Online Appendix to "Multi-Product Price and Assortment Competition" Preliminaries, Proofs and Examples.

A. Preliminaries

We use the following properties of ZP-matrices.

LEMMA A.1 (Properties of ZP-matrices). *Let X be a ZP-matrix and Y be a Z-matrix such that $X \leq Y$, i.e., $Y - X \geq 0$. Then*

(a) X^{-1} exists and $X^{-1} \geq 0$;

(b) Y is a ZP-matrix and $Y^{-1} \leq X^{-1}$;

(c) XY^{-1} and $Y^{-1}X$ are ZP-matrices; and

(d) If D is a positive diagonal matrix, then DX, XD and $X+D$ are ZP-matrices.

Proof of Lemma A.1. (a)-(d). By Horn and Johnson (1991, Theorem 2.5.3), a ZP-matrix is a nonsingular, so-called, M-matrix. Properties (a)-(d) of ZP-matrices can be found in Horn and Johnson (1991, Section 2.5) as properties of M-matrices. □

Moreover, we use the following lemma.

LEMMA A.2. *(a) Suppose X, Y and $X+Y$ are invertible matrices of the same order. Then $X^{-1}+Y^{-1}$ is nonsingular and $(X^{-1}+Y^{-1})^{-1} = X(Y+X)^{-1}Y = Y(X+Y)^{-1}X$.*

(b) If X is positive definite, then the bisymmetric matrix $\begin{pmatrix} X & -Y^T \\ Y & 0 \end{pmatrix}$ is positive semi-definite.

(c) If X is positive semi-definite and Y is a P_0-matrix, then $M = X+Y$ is a P_0-matrix.

Proof of Lemma A.2. (a) Since $X^{-1} + Y^{-1} = (\mathrm{I} + Y^{-1}X)X^{-1}$, $(X^{-1} + Y^{-1})^{-1} = X(\mathrm{I} + Y^{-1}X)^{-1} = X(Y+X)^{-1}Y$. The second equality follows from $X^{-1}+Y^{-1} = Y^{-1}+X^{-1}$ and the proven first equality.

(b) We verify this directly from the definition of positive semi-definiteness. For any z,

$$z^T \begin{pmatrix} X & -Y^T \\ Y & 0 \end{pmatrix} z = (z_1^T, z_2^T) \begin{pmatrix} X & -Y^T \\ Y & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1^T X z_1 + z_2^T Y z_1 - z_1^T Y^T z_2 = z_1^T X z_1 \geq 0,$$

since X is positive definite.

(c) Suppose M is not a P_0-matrix. By one of several equivalent definitions of a P_0-matrix (Theorem 3.4.2 (b) in Cottle et al. 1992), there exists $\tilde{z} \neq 0$, such that for all k, either $\tilde{z}_k = 0$ or $\tilde{z}_k(M\tilde{z})_k < 0$. In other words, there exists $\tilde{z} \neq 0$ such that $\tilde{z}_k(M\tilde{z})_k = \tilde{z}_k(X\tilde{z})_k + \tilde{z}_k(Y\tilde{z})_k < 0$ for all k satisfying $\tilde{z}_k \neq 0$. Since X is positive semi-definite, $\tilde{z}_k(X\tilde{z})_k \geq 0$. Hence, we have $\tilde{z} \neq 0$ such that $\tilde{z}_k(Y\tilde{z})_k < 0$ for all k satisfying $\tilde{z}_k \neq 0$, which contradicts the fact that Y is a P_0-matrix, thus proving the lemma by contradiction. □

B. Proofs for Sections 2 and 3

Proof of Proposition 1. (a)⇔(b). Under Assumptions (P) and (Z), for any $\tilde{\mathcal{N}} \subseteq \mathcal{N}$, $R_{\tilde{\mathcal{N}},\tilde{\mathcal{N}}}$ is a ZP-matrix and hence $R^{-1}_{\tilde{\mathcal{N}},\tilde{\mathcal{N}}} \geq 0$ by Lemma A.1(a). Since $a \geq 0$,

$$R^{-1}_{\tilde{\mathcal{N}},\tilde{\mathcal{N}}} a_{\tilde{\mathcal{N}}} \geq 0, \qquad \text{for all subsets } \tilde{\mathcal{N}} \subseteq \mathcal{N}. \tag{B.1}$$

By Lemma 6 and Theorem 4 in Soon et al. (2009), we have the desired result.

(c)⇔(b). The Lagrangian associated with (2) is given by:

$$(R^{-1}a - p)^T d - \frac{1}{2} d^T R^{-1} d + t^T d = [R^{-1}a - (p-t)]^T d - \frac{1}{2} d^T R^{-1} d.$$

Since R is symmetric, (3) represents the complementarity conditions for the quadratic program (2), which by Assumption (P) are both necessary and sufficient to characterize a unique optimum. □

Lemma B.1. *Under conditions*

$$\begin{aligned} &(D) \quad (\textit{strict row dominant diagonality}) &\quad R_{ik,ik} > \sum_{(i',k') \neq (i,k)} |R_{ik,i'k'}|, \quad \forall (i,k), \\ &(D') \; (\textit{strict column dominant diagonality}) &\quad R_{ik,ik} > \sum_{(i',k') \neq (i,k)} |R_{i'k',ik}|, \quad \forall (i,k), \end{aligned}$$

R is positive definite.

Proof of Lemma B.1. Since R is strictly row and column diagonally dominant with positive diagonal entries, $\frac{1}{2}(R+R^T)$ is symmetric and strictly row diagonally dominant with positive diagonal entries. By Horn and Johnson (1985, Corollary 7.2.3), $\frac{1}{2}(R+R^T)$ is positive definite. The desired result follows because R is positive definite if and only if $\frac{1}{2}(R+R^T)$ is positive definite (Horn and Johnson 1985, P. 399). □

Proof of Lemma 1. (a) By the definition of $\Omega(p) = p - t$, $\Omega(p)$ satisfies (3) so that $q(\Omega(p)) \geq 0$. Lemma 6 in Soon et al. (2009) shows that the necessary and sufficient condition for $\Omega(p) \geq 0$, for any $p \in \mathbb{R}^N_+$, is given by (B.1), which holds, as shown in the proof of Proposition 1.

(b) If $p \in P$, $t = 0$ is the unique solution to (3). □

Proposition B.1. *For any product $l = (i,k)$, $d_l(p)$ is decreasing in its own price and increasing in the price of any other product $l' \neq l$.*

Proof of Proposition B.1. The fact that $d_l(\cdot)$ is decreasing in p_l follows from Theorem 8 of Soon et al. (2009). The fact that $d_l(\cdot)$ is increasing in the prices of the other products was shown in Corollary 1 of Farahat and Perakis (2010), noting that the proof of that corollary does not depend on the matrix R being symmetric, but depend on the matrix R being a Z-matrix. □

Proposition B.2 (Non-negative profit margins in best responses). *Fix $w \geq 0$ and $i \in \mathcal{I}$. For any price choices $p_{-\mathcal{N}(i)}$ by retailer i's competitors, there exists a best response $p^*_{\mathcal{N}(i)}(p_{-\mathcal{N}(i)}) \geq w_{\mathcal{N}(i)}$.*

Proof of Proposition B.2. Assume for some product (i,k), $\hat{p}_{ik}(p_{-\mathcal{N}(i)}) < w_{ik}$. Increasing $\hat{p}_{ik}$ to a value $\geq w_{ik}$ improves the profit earned for *this* product, while, by Proposition B.1, increasing the sales volume and hence the profit earned for all other products sold by retailer i with a non-negative profit margin. Thus, sequentially increasing each of the prices $\hat{p}_{ik} < w_{ik}$ to the w_{ik}-level results in a profit improvement while ensuring that all profit margins are non-negative. □

