Carlton and Israel [21], and former FTC Commissioners Leary [48, 49] and Rosch [57] have argued that the above 2010 FTC and DOJ guidelines “do not go far in their recognition of non-price effects.” Finally, in 2016, the American Antitrust Institute devoted its annual symposium to the topic of “Non-Price Effects of Mergers” (see also Gundlach [43]).

Building on the two-echelon competition model introduced in Section 6.3, Federgruen and Hu [36] study the simultaneous price and variety effects of vertical mergers. In particular, Federgruen and Hu [36] consider a vertical merger between one of the suppliers and a group of retailers this supplier sells to, possibly along with other suppliers. (The supplier may, likewise, sell to other retailers outside of the group she merges with.) After the merger, any outside supplier continues to offer the merged company all the products she used to offer one of the merging retailers; similarly, the merging supplier continues to offer outside retailers any of the products she offered them before the merger. All postmerger performance measures of interest can be efficiently computed. These measures include the robustly and globally stable wholesale and retail price equilibria, their associated product assortment, sales volumes of the different products, firms’ profit values, and a consumer and social welfare measure. When characterizing the postmerger equilibrium behavior, it is reasonable to assume that the retail prices of the products sold by the merged company continue to be determined along with all other retail prices (i.e., as part of the second-stage retailer competition game). However, in some settings, these prices may be determined by the merged company during the first-stage competition game. The sequence in which price decisions are made is often of critical importance. Federgruen and Hu [36] therefore cover both settings. When the merged company discloses its retail prices up front during the first competition stage, this is equivalent to a setting where those products are sold *directly* from the (merging) supplier to the end consumer. This gives rise to a new type of sequential oligopoly, with “*direct sales*,” in contrast to the premerger, strictly hierarchical network structure.

In a general setting, it is impossible to show *categoric directional* changes for either the prices or the product variety: On the one hand, the vertical merger eliminates some of the double marginalization and hence tends, in and of itself, to reduce prices. However, when a *group* of retailers participates in the merger, the merger also has a horizontal component.

Horizontal mergers tend to result in *increased* prices, since they reduce the number of competing retail organizations. The net impact of these two conflicting forces depends on the specifics of the network structure, the demand functions, and, last but not least, the exogenously specified cost rates incurred by the suppliers.

To isolate the impact of a *vertical* merger, Federgruen and Hu [36] analyze the merger of a supplier with a single retailer with whom she has an exclusive relationship: the supplier sells exclusively via the retailer and the retailer sources all of his products from this supplier. In the more common setting where all retail prices continue to be determined during the second-stage competition game, let  $\hat{C}$  denote the postmerger effective cost rate polyhedron, and let  $\mathcal{N}^*$  ( $\hat{\mathcal{N}}^*$ ) denote the set of products sold in the premerger (postmerger) world. Recall that the *premerger* robust wholesale price equilibrium is given by  $w^*(\Gamma(c))$  and the robust retail price equilibrium by  $p^*(w^*(\Gamma(c)))$ . We denote the corresponding postmerger robust wholesale and retail price equilibria by  $\hat{w}^*(\hat{\Gamma}(c))$  and  $\hat{p}^*(\hat{w}^*(\hat{\Gamma}(c)))$ , respectively.

**Theorem 11 (Merger of a Supplier with an Exclusive Retailer; Federgruen and Hu [36]).** *Suppose Assumptions (P), (Z), and (S) hold.*

- (a) (Effective cost rate polyhedron)  $\hat{C} \subseteq C$ .
- (b) (Assortment)  $\hat{\mathcal{N}}^* \subseteq \mathcal{N}^*$ .
- (c) (Wholesale prices)  $\hat{w}^*(\hat{\Gamma}(c)) \leq w^*(\Gamma(c))$ .
- (d) (Retail prices)  $\hat{p}^*(\hat{w}^*(\hat{\Gamma}(c))) \leq p^*(w^*(\Gamma(c)))$ .