Proof of Theorem 1. We first prove parts (b) and (c): Suppose $p^o \in \mathbb{R}^N_+ \setminus P$ is an equilibrium. By Proposition 1(b), there exists a unique $t \geq 0$ such that $0 \leq d(p^o) = a - R\Omega(p^o) = a - R(p^o - t)$ and $t^T[a - R(p^o - t)] = 0$. Clearly, $t_l > 0$ for some product l; otherwise, $p^o \in P$. By the complementarity condition, $d_l(p^o) = 0$. Let $\hat{p} = \Omega(p^o) = p^o - t$. By Proposition 1(b), $0 \leq d(p^o) = q(\hat{p}) = d(\hat{p})$ and $\hat{p} \in P$. Clearly, $\hat{p} \leq p^o$ and $\hat{p} \neq p^o$. Then for any retailer i,

$$\begin{aligned} \pi_i(p^o) &= (p^o_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p^o) \\ &= (p^o_{\mathcal{N}(i)} - t_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p^o) \\ &= (\hat{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(\hat{p}) = \pi_i(\hat{p}), \end{aligned} \tag{B.2}$$

where the second equality is due to the complementarity of t and $d(p^o)$. By Proposition B.2, for given $\hat{p}_{-\mathcal{N}(i)}$, there exists a best response $\bar{p}_{\mathcal{N}(i)} \geq 0$ such that $\bar{p}_{\mathcal{N}(i)} \geq w_{\mathcal{N}(i)}$. Then

$$\begin{aligned} \pi_i(\hat{p}) &\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)})] \\ &= (\bar{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(\bar{p}_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)}) \\ &\leq (\bar{p}_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(\bar{p}_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)}) \\ &\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)})] \\ &= \pi_i(p^o) = \pi_i(\hat{p}), \end{aligned} \tag{B.3}$$

where the second inequality is due to $\bar{p}_{\mathcal{N}(i)} \geq w_{\mathcal{N}(i)}$ and Proposition B.1; the latter guarantees that $0 \leq d_{\mathcal{N}(i)}(\bar{p}_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)}) \leq d_{\mathcal{N}(i)}(\bar{p}_{\mathcal{N}(i)}, p^o_{-\mathcal{N}(i)})$. The last equality follows from (B.2). Thus all inequalities in (B.3) hold as equalities and in particular, $\pi_i(\hat{p}) = \max_{p_{\mathcal{N}(i)} \geq 0}[(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, \hat{p}_{-\mathcal{N}(i)})]$. Hence $\hat{p} = \Omega(p^o)$ is another equilibrium of the retailers' price competition game, and $d(\Omega(p^o)) = d(p^o)$, $\pi(\Omega(p^o)) = \pi(p^o)$. By part (a), this implies that a unique equilibrium $p^* \in P$ exists and $\Omega(p^o) = p^*$. Moreover, p^o and p^* are equivalent.

(a) Soon et al. (2009, Theorem 15) showed that there exists at least one equilibrium p^o. In view of part (b), this implies that an equilibrium can be found in P, since $\Omega(p^o) \in P$, for all $p^o \in \mathbb{R}^N_+$. It remains to show that, within P, no alternative equilibria exist.

In conjunction with the *full* competition game in which each retailer is able to select an arbitrary price vector, we consider a *restricted* game in which the industry-wide price vector p must be selected within the polyhedron P. This is a *generalized* Nash game with coupled constraints, a term coined by Rosen (1965), i.e., even the feasible price range for any retailer i depends on the price choices made by the competitors; see also Topkis (1998) for a treatment of such generalized games.

While the structure of the feasible strategy space is more complex in this restricted game, it has the advantage that the profit functions are simple quadratic functions, because for $p \in P$, $d(p) = q(p) = a - Rp$ is affine.

We prove a stronger result, namely that even the restricted game has at most a single equilibrium in P. (If $p^* \in P$ is an equilibrium of the full price game, it is, a fortiori, an equilibrium of the restricted game.) In the restricted game, all feasible price vectors $p \in P$, so that $d(p) = q(p) = a - Rp$. For any equilibrium p^o in the restricted game and any retailer i, $p^o_{N(i)}$ must solve the quadratic program:

$$\begin{aligned} \max_{p_{\mathcal{N}(i)}} \; & (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^o_{-\mathcal{N}(i)}) \\ \text{s.t.} \quad & a - Rp \geq 0 \quad \text{and} \quad p_{\mathcal{N}(i)} \geq 0. \end{aligned}$$

This quadratic program may be formulated as

$$\begin{aligned} \min_{p_{\mathcal{N}(i)}} \; & -(w^T_{\mathcal{N}(i)} R_{\mathcal{N}(i),\mathcal{N}(i)} + a^T_{\mathcal{N}(i)} - (p^o_{-\mathcal{N}(i)})^T R^T_{\mathcal{N}(i),-\mathcal{N}(i)}) p_{\mathcal{N}(i)} + \frac{1}{2} p^T_{\mathcal{N}(i)} (2R_{\mathcal{N}(i),\mathcal{N}(i)}) p_{\mathcal{N}(i)} \\ & + w^T_{\mathcal{N}(i)} (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^o_{-\mathcal{N}(i)}) \\ \text{s.t.} \quad & -Rp \geq -a, \qquad \text{(B.4)} \\ & p_{\mathcal{N}(i)} \geq 0. \qquad \text{(B.5)} \end{aligned}$$

Since R is positive definite, $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is positive definite, as well. Let $y^i \geq 0$ and $s_{\mathcal{N}(i)} \geq 0$ denote the Lagrange multipliers associated with the constraint sets (B.4) and (B.5), respectively. Also let $t^i = (t^i_{\mathcal{N}(i)}, t^i_{-\mathcal{N}(i)}) \geq 0$ denote the surplus variables of constraint set (B.4). Since $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is positive definite, the optimal solution to this quadratic program is the unique solution to the complementarity conditions:

$$\begin{pmatrix} s_{\mathcal{N}(i)} \\ t^i_{\mathcal{N}(i)} \\ t^i_{-\mathcal{N}(i)} \end{pmatrix} - \begin{pmatrix} R_{\mathcal{N}(i),\mathcal{N}(i)} + R^T_{\mathcal{N}(i),\mathcal{N}(i)} & R_{\mathcal{N}(i),-\mathcal{N}(i)} & R^T_{\mathcal{N}(i),\mathcal{N}(i)} & R^T_{-\mathcal{N}(i),\mathcal{N}(i)} \\ -R_{\mathcal{N}(i),\mathcal{N}(i)} & -R_{\mathcal{N}(i),-\mathcal{N}(i)} & 0 & 0 \\ -R_{-\mathcal{N}(i),\mathcal{N}(i)} & -R_{-\mathcal{N}(i),-\mathcal{N}(i)} & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{\mathcal{N}(i)} \\ p^o_{-\mathcal{N}(i)} \\ y^i_{\mathcal{N}(i)} \\ y^i_{-\mathcal{N}(i)} \end{pmatrix}$$

$$= \begin{pmatrix} -(R^T_{\mathcal{N}(i),\mathcal{N}(i)} w_{\mathcal{N}(i)} + a_{\mathcal{N}(i)}) \\ a_{\mathcal{N}(i)} \\ a_{-\mathcal{N}(i)} \end{pmatrix}, \quad \text{(B.6)}$$

and

$$\begin{aligned} s_{\mathcal{N}(i)} \geq 0, \quad p_{\mathcal{N}(i)} \geq 0, \quad s^T_{\mathcal{N}(i)} p_{\mathcal{N}(i)} = 0, \\ t^i = (t^i_{\mathcal{N}(i)}, t^i_{-\mathcal{N}(i)}) \geq 0, \quad y^i = (y^i_{\mathcal{N}(i)}, y^i_{-\mathcal{N}(i)}) \geq 0, \quad (t^i)^T y^i = 0. \end{aligned} \quad \text{(B.7)}$$