Theorem 11 shows that after the vertical merger, the robust and globally stable wholesale price equilibrium decreases, as does the robust and globally stable retail price equilibrium. Moreover, the postmerger equilibrium product assortment is a subset of its premerger counterpart. In other words, any product that failed to be competitive in the premerger market continues to be uncompetitive after the merger; moreover, the merger may “elbow” some of the products out of the market. The intuition is as follows. Because a vertical merger eliminates double marginalization for the integrated firm, it leads to lower equilibrium prices. Some products that have high marginal costs with positive demand levels in the premerger world can be priced out of the market after the merger, as the vertical merger drives down the equilibrium prices. This result implies that vertical mergers increase consumer welfare as the market prices decrease; however, this may be achieved at the cost of a reduction of product variety in the market.

In the alternative setting where the (retail) prices of the merged firm’s products are determined and disclosed in the *first*-stage game, Federgruen and Hu [36] obtain somewhat more restricted comparison results: for any given vector of wholesale prices, the corresponding robust and globally stable retail price equilibrium is smaller than before the merger, and the equilibrium product variety is (weakly) reduced as a result of the merger. When the cost rate vector belongs to the above *premerger* effective cost rate polyhedron  $C$ , *all* robust equilibrium wholesale and retail prices weakly *decline*, as does the product assortment. However, price increases or expansions of the product assortment may arise for other cost rate vectors.

## 6.7. Coexistence of Substitutable and Complementary Products

In this subsection, we extend Section 6.2 to settings with complementary products in addition to pairs of products that are substitutes for each other. A pair of products  $(l, l')$  are complements if their cross-price elasticities of demand are *negative*, such that a price increase for product  $l$  reduces the demand for product  $l'$ , and vice versa.

Little has been established in the general literature on price competition with complementary products, even with demand systems under which *all* potential products are sold in the market, for *any* possible price combination in the feasible price region. Consider, for

example, the transcendental logarithmic demand functions, first introduced by Christensen et al. [24] (see also Milgrom and Roberts [54]):

$$\log d_n(p) = \alpha^n + \sum_{j \in \mathcal{N}} \beta_j^n \log(p_j) + \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N}} \gamma_{ij}^n \log(p_i) \log(p_j), \quad \text{where } \beta_n^n < 0, \gamma_{nn}^n < 0.$$

Note that in this type of model, every product  $n$  is sold in the market, irrespective of what price combinations are chosen. If  $\beta_j^n > 0$  and  $\gamma_{nj}^n \geq 0$  for all  $j \neq n$ , it is easily verified that all products are substitutes. Moreover, the price competition game is supermodular (see Milgrom and Roberts [54]), guaranteeing the existence of a Nash equilibrium, along with several other equilibrium properties. On the other hand, if  $\beta_j^n < 0$  and  $\gamma_{nj}^n < 0$  for all  $j \neq n$ , all products are complements. In this case, it is easily verified that the profit functions are *log-submodular*. This implies that the best response functions are decreasing; in such games, it is usually impossible to establish that a pure Nash equilibrium exists; see, e.g., Vives [64, Section 2.3.2]. (Note also that if some or all of the coefficients  $\beta_j^n < 0$  and  $\gamma_{nj}^n \leq 0$ , the items' profit functions also fail to be quasiconcave in their own price variables, so that the existence of a pure-strategy equilibrium cannot be established on the basis of the Debreu theorem, either.)

Clearly, treatment of complementary products is even considerably more challenging in a demand system that allows retailers (and wholesalers) to select among the potential products that they want to carry, based on the various price choices by all members of the supply chain. Nevertheless, we have been able to extend our results to allow for systems with complementary products (i.e., a matrix  $R$  that fails to be a  $Z$ -matrix) under the following relaxation of the  $Z$ -property.