This implies that a price vector p is a generalized Nash equilibrium if and only if vectors $s, y^i, t^i \in \mathbb{R}^N_+$ can be found, for all i, such that (B.6) and (B.7) are satisfied for all i, *simultaneously*. In other words, the price vector p is a generalized Nash equilibrium if and only if the extended vector $(p, y^1, y^2, \ldots, y^{|\mathcal{I}|}) \in \mathbb{R}^{N(|\mathcal{I}|+1)}_+$ is a solution to a specific *master* LCP that takes the following form:

$$\begin{aligned} \left(s, t^1, \ldots, t^{|\mathcal{I}|}\right)^T - \tilde{R}\left(p, y^1, \ldots, y^{|\mathcal{I}|}\right)^T = \left(-T(R)w + a, a, \ldots, a\right)^T, \\ \left(s, t^1, \ldots, t^{|\mathcal{I}|}\right) \geq 0, \quad \left(p, y^1, \ldots, y^{|\mathcal{I}|}\right) \geq 0, \\ \left(s^T, (t^1)^T, \ldots, (t^{|\mathcal{I}|})^T\right) \begin{pmatrix} p \\ y^1 \\ \vdots \\ y^{|\mathcal{I}|} \end{pmatrix} = 0, \end{aligned}$$

where

$$\tilde{R} \equiv \begin{pmatrix} R + T(R) & \mathring{R}_{\mathcal{N}(1)} & \cdots & \mathring{R}_{\mathcal{N}(|\mathcal{I}|)} \\ -R & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N(|\mathcal{I}|+1) \times N(|\mathcal{I}|+1)},$$

$$\mathring{R}_{\mathcal{N}(i)} \equiv \begin{pmatrix} 0 \\ \vdots \\ R^T_{\mathcal{N},\mathcal{N}(i)} \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{N \times N},$$

$$T(R) \equiv \begin{pmatrix} R^T_{\mathcal{N}(1),\mathcal{N}(1)} & 0 & \cdots & 0 \\ 0 & R^T_{\mathcal{N}(2),\mathcal{N}(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R^T_{\mathcal{N}(|\mathcal{I}|),\mathcal{N}(|\mathcal{I}|)} \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

We now show that the matrix $\tilde{R}$ is a P_0-matrix. To this end, write

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2 \equiv \begin{pmatrix} R & \mathring{R}_{\mathcal{N}(1)} & \cdots & \mathring{R}_{\mathcal{N}(|\mathcal{I}|)} \\ -\mathring{R}^T_{\mathcal{N}(1)} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\mathring{R}^T_{\mathcal{N}(|\mathcal{I}|)} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} T(R) & 0 & \cdots & 0 \\ -R + \mathring{R}^T_{\mathcal{N}(1)} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -R + \mathring{R}^T_{\mathcal{N}(|\mathcal{I}|)} & 0 & 0 & 0 \end{pmatrix}.$$

Applying Lemma A.2(b) with $X = R$, $\tilde{R}_1$ is positive semi-definite. Since R is positive definite, it is easily verified that $T(R)$ is positive definite as well, and for sure, a P_0-matrix: let $z \in \mathbb{R}^N \neq 0$, and note that $z^T T(R) z = z^T_{\mathcal{N}(1)} R^T_{\mathcal{N}(1),\mathcal{N}(1)} z_{\mathcal{N}(1)} + \cdots + z^T_{\mathcal{N}(|\mathcal{I}|)} R^T_{\mathcal{N}(|\mathcal{I}|),\mathcal{N}(|\mathcal{I}|)} z_{\mathcal{N}(|\mathcal{I}|)} > 0$, since each of the $|\mathcal{I}|$ terms

$$z^T_{\mathcal{N}(i)} R^T_{\mathcal{N}(i),\mathcal{N}(i)} z_{\mathcal{N}(i)} \geq 0, \quad \text{for all } \ i, \tag{B.8}$$

with strict inequality for at least one of the terms. (To verify the inequality in (B.8), define $z^{(i)} \in \mathbb{R}^N$ as follows: $z^{(i)}_{i'k} = z_{ik}$, if $i' = i$ and $z^{(i)}_{i'k} = 0$, otherwise. Then, $z^T_{\mathcal{N}(i)} R^T_{\mathcal{N}(i),\mathcal{N}(i)} z_{\mathcal{N}(i)} = {z^{(i)}}^T R^T z^{(i)} \geq 0$.) For any principal minor of $\tilde{R}_2$, if the minor is a sub-matrix of $T(R)$, then such a minor is nonnegative since $T(R)$ is a P_0-matrix; otherwise, the minor must involve a matrix with a full column of zeros, so that the minor equals zero. Hence $\tilde{R}_2$ is a P_0-matrix. It follows from Lemma A.2(c) that $\tilde{R} = \tilde{R}_1 + \tilde{R}_2$ is a P_0-matrix. By Theorem 3.4.4 (a) in Cottle et al. (1992), this implies that the vector $(s, t^1, \ldots, t^{|\mathcal{I}|})$ is *unique* among any and all solutions to the master LCP. This implies, in particular, $a - Rp^* = t^1 = \cdots = t^{|\mathcal{I}|} \equiv \hat{t}$ is unique among any and all generalized Nash equilibria p^* in P as a solution to the master LCP. Since R is invertible, $p^* = R^{-1}(a - \hat{t})$. Since $\hat{t}$ is unique, there exists at most one generalized Nash equilibrium in P. $\square$

Proof of Proposition 2. (a) In the proof of Theorem 1, we noted that $T(R)$ is positive definite. It follows that $R + T(R)$ is positive definite, and hence is invertible, so that $p^*(w)$ is the unique solution to the FOC (6).

(b) Note that

$$\begin{aligned} q(p^*(w)) &= a - Rw - R[R + T(R)]^{-1} q(w) \\ &= \{I - R[R + T(R)]^{-1}\} q(w) \\ &= \left\{[R + T(R)][R + T(R)]^{-1} - R[R + T(R)]^{-1}\right\} q(w) \\ &= T(R)[R + T(R)]^{-1} q(w) = \Psi(R) q(w). \end{aligned}$$

Moreover, since R is a positive-definite Z-matrix, $T(R)$ is a positive-definite Z-matrix, hence a ZP-matrix, therefore $T(R)^{-1} \geq 0$. Hence $w \in W \Leftrightarrow w \geq 0$ and $\Psi(R) q(w) \geq 0 \Rightarrow p^*(w) = w + T(R)^{-1} \Psi(R) q(w) \geq w \geq 0$. In summary, if $w \in W$, then $q(p^*(w)) \geq 0$ and $p^*(w) \geq 0$, i.e., $p^*(w) \in P$. Conversely, assume $w \geq 0$ and $p^*(w) \in P$. Then $q(p^*(w)) = \Psi(R) q(w) \geq 0$, i.e., $w \in W$.

(c) $w^o = R^{-1} a \geq 0$ since $R^{-1} \geq 0$ and $a \geq 0$; moreover, $\Psi(R) q(w^o) = \Psi(R) a - \Psi(R) a = 0$.

(d) $\Psi(R) R = T(R)[R + T(R)]^{-1} R = [R^{-1} + T(R)^{-1}]^{-1}$, by Lemma A.2(a). Since both R and $T(R)$ are positive definite, the same property applies to their inverses and hence to $\Psi(R) R$.

(e) We first show that $\Psi(R) = T(R)[R + T(R)]^{-1} = [RT(R)^{-1} + I]^{-1} \geq 0$. Since R has non-positive off-diagonal elements, it follows from the definition and symmetry of $T(R)$ that $R \leq T(R)$.

Since R is a ZP-matrix and $T(R)$ is a Z-matrix, it follows from Lemma A.1(c) that $RT(R)^{-1}$ is a ZP-matrix. By Lemma A.1(d), $[RT(R)^{-1}+I]$ is a ZP-matrix, so that, by Lemma A.1(a), $\Psi(R)=[RT(R)^{-1}+I]^{-1}\geq 0$. Finally, if $w\in P$, $q(w)\geq 0$ and $\Psi(R)q(w)\geq 0$, i.e., $P\subseteq W$. □

Proof of Theorem 2. (a) Given Theorem 1, it suffices to verify that $p^*=p^*(w)$ is indeed an equilibrium retail price vector over the full strategy space $p\geq 0$. For any retailer i and any $p\geq 0$ such that $p_{-\mathcal{N}(i)}=p^*_{-\mathcal{N}(i)}$, there exists a unique vector $t\geq 0$ such that $d(p)=a-R(p-t)\geq 0$ and $t^T d(p)=0$, by Proposition 1(b). Since $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is a positive definite Z-matrix, $R_{\mathcal{N}(i),\mathcal{N}(i)}$ is a ZP-matrix, hence $R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}\geq 0$. Since R is a Z-matrix, $R_{\mathcal{N}(i),-\mathcal{N}(i)}\leq 0$. Then $-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)}\geq 0$. Hence,

$$
\begin{aligned}
\pi_i(p) &= (p_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p)\\
&= (p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p)\\
&= (p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T [a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)})-R_{\mathcal{N}(i),-\mathcal{N}(i)}(p^*_{-\mathcal{N}(i)}-t_{-\mathcal{N}(i)})]\\
&= (p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T\\
&\qquad\cdot [a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)})-R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}]\\
&\leq [(p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)})-w_{\mathcal{N}(i)}]^T\\
&\qquad\cdot [a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}(p_{\mathcal{N}(i)}-t_{\mathcal{N}(i)}-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)})-R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)}]\\
&\equiv (\tilde{p}_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T(a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}\tilde{p}_{\mathcal{N}(i)}-R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)})\\
&\leq (p^*_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T(a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}p^*_{\mathcal{N}(i)}-R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)})\\
&= (p^*_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p^*)=\pi_i(p^*).
\end{aligned}
$$

The second equality is due to the complementarity of t and $d(p)$. The first inequality is due to adding the term $[-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)})]d_{\mathcal{N}(i)}(p)\geq 0$ to the right-hand side of the inequality, since both $-R^{-1}_{\mathcal{N}(i),\mathcal{N}(i)}R_{\mathcal{N}(i),-\mathcal{N}(i)}t_{-\mathcal{N}(i)}\geq 0$ and $d_{\mathcal{N}(i)}(p)\geq 0$. The second inequality is due to the way $p^*=p^*(w)$ is determined: p^* satisfies the first-order conditions (6), hence $p^*_{\mathcal{N}(i)}$ is the maximizer of the quadratic concave function $\pi_i(p_{\mathcal{N}(i)},p^*_{-\mathcal{N}(i)})=(p_{\mathcal{N}(i)}-w_{\mathcal{N}(i)})^T(a_{\mathcal{N}(i)}-R_{\mathcal{N}(i),\mathcal{N}(i)}p_{\mathcal{N}(i)}-R_{\mathcal{N}(i),-\mathcal{N}(i)}p^*_{-\mathcal{N}(i)})$ among all $p_{\mathcal{N}(i)}$.

(b) By part (a), $p^*(w)$ is the unique equilibrium in P. If there were an additional equilibrium $p^o\notin P$, its projection $\Omega(p^o)$ would, by Theorem 1, also be an equilibrium; but $\Omega(p^o)$ is on the boundary of P and P contains $p^*(w)\in P^o$ as its *unique* equilibrium, see part (a). This is a contradiction. □

Proof of Proposition 3. (a) "(WRS) ⇒ (NPW)". The proof is analogous to that of Lemma 1.

(b) "(IS) ⇒ (WRS)". Let $E=T(R)-R$. Then $R=T(R)-E$. By the symmetry of $T(R)$, $E\geq 0$. Then $S=\Psi(R)R=\Psi(R)[T(R)-E]\leq\Psi(R)\frac{1}{2}[2T(R)-E]=\frac{1}{2}\Psi(R)[T(R)+R]=\frac{1}{2}T(R)$, where the inequality follows from $\Psi(R) \geq 0$, see Proposition 2(e) and $E \geq 0$. Thus, the off-diagonal elements of S are bounded from above by non-positive numbers, i.e., S is a Z-matrix. Moreover, in Proposition 2(e), we showed that $\Psi(R) = [I + RT(R)^{-1}]^{-1} \geq 0$. Thus, $b = \Psi(R)a \geq 0$. (If $a > 0$, $b = \Psi(R)a > 0$. This is because, assume, to the contrary that for some product (i,k), $[\Psi(R)a]_{ik} = 0$. Since $a > 0$, this implies that the $(i,k)^{th}$ row of $\Psi(R)$ is a row of zero's, which contradicts the fact that $\Psi(R)$ has an inverse.) $\square$

Proof of Theorem 3. In view of Theorem 1, it suffices to show that $p^*(w')$ is a price equilibrium. Note that $w' = \Theta(w)$ such that $w' \neq w$, $w' \leq w$ and $Q(w')^T(w - w') = 0$. Moreover, under Assumption (NPW), $w' \geq 0$, so that $w' \in W$. By Theorem 2, $p^*(w') \in P$ is the unique equilibrium in P for the retailers' competition game under the wholesale price vector w'. Thus, for any retailer i,

$$\begin{aligned}
\pi_i(p^*(w'); w) &= [p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}]^T d_{\mathcal{N}(i)}(p^*(w')) \\
&= [p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}]^T q_{\mathcal{N}(i)}(p^*(w')) \\
&= [p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)}]^T Q_{\mathcal{N}(i)}(w') \\
&= [p^*_{\mathcal{N}(i)}(w') - w_{\mathcal{N}(i)} + (w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})]^T Q_{\mathcal{N}(i)}(w') \\
&= [p^*_{\mathcal{N}(i)}(w') - w'_{\mathcal{N}(i)}]^T d_{\mathcal{N}(i)}(p^*(w')) = \pi_i(p^*(w'); w').
\end{aligned} \tag{B.9}$$

The second equality follows from $p^*(w') \in P$, since $w' \in W$. The third equality follows from Proposition 2(b). The fourth equality follows from $(w_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})^T Q_{\mathcal{N}(i)}(w') = 0$, since $w' = w - t$ is the solution to the LCP (8). For any retailer i,

$$\begin{aligned}
\pi_i(p^*(w'); w) &\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w'))] \\
&\leq \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w'_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w'))] \\
&= \pi_i(p^*(w'); w'),
\end{aligned} \tag{B.10}$$

where the second inequality is due to $0 \leq w' \leq w$. By Equation (B.9), all inequalities in (B.10) hold as equalities and in particular, $\pi_i(p^*(w'); w) = \max_{p_{\mathcal{N}(i)} \geq 0} [(p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^T d_{\mathcal{N}(i)}(p_{\mathcal{N}(i)}, p^*_{-\mathcal{N}(i)}(w'))]$ for any retailer i. Hence $p^*(w') \in P$ is an equilibrium in the retailers' competition game under the wholesale price vector $w \notin W$. $\square$

Proof of Proposition 4. (a) By Theorem 2, if $w \in W^o$, there exists a ball around the vector w which is contained within W, and $p^* \in P^o$ is the unique Nash equilibrium in the form of (7). The marginal pass-through rates of wholesale price changes are immediate from (7).

(b) We write

$$[R+T(R)]^{-1}T(R)=[[T(R)]^{-1}R+I]^{-1}.$$

Since $T(R)$ is symmetric and then $T(R)\geq R$. By Lemma A.1(c), since $T(R)$ is a ZP-matrix and R is a Z-matrix, $[T(R)]^{-1}R$ is a ZP-matrix. By Lemma A.1(d), $[T(R)]^{-1}R+I$ is a ZP-matrix, as well. By Lemma A.1(a),

$$[[T(R)]^{-1}R+I]^{-1}\geq 0. \tag{B.11}$$

Since $T(R)$ is a ZP matrix, hence $[T(R)]^{-1}\geq 0$ by Lemma A.1(a). Then because $R\leq T(R)$, $[T(R)]^{-1}R\leq I$ and hence $[T(R)]^{-1}R+I\leq 2I$. Since $[T(R)]^{-1}R+I$ is a ZP-matrix, we have by Lemma A.1(b), $[R+T(R)]^{-1}T(R)=[[T(R)]^{-1}R+I]^{-1}\geq \frac{I}{2}$.

To prove the upper bound, let $\Delta\equiv T(R)-R$. Then $R=T(R)-\Delta$. By the symmetry of $T(R)$, $\Delta\geq 0$. Then

$$\begin{aligned} R[R+T(R)]^{-1}T(R) &= [T(R)-\Delta][R+T(R)]^{-1}T(R) \\ &\leq \frac{1}{2}[2T(R)-\Delta][R+T(R)]^{-1}T(R) \\ &= \frac{1}{2}[T(R)+R][R+T(R)]^{-1}T(R)=\frac{1}{2}T(R), \end{aligned}$$

where the inequality is due to $\Delta\geq 0$ and $[R+T(R)]^{-1}T(R)\geq 0$, by (B.11). Since $R^{-1}\geq 0$, we have the desired upper bound.