**Assumption (C) (Substitutes and Complements).** *The matrix  $R$  satisfies*

- (a)  $R_{\mathcal{N}(i), -\mathcal{N}(i)} \leq 0$  for any retailer  $i$ ; and
- (b)  $R_{\mathcal{N}(i), \mathcal{N}(i)}^{-1} \geq 0$  for any retailer  $i$ .

Assumption (C)(a) stipulates that products carried by *distinct* retailers are gross substitutes. Products carried by the same retailer may be complements, as long as the inverse of each of the retailers' matrices  $R_{\mathcal{N}(i), \mathcal{N}(i)}$  is nonnegative. (If  $R$  is positive definite, each of the submatrices  $R_{\mathcal{N}(i), \mathcal{N}(i)}$  is positive definite and hence invertible as well.) Note that Assumption (C) is a significant generalization of Assumption (Z). While the former requires that *all* off-diagonal elements of the  $R$ -matrix—hence, all cross-elasticities—are nonpositive, the requirement is now restricted to products (potentially) sold by different retailers (Assumption (C)(a)). Note also that Assumption (C)(b) is easily satisfied when  $R$  is a  $Z$ -matrix. However, the condition is considerably more general.

We first characterize the equilibrium behavior in the retailer competition model, when relaxing Assumption (Z) to Assumption (C).

Our main result in characterizing the equilibrium behavior of the retailer competition game continues to apply; i.e., there exists a pure Nash equilibrium. Moreover, the effective retail price polyhedron  $P$  contains a *unique* equilibrium that is globally stable under the robust best response mapping.

**Theorem 12 (Existence of Price Equilibrium in Retailer Competition Model; Federgruen and Hu [33]).** *Suppose Assumptions (P) and (C) hold.*

- (a) *If  $w \in W$ ,  $p^*(w) = w + [R + T^r(R)]^{-1}q(w)$  is a unique equilibrium in  $P$  that is globally stable under the robust best response mapping.*
- (b) *If  $w \notin W$ , assume Assumption (NPW) (a weaker condition than Assumption (S); see Federgruen and Hu [34]) holds, then  $p^*(w') = w' + [R + T^r(R)]^{-1}q(w')$  is a unique equilibrium in  $P$  that is globally stable (under the robust best response mapping), where  $w' = \Theta(w)$  is the projection of  $w$  onto  $W$ .*

Even in Section 6.2, where all products are substitutes (i.e.,  $R$  is a  $Z$ -matrix), there may be any number of equilibria outside of  $P$ . However, in Section 6.2, any equilibrium outside  $P$  is equivalent to one on the boundary of  $P$  in the sense of generating the identical sales volumes and profit levels for all retailers and products. In the generalized model where Assumption (Z) is relaxed to Assumption (C), this equivalency result may fail to hold. We now develop sufficient conditions for such an equivalency. It amounts to establishing the following two results:

- (a) If  $p \notin P$ , the projection  $\Omega(p) \in P$ .
- (b) If  $p^* \notin P$  is an equilibrium, then  $\Omega(p^*)$  is an equivalent equilibrium.

To establish (b), two auxiliary results are needed:

- (b1) The retailers' market is *competitive*; i.e., for any pair of products  $l$  and  $l'$  that are sold by different retailers,  $d_l(p)$  increases when  $p_{l'}$  increases.
- (b2) For any firm, there exists a best response to a given competitors' price vector such that it employs prices at levels greater than or equal to the corresponding wholesale prices.

As mentioned, (b1) appears to be an immediate consequence of Assumption (C)(a), guaranteeing that cross-elasticities of the "raw" demand for pairs of products  $(l, l')$  sold by different retailers are nonnegative; i.e.,  $q_l(p)$  increases when  $p_{l'}$  increases. However, when  $R$  fails to be a  $Z$ -matrix, the *actual* demand function  $d(\cdot)$  may fail to inherit the "competitiveness property" of the raw demand functions. Several sufficient conditions *do* guarantee (b1), but the simplest such condition is symmetry for the matrix  $R$ . We therefore assume symmetry going forward (i.e., Assumption (S)).