Under a monopoly, $T(R)=R$ and $\partial p^*(w)/\partial w=I/2$, i.e., the lower bound is tight. □

Proof of Proposition 5. (a) If products l and l' are sold by the same retailer, we write $[\tilde{R}+T(\tilde{R})]^{-1}=[R+\delta E_{l,l'}+T(R+\delta E_{l,l'})]^{-1}=[R+T(R)+\delta E_{l,l'}+\delta E_{l',l}]^{-1}$. By Chang (2006, Eq. (6) and (7)), we can write $[\tilde{R}+T(\tilde{R})]^{-1}=[R+T(R)]^{-1}+H=\Xi(R)+H$, where

$$\begin{aligned} H &= -\begin{pmatrix}\Xi_{\mathcal{N},l} & \Xi_{\mathcal{N},l'}\end{pmatrix}\begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}\Xi_{ll} & \Xi_{ll'}\\ \Xi_{l'l} & \Xi_{l'l'}\end{pmatrix}\begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix}\right]^{-1}\begin{pmatrix}\Xi_{l,\mathcal{N}}\\ \Xi_{l',\mathcal{N}}\end{pmatrix} \\ &= -\delta\begin{pmatrix}\Xi_{\mathcal{N},l} & \Xi_{\mathcal{N},l'}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}1+\Xi_{ll'}\delta & \Xi_{ll}\delta\\ \Xi_{l'l'}\delta & 1+\Xi_{l'l}\delta\end{pmatrix}\right]^{-1}\begin{pmatrix}\Xi_{l,\mathcal{N}}\\ \Xi_{l',\mathcal{N}}\end{pmatrix}=\Gamma(R,\delta). \end{aligned}$$

By part (a) of Proposition 2,

$$\begin{aligned} \frac{\partial\tilde{p}^*(w)}{\partial w}-\frac{\partial p^*(w)}{\partial w} &= [\tilde{R}+T(\tilde{R})]^{-1}T(\tilde{R})-[R+T(R)]^{-1}T(R) \\ &= [\Xi(R)+\Gamma(R,\delta)][T(R)+\delta E_{l',l}]-\Xi(R)T(R) \\ &= \Gamma(R,\delta)T(R)+\delta[\Xi(R)+\Gamma(R,\delta)]E_{l',l} \\ &= \Gamma(R,\delta)T(R)+\delta\Upsilon_{l'l}\big(\Xi(R)+\Gamma(R,\delta)\big), \end{aligned}$$

where the last equality is due to the fact that $M \cdot E_{l',l}$ is equivalent to applying the matrix operator $\Upsilon_{l'l}(\cdot)$ to M. Since $\Gamma(R,\delta)$ is a rational function in δ, $\frac{\partial \tilde{p}^*(w)}{\partial w} - \frac{\partial p^*(w)}{\partial w}$ is also a rational function in δ.

If products l and l' are not sold by the same retailer, $T(\tilde{R}) = T(R)$. Then we write $[\tilde{R}+T(\tilde{R})]^{-1} = [R+T(R)+\delta E_{l,l'}]^{-1}$. By Chang (2006, Eq. (6) and (7)), we can write $[\tilde{R}+T(\tilde{R})]^{-1} = [R+T(R)+\delta E_{l,l'}]^{-1} = \Xi(R) - \frac{\delta \Xi_{\mathcal{N},l'} \Xi_{l,\mathcal{N}}}{1+\Xi_{ll'}\delta}$.

By part (a) of Proposition 2,

$$\begin{aligned}
\frac{\partial \tilde{p}^*(w)}{\partial w} - \frac{\partial p^*(w)}{\partial w} &= [\tilde{R}+T(\tilde{R})]^{-1}T(\tilde{R}) - [R+T(R)]^{-1}T(R) \\
&= \left[\Xi(R) - \frac{\delta \Xi_{\mathcal{N},l'} \Xi_{l,\mathcal{N}}}{1+\Xi_{ll'}\delta}\right] T(R) - \Xi(R)T(R) \\
&= -\frac{\delta \Xi_{\mathcal{N},l'} \Xi_{l,\mathcal{N}}}{1+\Xi_{ll'}\delta} T(R).
\end{aligned}$$

(b) If products l and l' are not sold by the same retailer, $T(\tilde{R}) = T(R+\delta E_{l,l'}) = T(R)$ which is symmetric by stipulation. By part (b) of Proposition 4, $\frac{\partial \tilde{p}^*(w)}{\partial w} \geq I/2 \geq 0$. □

Proof of Proposition 6. The existence of a unique Cholesky factorization for the matrix $[\frac{\Phi(R)+\Phi(R)^T}{2}]$ is guaranteed by the fact that it is symmetric and positive definite. The matrix L is lower triangular with positive diagonal elements, and therefore has an inverse L^{-1}. Thus, the matrix G is well defined. Since the matrix $[\frac{\Pi(R)+\Pi(R)^T}{2}]$ is positive definite and symmetric, it is easily verified that G has the same two properties. It is therefore possible to write $G \equiv UDU^T$, with D a diagonal matrix with the eigenvalues of G as the diagonal elements. Since G is symmetric and positive definite, all the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are positive. Moreover, the matrix U is orthogonal, i.e., $(UU^T) = (U^TU) = I$.

Let $V \equiv D^{-\frac{1}{2}} U^T L^{-1}$, so that $V^T = L^{-T} U D^{-\frac{1}{2}}$. We show that $V[\frac{\Pi(R)+\Pi(R)^T}{2}]V^T = I$ and $V[\frac{\Phi(R)+\Phi(R)^T}{2}]V^T = D^{-1}$. To see this, we write

$$\begin{aligned}
V\left[\frac{\Pi(R)+\Pi(R)^T}{2}\right]V^T &= D^{-\frac{1}{2}} U^T L^{-1} \left[\frac{\Pi(R)+\Pi(R)^T}{2}\right] L^{-T} U D^{-\frac{1}{2}} \\
&= D^{-\frac{1}{2}} U^T G U D^{-\frac{1}{2}} \\
&= D^{-\frac{1}{2}} D D^{-\frac{1}{2}} = I,
\end{aligned}$$

where the second-to-last equality is due to $G = UDU^T$ so $U^T G U = (U^T U) D (U^T U) = D$.

Also,

$$\begin{aligned}
V\left[\frac{\Phi(R)+\Phi(R)^T}{2}\right]V^T &= D^{-\frac{1}{2}} U^T L^{-1} \left[\frac{\Phi(R)+\Phi(R)^T}{2}\right] L^{-T} U D^{-\frac{1}{2}} \\
&= D^{-\frac{1}{2}} U^T L^{-1} (LL^T) L^{-T} U D^{-\frac{1}{2}}
\end{aligned}$$

$$= D^{-\frac{1}{2}} U^T U D^{-\frac{1}{2}} = D^{-\frac{1}{2}} D^{-\frac{1}{2}} = D^{-1},$$

where the second-to-last equality is again due to $U^T U = I$. Thus,

$$\frac{1}{\lambda_{\max}} = \min_i \left(\frac{1}{\lambda_i}\right) \leq \frac{\pi^d(w)}{\pi^c(w)} = \frac{q(w)^T [\frac{\Phi(R)+\Phi(R)^T}{2}] q(w)}{q(w)^T [\frac{\Pi(R)+\Pi(R)^T}{2}] q(w)} = \frac{y^T D^{-1} y}{y^T y} = \frac{\sum_{i=1}^N \left(\frac{1}{\lambda_i}\right) y_i^2}{\sum_{i=1}^N y_i^2} \leq \max_i \left(\frac{1}{\lambda_i}\right) = \frac{1}{\lambda_{\min}},$$

where $q(w) = V^T y$. $\square$

C. Example That S Fails to Be a Z-Matrix

Example C.1. Consider an industry $I = 2$ retailers. Retailer 1 potentially can carry product 1, 2, 3 and retailer 2 potentially can carry product 4, 5, 6. The R-matrix is given by

$$R = \begin{pmatrix} 5.5 & -0.7 & -0.62 & -0.8 & -0.19 & -0.93 \\ -0.4 & 5.35 & -0.73 & -0.92 & -0.90 & -0.33 \\ -0.9 & -0.98 & 5.62 & -0.84 & -0.57 & -0.66 \\ -0.01 & -0.55 & -0.01 & 5.63 & -0.63 & -0.39 \\ -0.3 & -0.4 & -0.42 & -0.62 & 5.76 & -0.63 \\ -0.04 & -0.2 & -0.75 & -0.73 & -0.55 & 5.3 \end{pmatrix}.$$