**Proposition 4 (A Competitive Retailer Market; Federgruen and Hu [33]).** *Suppose Assumptions (P), (C), and (S) hold. For any pair of products  $(l, l')$  sold via different retailers,  $d_l$ , the sales volume of product  $l$  increases with the price  $p_{l'}$  of product  $l'$ .*

We establish result (b2) under the following condition.

**Assumption (C<sup>+</sup>).** *The matrix  $R$  satisfies*

- (a)  $R_{\tilde{\mathcal{N}}, \tilde{\mathcal{N}}}^{-1} a_{\tilde{\mathcal{N}}} \geq 0$  for any  $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ , and
- (b)  $R_{\tilde{\mathcal{N}}, \tilde{\mathcal{N}}}^{-1} R_{\tilde{\mathcal{N}}, -\mathcal{N}(i)} \leq 0$  for any  $\tilde{\mathcal{N}} \subseteq \mathcal{N}(i)$  and any retailer  $i$ .

As with Assumption (C), Assumption (C<sup>+</sup>) is a relaxation of the requirement that  $R$  is a  $Z$ -matrix: when  $R$  is a  $Z$ -matrix,  $R_{\tilde{\mathcal{N}}, \tilde{\mathcal{N}}}$  is both positive definite and a  $Z$ -matrix for any  $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ . This implies that  $R_{\tilde{\mathcal{N}}, \tilde{\mathcal{N}}}^{-1} \geq 0$ . Combined with the assumption  $a \geq 0$ , this completes the verification of Assumptions (C<sup>+</sup>)(a) and (C<sup>+</sup>)(b).

**Proposition 5 (Federgruen and Hu [33]).** *Suppose Assumptions (P), (C), (C<sup>+</sup>), and (S) hold. For any  $w \geq 0$ , there exists a best response vector  $\bar{p} \geq w$ .*

Finally, we turn to condition (a). When the matrix  $R$  fails to be a  $Z$ -matrix, the projection  $\hat{p} = \Omega(p)$  onto the polyhedron  $P$  of a vector  $p \notin P$  may fail to belong to  $P$ : while  $\hat{p}$  satisfies the inequalities  $q(\hat{p}) \geq 0$ , the condition  $\hat{p} \geq 0$  fails to be guaranteed. Soon et al. [60, Lemma 6] shows that Assumption (C<sup>+</sup>)(a) is the necessary and sufficient condition to guarantee that  $\Omega(p) \in P$  for all  $p \in \mathbb{R}_+^N$ . We are now ready to provide a set of sufficient conditions under which any equilibrium  $\hat{p}$  outside of  $P$ ,  $\Omega(\hat{p})$  is an equivalent equilibrium, and as a result, there exists an essentially unique equilibrium which is globally stable under the robust best response mapping.

**Theorem 13 (Equivalent Equilibria; Federgruen and Hu [33]).** *Suppose Assumptions (P), (C), (C<sup>+</sup>), and (S) hold. Fix  $w \geq 0$ . Any equilibrium  $p^o$  of the retailer competition model is equivalent to  $\hat{p} = \Omega(p^o)$  and  $p^*(w')$  is the essentially unique equilibrium which is globally stable under the robust best response mapping, where  $w' = \Theta(w)$ .*

We conclude that, as in Section 6.2, with *substitutable products only*, the equilibrium in the retailer competition model is *essentially* unique under the robust best response mapping with the conditions of Theorem 13. Moreover, in view of Section 6.3, the induced demand functions for the suppliers can be structurally identical to the demand functions  $d(\cdot)$  faced by the retailers. This implies that we can extend the characterization of the equilibria to the first-stage supplier competition game, as we did in Section 6.3.