This matrix is both row- and column-diagonally dominant and hence positive definite and a ZP-matrix. Yet

$$S = \Psi(R)R = \begin{pmatrix} 2.7492 & -0.2956 & -0.4010 & -0.2149 & -0.0521 & -0.2277 \\ -0.2811 & 2.6537 & -0.4480 & -0.2254 & -0.2211 & -0.0674 \\ -0.3850 & -0.4565 & 2.7907 & -0.2372 & -0.1543 & -0.1725 \\ \boxed{0.0010} & -0.1378 & \boxed{0.0064} & 2.8020 & -0.3266 & -0.2873 \\ -0.0758 & -0.1100 & -0.1036 & -0.3448 & 2.8595 & -0.3162 \\ -0.0133 & -0.0630 & -0.1871 & -0.3086 & -0.3144 & 2.6339 \end{pmatrix}$$

has two positive off-diagonal elements. $\square$

D. Auxiliary Calculations for Example 1

When $w \in W(\text{I})$, the unique retail price equilibrium in P is on the edge BC between $P(\text{I})$ and P. The effective demand of retailer 1 for any retail price $p \in P(\text{I})$ can be expressed as an affine function of p_1 only as $d_1(p_1) = (1+\gamma_1) - (1-\gamma_1\gamma_2)p_1$ for $p \in P(\text{I})$. Given a wholesale price w_1, the optimal monopoly price of retailer 1 is $\tilde{p}_1^*(w_1) = \frac{1+\gamma_1}{2(1-\gamma_1\gamma_2)} + \frac{1}{2}w_1$, as the solution to the optimization problem $\max_{p_1 \geq 0}(p_1 - w_1)d_1(p_1)$. For $w \in W(\text{I})$,

$$\tilde{p}_1^*(w_1) = \frac{1+\gamma_1}{2(1-\gamma_1\gamma_2)} + \frac{1}{2}w_1 \geq p_1^*(w') = \frac{1+\gamma_1}{2-\gamma_1\gamma_2} + \frac{1}{2-\gamma_1\gamma_2}w_1,$$

because $w_1 \leq \frac{1+\gamma_1}{1-\gamma_1\gamma_2}$ for $w \in W(\text{I})$. In other words, the equilibrium price $p_1^*(w')$ of retailer 1, under competition and the possibility of retailer 2 getting back into the market, is lower than the optimal monopoly price $\tilde{p}_1^*(w_1)$, which applies when retailer 2 has exited the market permanently, i.e., the price range is confined to $P(\mathrm{I})$.

We verify that when $\gamma_1, \gamma_2 > 0$, $p^*(w')$ is the *unique* equilibrium, altogether. To verify this, by Theorem 1, the only other equilibrium candidates are points p^o such that $\Omega(p^o) = p^*(w')$, i.e., the points on the vertical half line above $p^*(w')$ in Figure 1(a). However, an arbitrary point on this half line fails to be an equilibrium, since retailer 1 can improve its profit by moving to the right: during this horizontal move, the price vector remains in $P(\mathrm{I})$, where the effective demand for product 1 is given by $d_1(p) = (1+\gamma_1) - (1-\gamma_1\gamma_2)p_1$ (see above), and the profit function is quadratic in p_1, hence unimodal with its peak possibly at $p_1 = \tilde{p}_1^*(w_1)$.

When $\gamma_1 = 0$ or $\gamma_2 = 0$, $p_1^*(w') = \tilde{p}_1^*(w_1)$ and all points on the vertical half line above $p^*(w')$ are equilibria: if retailer 1 deviates from the price $p_1^*(w') = \tilde{p}_1^*(w_1)$, he decreases his profit, see above; similarly, retailer 2's profit increase by unilaterally switching to a different price level, contradicts the fact that $p^*(w')$ is an equilibrium.

Finally, consider the case where $w \in W(\mathrm{III})$. It is easily verified that all points in $W(\mathrm{III})$ have $w' = \Theta(w) = C$ and that $p^*(C) = C$: in other words, when $w \in W(\mathrm{III})$, the unique retailer equilibrium in P is for both firms to exit the market by setting $p^*(w') = C$. At the same time all other points $p^o \in W(\mathrm{III}) = P(\mathrm{III})$ are equilibria as well: retailer 1 may also generate a positive demand by decreasing its price sufficiently so as to move into $P(\mathrm{I})$, however since $w \in W(\mathrm{III})$, this is accompanied by a negative profit margin; similarly retailer 2 cannot improve his profit by changing his price.

Consider the distribution structure analyzed in McGuire and Staelin (2008), where supplier i, $i = 1, 2$, sells product i exclusively through retailer i. Clearly, $T(R) = I$ and

$$\Psi(R) = [I+R]^{-1} = \frac{1}{4-\gamma_1\gamma_2}\begin{pmatrix} 2 & \gamma_1 \\ \gamma_2 & 2 \end{pmatrix}.$$

Then we have

$$S = \Psi(R)R = \frac{1}{4-\gamma_1\gamma_2}\begin{pmatrix} 2-\gamma_1\gamma_2 & -\gamma_1 \\ -\gamma_2 & 2-\gamma_1\gamma_2 \end{pmatrix}.$$

By Federgruen and Hu (2013),

$$\Psi(S) = T(S)[S+T(S)]^{-1} = \frac{2-\gamma_1\gamma_2}{4(2-\gamma_1\gamma_2)^2 - \gamma_1\gamma_2}\begin{pmatrix} 2(2-\gamma_1\gamma_2) & \gamma_1 \\ \gamma_2 & 2(2-\gamma_1\gamma_2) \end{pmatrix},$$

and the effective supply cost polyhedron

$$C = \left\{ c \geq 0 \;\middle|\; \begin{array}{l} (8+6\gamma_1-3\gamma_1\gamma_2-2\gamma_1^2\gamma_2) - (8-9\gamma_1\gamma_2+2\gamma_1^2\gamma_2^2)c_1 + \gamma_1(2-\gamma_1\gamma_2)c_2 \geq 0 \\ (8+6\gamma_2-3\gamma_1\gamma_2-2\gamma_1\gamma_2^2) + \gamma_2(2-\gamma_1\gamma_2)c_1 - (8-9\gamma_1\gamma_2+2\gamma_1^2\gamma_2^2)c_2 \geq 0 \end{array} \right\}.$$

We provide an example where $c \in C$ and $w^*(c) \in (W^o \setminus P)$. Let $\gamma_1 = 0.7, \gamma_2 = 0.3$. Then, with

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -0.7 \\ -0.3 & 1 \end{pmatrix},$$

it is easily verified that

$$b = \Psi(R)a = \begin{pmatrix} 0.7124 \\ 0.6069 \end{pmatrix} \quad \text{and} \quad S = \Psi(R)R = \begin{pmatrix} 0.4723 & -0.1847 \\ -0.0792 & 0.4723 \end{pmatrix},$$

and moreover,

$$\Psi(S) = T(S)[S + T(S)]^{-1} = \begin{pmatrix} 0.5083 & 0.0994 \\ 0.0426 & 0.5083 \end{pmatrix}.$$

Consider $c = (1, 1.5)^T$. It is easily verified that

$$\Psi(S)Q(c) = \Psi(S)(b - Sc) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0,$$

i.e., $c \in C^o$. By Theorem 2 in Federgruen and Hu (2013),

$$w^*(c) = c + [S + T(S)]^{-1}Q(c) = \begin{pmatrix} 1.5519 \\ 1.5225 \end{pmatrix} \in W^o.$$

By Theorem 2(b),

$$p^*(w^*(c)) = w^*(c) + [R + T(R)]^{-1}q(w^*(c)) = \begin{pmatrix} 1.8125 \\ 1.5331 \end{pmatrix} \in P^o$$

and

$$d(p^*(w^*(c))) = a - Rp^*(w^*(c)) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0.$$

However, note that

$$a - Rw^*(c) = \begin{pmatrix} 0.5139 \\ -0.0569 \end{pmatrix},$$

i.e., $w^*(c) \notin P$. $\square$

E. A 2-Firm 2-Product Example

Example E.1 (2 Firms, 2 Products). As in Example 1, consider a duopoly of retailers $i = 1, 2$, each offering a single product $i = 1, 2$, with raw demand functions specified as:

$$a = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & -1 + \delta \\ -1 - \delta & 4 \end{pmatrix}.$$

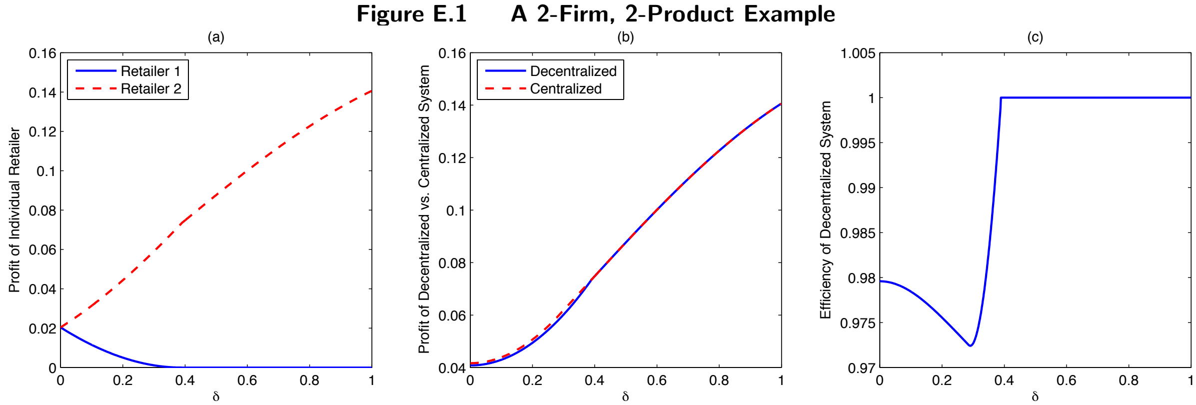
Regardless of the value of $\delta \geq 0$, the symmetrized matrix $\tilde{R} \equiv (R + R^T)/2 = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$. As δ increases, the matrix R becomes increasingly asymmetric. Consider the wholesale price vector $(w_1, w_2) = (1.5, 1.5)^T$. Figure E.1(a) plots the profits for retailers 1 and 2, respectively, as δ increases from 0 to 1, its maximum value before R ceases to be a Z-matrix, i.e., before the products cease to be substitutes.

Clearly, retailer 1 (2) suffers (benefits) when δ increases: the raw demand for its product decreases (increases) under any given price vector p. This is reflected in the equilibrium profit function for retailer 1 (2) being decreasing (increasing) with very significant bottom line changes as δ increases from 0 to 1. Moreover, the degree of asymmetry has a major impact on the market structure: as long as $\delta < 0.3885$, both retailers maintain a market share; when $\delta \geq 0.3885$, retailer 1 is unable to compete, with retailer 2 remaining as a monopolist.

Case (i). Consider $\delta \in [0, \frac{-11+2\sqrt{37}}{3} \approx 0.3885)$. In this case, both retailers enjoy positive demand in equilibrium. The equilibrium prices are $p_1^* = \frac{11(9-\delta)}{63+\delta^2}$, $p_2^* = \frac{11(9+\delta)}{63+\delta^2}$, and the equilibrium demand volumes are $d_1^* = \frac{44(9-\delta)}{63+\delta^2} - 6$ and $d_2^* = \frac{44(9+\delta)}{63+\delta^2} - 6$. Thus, both the price and the sales volume of retailer 1 (2) decreases (increases) with δ.

Case (ii). Consider $\delta \in [\frac{-11+2\sqrt{37}}{3}, 1]$. In this case, retailer 2 has a monopoly. As δ increases, its profit increases, see Figure E.1(a), along with the retail price $p_2^* = 0.75 + \frac{5(5+\delta)}{2(15+\delta^2)}$.

Figure E.1 A 2-Firm, 2-Product Example



Note that the wholesale price vector $w^o = (1.5, 1.5)^T$ falls *outside* of the effective retail price polyhedron P when $\delta > \frac{1}{3}$: after all, $q^\delta(w^o) = \begin{pmatrix} 5 - 4 \times 1.5 + (1-\delta)1.5 \\ 5 - 4 \times 1.5 + (1+\delta)1.5 \end{pmatrix}$, so that $q_1^\delta(w^o) = 0.50 - 1.5\delta < 0 \Leftrightarrow \delta > \frac{1}{3}$. In other words, when $\delta > \frac{1}{3}$, retailer 1 is driven out of the market even if she is willing to operate without *any* markup, i.e., even when setting $p_1 = w_1 = 1.5$ (, as long as retailer 2 does the same). Nevertheless, as shown in case (i), as long as $0.333 < \delta < 0.3885$, retailer 1 maintains, in its unique equilibrium, a positive market share and adopts a positive markup.

For example, when $\delta = 0.36$, $p_1^* = 1.51$ and $p_2^* = 1.63$. To show that the wholesale price vector $w^o = (1.5, 1.5)^T$ may well arise in this market (, with $\delta = 0.36$), even though $w^o \notin P$, consider the basic channel structure studied in McGuire and Staelin (2008) where retailer 1 (2) uniquely procures from a dedicated supplier 1 (2), operating with a marginal cost rate vector $c^o = (1.4889, 1.2345)^T$. Assume the market operates as a sequential oligopoly: first the two suppliers, non-cooperatively, select their wholesale prices, accounting for the retailers' equilibrium price responses. Then, the retailers follow and select their prices. Following the results in Federgruen and Hu (2013), one can show that the vector $w^o = (1.5, 1.5)^T$ arises as part of the unique supply-chain-wide equilibrium. See Appendix E.1.1 for the auxiliary calculations to verify this result. This example shows that characterizing the equilibrium in the retailer competition model, when $w \notin P^o$ or even when $w \notin P$ is of practical importance, because an actually observed market price vector $w \notin P$ may easily arise. In addition, to enable the analysis of the competition game among the suppliers in a two-stage sequential oligopoly, it is necessary to characterize the equilibrium behavior in the retailer game, for an *arbitrary* vector of wholesale prices, even if in the end certain price vectors do not arise as equilibria.

How does the efficiency ratio depend on δ, the degree of asymmetry in the matrix R? Figures E.1(b) and E.1(c) display the aggregate profits in the oligopoly, those in the centralized solution and the efficiency ratio as the asymmetry index δ varies between $\delta = 0$ to $\delta = 1$. The efficiency ratio first declines until $\delta \leq \frac{2\sqrt{190}-25}{9} \approx 0.2853$, the point where in the centralized solution, it becomes optimal to sell only product 2, as opposed to both products. Recall that, under competition, both products continue to be sold in equilibrium, under larger degrees of δ, i.e., even for $\delta \in [0.2853, 0.3885)$. On this interval, the efficiency ratio *increases* until it reaches the value $\delta = 0.3885$ where retailer 1 and its product 1 are driven out of the market. When $\delta > 0.3885$, only product 1 is sold both in the oligopoly and the centralized solution. Thus, the competitive and centralized solutions coincide for $\delta > 0.3885$, resulting in an efficiency ratio of 1. The high efficiency ratios in this example are not representative, see Example 2 below. $\square$

E.1. Auxiliary Calculations

E.1.1. The Example of $w \in W^o \setminus P$.