## 6.8. Summary and Future Work

The demand model introduced in Section 6.1 has many advantages: First, it is compact and characterized by a single  $N \times N$  matrix,  $R$ , of price sensitivity coefficients along with a single intercept vector for the affine part of the demand functions. Second, depending on what prices are selected, a different subset of all potential products is offered in the market. Thus, the model specifies a *product assortment*, along with specifically associated demand volumes. This is in sharp contrast to all other commonly used demand models. For example, under the various variants of the MNL model discussed in Section 2, *all* products attain some market share, irrespective of their absolute and relative price levels. Under the demand model of this section, we are able to obtain a characterization of the equilibrium behavior of multiechelon competition with a plethora of analytical comparative statics results. Moreover, the equilibria can be computed with only a few matrix multiplications and inverses, in some cases combined with the solution of a single linear program in  $N$  variables and  $N$  constraints. This, in turn, enables computational comparative statics if analytical comparisons are intractable. We expect to use the framework as a workhorse model to answer other questions of managerial importance, such as the impact of horizontal mergers and more general network structure changes. Future research may also focus on extending the framework to account for demand uncertainty, capacity constraints, and dynamic interactions.

## References

- [1] E. Adida and V. DeMiguel. Supply chain competition with multiple manufacturers and retailers. *Operations Research* 59(1):156–172, 2011.
- [2] K. Ailawadi, P. Farris, and E. Shames. Trade promotion: Essential to selling through resellers. *Sloan Management Review* 41(1):83–92, 1999.
- [3] M. Aksoy-Pierson, G. Allon, and A. Federgruen. Price competition under mixed multinomial logit demand functions. *Management Science* 59(8):1817–1835, 2013.
- [4] G. Allon and A. Federgruen. Competition in service industries. *Operations Research* 55(1):37–55, 2007.
- [5] S. P. Anderson, A. de Palma, and J.-F. Thisse. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge, MA, 1992.
- [6] N. W. Averitt and R. H. Lande. Consumer sovereignty: A unified theory of antitrust and consumer protection law. *Antitrust Law Journal* 65(3):713–756, 1997.
- [7] N. W. Averitt and R. H. Lande. Consumer choice: The practical reason for both antitrust and consumer protection law. *Loyola Consumer Law Review* 10(1):44–63, 1998.
- [8] M. E. Ben-Akiva. Structure of passenger travel demand models. PhD thesis, Massachusetts Institute of Technology, Cambridge, 1973.
- [9] F. Bernstein and A. Federgruen. Pricing and replenishment strategies in a distribution system with competing retailers. *Operations Research* 51(3):409–426, 2003.
- [10] F. Bernstein and A. Federgruen. A general equilibrium model for industries with price and service competition. *Operations Research* 52(6):868–886, 2004.
- [11] F. Bernstein and A. Federgruen. Decentralized supply chains with competing retailers under demand uncertainty. *Management Science* 51(1):18–29, 2005.
- [12] F. Bernstein, F. Chen, and A. Federgruen. Coordinating supply chains with simple pricing schemes: The role of vendor-managed inventories. *Management Science* 52(10):1483–1492, 2006.
- [13] S. Berry, J. Levinsohn, and A. Pakes. Automobile prices in market equilibrium. *Econometrica* 63(4):841–890, 1995.



- [14] J. Bertrand. Review of “Théorie mathématique de la richesse sociale” and “Recherches sur les principes mathématiques de la théorie des richesses.” *Journal de Savants* 67:499–508, 1883.
- [15] K. Bimpikis, S. Ehsani, and R. Ilklic. Cournot competition in networked markets. Working paper, Stanford University, Stanford, CA, 2016.
- [16] J. H. Boyd and R. E. Mellman. The effect of fuel economy standards on the US automotive market: A hedonic demand analysis. *Transportation Research A* 14(5):367–378, 1980.
- [17] C. Brown and J. L. Medoff. Cheaper by the dozen. Technical report, Harvard Institute of Economic Research, Cambridge, MA, 1991.
- [18] G. P. Cachon. Competitive supply chain inventory management. S. Tayur, M. Magazine, and R. Ganeshan, eds. *Quantitative Models for Supply Chain Management*. Kluwer Academic Publishers, Boston, 111–146, 1999.
- [19] G. P. Cachon and M. A. Lariviere. Supply chain coordination with revenue-sharing contracts: Strengths and limitations. *Management Science* 51(1):30–44, 2005.
- [20] N. S. Cardell and F. C. Dunbar. Measuring the societal impacts of automobile downsizing. *Transportation Research Part A* 14(5):423–434, 1980.
- [21] D. W. Carlton and M. Israel. Will the new guidelines clarify or obscure antitrust policy? *Antitrust Source* October, 2010.
- [22] F. Chen, A. Federgruen, and Y.-S. Zheng. Coordination mechanisms for a distribution system with one supplier and multiple retailers. *Management Science* 47(5):693–708, 2001.
- [23] S.-H. Cho. Horizontal mergers in multitier decentralized supply chains. *Management Science* 60(2):356–379, 2014.
- [24] L. R. Christensen, D. W. Jorgenson, and L. J. Lau. Transcendental logarithmic production frontiers. *Review of Economics and Statistics* 55(1):28–45, 1973.
- [25] C. Corbett and U. Karmarkar. Competition and structure in serial supply chains with deterministic demand. *Management Science* 47(7):966–978, 2001.
- [26] R. W. Cottle, J. S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Boston, 1992.
- [27] G. Debreu. Review: *Individual choice behavior: A theoretical analysis* by R. Duncan Luce. *American Economic Review* 50(1):186–188, 1960.
- [28] V. DeMiguel and H. Xu. A stochastic multiple-leader Stackelberg model: Analysis, computation, and application. *Operations Research* 57(5):1220–1235, 2009.
- [29] J.-P. Dubé and P. Manchanda. Differences in dynamic brand competition across markets: An empirical analysis. *Marketing Science* 24(1):81–95, 2005.
- [30] Dun & Bradstreet. *Industry Norms and Key Business Ratios (2000–2001)*. Dun & Bradstreet, Inc., New York, 2001.
- [31] A. Farahat and G. Perakis. Profit loss in differentiated oligopolies. *Operations Research Letters* 37(1):43–46, 2009.
- [32] A. Farahat and G. Perakis. A nonnegative extension of the affine demand function and equilibrium analysis for multiproduct price competition. *Operations Research Letters* 38(4):280–286, 2010.
- [33] A. Federgruen and M. Hu. Price competition in sequential multi-product oligopolies. Unabridged memo, Columbia University, New York, <https://ssrn.com/abstract=2437032>, 2014.
- [34] A. Federgruen and M. Hu. Multi-product price and assortment competition. *Operations Research* 63(3):572–584, 2015.
- [35] A. Federgruen and M. Hu. Sequential multiproduct price competition in supply chain networks. *Operations Research* 64(1):135–149, 2016.
- [36] A. Federgruen and M. Hu. The price and variety effects of vertical mergers. Working paper, Columbia University, New York, <https://ssrn.com/abstract=2830156>, 2016.
- [37] A. Federgruen and M. Hu. Global and robust stability in a general price and assortment competition model. Working paper, Columbia University, New York, <https://ssrn.com/abstract=3010503>, 2017.
- [38] D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, Cambridge, MA, 1991.
- [39] G. Gallego and M. Hu. Dynamic pricing of perishable assets under competition. *Management Science* 60(5):1241–1259, 2014.
- [40] G. Gallego and R. Wang. Multi-product price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research* 62(2):450–461, 2014.

- [41] G. Gallego, W. T. Huh, W. Kang, and R. Phillips. Price competition with the attraction demand model: Existence of unique equilibrium and its stability. *Manufacturing & Service Operations Management* 8(4):359–375, 2006.
- [42] A. Goldfarb, Q. Lu, and S. Moorthy. Measuring brand value in an equilibrium framework. *Marketing Science* 28(1):69–86, 2009.
- [43] G. Gundlach. Non-price effects of mergers: A primer. Presentation, American Antitrust Institute Invitational Symposium on the Non-Price Effects of Mergers, June 15, National Press Club, Washington, DC, 2016.
- [44] F. W. Harris. What quantity to make at once. *Factory Management Series: Operation and Costs*, Library of Factory Management, Vol. V. A. W. Shaw Company, Chicago, 47–52, 1915.
- [45] C. A. Holt and A. E. Roth. The Nash equilibrium: A perspective. *Proceedings of the National Academy of Sciences*, 101(12):3999–4002, 2004.
- [46] R. Lal, J. D. C. Little, and J. M. Villas-Boas. A theory of forward buying, merchandising, and trade deals. *Marketing Science* 15(1):21–37, 1996.
- [47] M. A. Lariviere. Supply chain contracting and coordination with stochastic demand. S. Tayur, M. Magazine, and R. Ganeshan, eds. *Quantitative Models for Supply Chain Management*. Kluwer Academic Publishers, Boston, 233–268, 1999.
- [48] T. B. Leary. The significance of variety in antitrust analysis. Speech, Steptoe & Johnson 2000 Antitrust Conference, May 18, Federal Trade Commission, Washington, DC, 2000.
- [49] T. B. Leary. The significance of variety in antitrust analysis. *Antitrust Law Journal* 68(3): 1007–1022, 2001.
- [50] D. R. Luce. *Individual Choice Behavior: A Theoretical Analysis*. John Wiley & Sons, New York, 1959.
- [51] R. D. Luce and P. Suppes. Preference, utility, and subjective probability. R. D. Luce, R. R. Bush, and E. Galanter, eds. *Handbook of Mathematical Psychology*, Vol. III. John Wiley & Sons, New York, 252–410, 1965.
- [52] D. McFadden. Conditional logit analysis of qualitative choice behavior. P. Zarembka, ed. *Frontiers in Econometrics*. Academic Press, New York, 105–142, 1974.
- [53] D. McFadden and K. Train. Mixed MNL models for discrete response. *Journal of Applied Econometrics* 15(5):447–470, 2000.
- [54] P. Milgrom and J. Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* 58(6):1255–1277, 1990.
- [55] C. L. Munson and M. J. Rosenblatt. Theories and realities of quantity discounts: An exploratory study. *Production and Operations Management* 7(4):352–369, 1998.
- [56] M. H. Riordan. Competitive effects of vertical integration. P. Buccirossi, ed. *Handbook of Antitrust Economics*. Massachusetts Institute of Technology, Cambridge, 145–182, 2008.
- [57] J. T. Rosch. Economics in merger analysis. *Antitrust Bulletin* 59(1):111–128, 2014.
- [58] R. Roundy. 98%-effective integer-ratio lot-sizing for one-warehouse multi-retailer systems. *Management Science* 31(11):1416–1430, 1985.
- [59] M. Shubik and R. Levitan. *Market Structure and Behavior*. Harvard University Press, Cambridge, MA, 1980.
- [60] W. Soon, G. Zhao, and J. Zhang. Complementarity demand functions and pricing models for multi-product markets. *European Journal of Applied Mathematics* 20(5):399–430, 2009.
- [61] G. J. Tellis. The price sensitivity of competitive demand: A meta-analysis of sales response models. *Journal of Marketing Research* 15(3):331–341, 1988.
- [62] U.S. Department of Justice and Federal Trade Commission. Horizontal merger guidelines. Guidelines, U.S. Department of Justice, Washington, DC, 2010.
- [63] N. J. Vilcassim, V. Kadiyali, and P. K. Chintagunta. Investigating dynamic multifirm market interactions in price and advertising. *Management Science* 45(4):499–518, 1999.
- [64] X. Vives. *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press, Cambridge, MA, 2001.
- [65] R. B. Westin. Predictions from binary choice models. *Journal of Econometrics* 2(1):1–16, 1974.