Clearly, $T(R) = 4I$ and

$$\Psi(R) = \frac{4}{\delta^2 + 63} \begin{pmatrix} 8 & 1-\delta \\ 1+\delta & 8 \end{pmatrix}.$$

Then we have

$$b = \Psi(R)a = \frac{20}{\delta^2 + 63} \begin{pmatrix} 9-\delta \\ 9+\delta \end{pmatrix}$$

and

$$S = \Psi(R)R = \frac{1}{\delta^2 + 63} \begin{pmatrix} 4\delta^2 + 124 & 16\delta - 16 \\ -16\delta - 16 & 4\delta^2 + 124 \end{pmatrix}$$

We verify that $w^o = (1.5, 1.5)^T \in W^o \setminus P$ and w^o arises as part of the unique supply-chain-wide equilibrium with the supply cost vector $c^o = (1.4889, 1.2345)^T$. Let $\delta = 0.36$. Then, with

$$a = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 4 & -0.64 \\ -1.36 & 4 \end{pmatrix},$$

it is easily verified that

$$b = \Psi(R)a = \begin{pmatrix} 2.7372 \\ 2.9653 \end{pmatrix} \quad \text{and} \quad S = \Psi(R)R = \begin{pmatrix} 1.9724 & -0.1622 \\ -0.3447 & 1.9724 \end{pmatrix},$$

and moreover,

$$\Psi(S) = T(S)[S+T(S)]^{-1} = \begin{pmatrix} 0.5018 & 0.0206 \\ 0.0438 & 0.5018 \end{pmatrix}.$$

Consider $c = (1.4889, 1.2345)^T$. It is easily verified that

$$\Psi(S)Q(c) = \Psi(S)(b - Sc) = \begin{pmatrix} 0.0219 \\ 0.5237 \end{pmatrix} > 0,$$

i.e., $c \in C^o = \{c \geq 0 \mid \Psi(S)(b - Sc) > 0\}$, where C is the effective supply cost polyhedron (as long as $c \in C^o$, all products enjoy positive market shares in equilibrium in the sequential oligopoly), see Federgruen and Hu (2013). It is also easily verified that

$$w^*(c) = c + [S + T(S)]^{-1}Q(c) = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix},$$

This verifies that $w = (1.5, 1.5)^T$ arises as part of the unique supply-chain-wide equilibrium with the supply cost vector $c = (1.4889, 1.2345)^T$. Since $c \in C^o$, $w^*(c) = (1.5, 1.5)^T \in W^o$. This also can be seen because for $\delta \in [0, 0.3885)$, both products are provided in equilibrium in the market and here, $\delta = 0.36$ is in this range. Then

$$p^*(w^*(c)) = w^*(c) + [R + T(R)]^{-1}q(w^*(c)) = \begin{pmatrix} 1.5055 \\ 1.6309 \end{pmatrix} \in P^o$$

and

$$d(p^*(w^*(c))) = a - Rp^*(w^*(c)) = \begin{pmatrix} 0.0219 \\ 0.5237 \end{pmatrix} > 0.$$

However, note that

$$a - Rw^*(c) = \begin{pmatrix} -0.04 \\ 1.04 \end{pmatrix},$$

i.e., $w^*(c) \notin P$.

E.1.2. Decentralized System. Case (i). Consider $\delta \in [0, \frac{-11+2\sqrt{37}}{3} \approx 0.3885)$. In this case, both retailers have positive demand in equilibrium. The equilibrium prices are $p_1^* = \frac{11(9-\delta)}{63+\delta^2}$, $p_2^* = \frac{11(9+\delta)}{63+\delta^2}$. The equilibrium demand volumes are $d_1^* = \frac{44(9-\delta)}{63+\delta^2} - 6$, $d_2^* = \frac{44(9+\delta)}{63+\delta^2} - 6$. The equilibrium profits are $\pi_1^* = \frac{(3\delta^2+22\delta-9)^2}{(63+\delta^2)^2}$ and $\pi_2^* = \frac{(-3\delta^2+22\delta+9)^2}{(63+\delta^2)^2}$.

Case (ii). Consider $\delta \in [\frac{-11+2\sqrt{37}}{3}, 1]$. In this case, retailer 2 is the monopoly in the market. The equilibrium price of retailer 2 is $p_2^* = \frac{3}{4} + \frac{5(5+\delta)}{2(15+\delta^2)}$. The smallest equilibrium price for retailer 1 to shut down his demand is $p_1^* = \frac{13-3\delta}{16} + \frac{5(5-\delta)}{2(15+\delta^2)}$. The equilibrium demand of retailer 2 is $d_2^* = \frac{-3\delta^2+10\delta+5}{16}$. The equilibrium demand of retailer 1 is 0. The equilibrium profit of retailer 2 is $\pi_2^* = \frac{(-3\delta^2+10\delta+5)^2}{64(15+\delta^2)}$. The equilibrium profit of retailer 1 is 0.

E.1.3. Centralized Solution. Case (i). Consider $\delta \in [0, \frac{2\sqrt{190}-25}{9} \approx 0.2853)$. Both products are offered in the market in the optimal solution. The optimal prices are

$$p = w + (R + R^T)^{-1}(a - Rw) = \begin{pmatrix} \frac{19}{12} - \frac{3\delta}{20} \\ \frac{19}{12} + \frac{3\delta}{20} \end{pmatrix}.$$

The optimal demand volumes are

$$\begin{pmatrix} -\frac{3\delta^2}{20} - \frac{5\delta}{6} + \frac{1}{4} \\ -\frac{3\delta^2}{20} + \frac{5\delta}{6} + \frac{1}{4} \end{pmatrix}.$$

The optimal profits are

$$\begin{pmatrix} \frac{(9\delta-5)(9\delta^2+50\delta-15)}{3600} \\ \frac{(9\delta+5)(-9\delta^2+50\delta+15)}{3600} \end{pmatrix}.$$

The optimal profit is $\frac{9\delta^2}{40} + \frac{1}{24}$.

Case (ii). Consider $\delta \in [0.2853, 1]$. The form of equilibria is the same as case (ii) of the decentralized system. This is because in this case, product B is the only product in the market. The optimal prices are $p_1^* = \frac{13-3\delta}{16} + \frac{5(5-\delta)}{2(15+\delta^2)}$ and $p_2^* = \frac{3}{4} + \frac{5(5+\delta)}{2(15+\delta^2)}$. The optimal demands are $d_1^* = 0$ and $d_2^* = \frac{-3\delta^2+10\delta+5}{16}$. The optimal profit is $\pi_1^* + \pi_2^* = \frac{(-3\delta^2+10\delta+5)^2}{64(15+\delta^2)}$.

F. Auxiliary Calculations for Example 2

F.1. Decentralized System

Case (i). Consider $\delta \in \left[0, \frac{3\sqrt{17}-11}{4} \approx 0.3423\right)$. In this case, all three products have positive demand in equilibrium. The equilibrium prices are

$$\begin{pmatrix} p_A^* \\ p_B^* \\ p_C^* \end{pmatrix} = \begin{pmatrix} \frac{50-11\delta}{\delta^2+23} \\ \frac{13\delta/2+101/2}{\delta^2+23} - \delta/5 \\ \frac{13\delta/2+101/2}{\delta^2+23} + \delta/5 \end{pmatrix}.$$

The equilibrium demand volumes are

$$\begin{pmatrix} d_A^* \\ d_B^* \\ d_C^* \end{pmatrix} = \begin{pmatrix} \frac{200-44\delta}{\delta^2+23} - 8 \\ \frac{301-31\delta}{\delta^2+23} + \delta - \frac{\delta^2}{5} - \frac{25}{2} \\ \frac{70\delta+2}{\delta^2+23} - \delta - \frac{\delta^2}{5} + \frac{1}{2} \end{pmatrix}.$$

The equilibrium profits are

$$\begin{pmatrix} \pi_A^* \\ \pi_B^* \\ \pi_C^* \end{pmatrix} = \begin{pmatrix} \frac{4(2\delta^2+11\delta-4)^2}{(\delta^2+23)^2} \\ \frac{4\delta^7+20\delta^6+104\delta^5+3680\delta^4-1469\delta^3-11915\delta^2-1035\delta+6075}{100(\delta^2+23)^2} \\ \frac{-4\delta^7+20\delta^6-104\delta^5+560\delta^4-14131\delta^3+47625\delta^2+36135\delta+6075}{100(\delta^2+23)^2} \end{pmatrix}.$$

The equilibrium profits for retailers are

$$\begin{pmatrix} \pi_1^* \\ \pi_2^* \end{pmatrix} = \begin{pmatrix} \pi_A^* \\ \pi_B^* + \pi_C^* \end{pmatrix} = \begin{pmatrix} \frac{4(2\delta^2+11\delta-4)^2}{(\delta^2+23)^2} \\ \frac{2\delta^2}{5} - \frac{156\delta^3+(1917\delta^2)/2-351\delta+25149/2}{(\delta^2+23)^2} + 24 \end{pmatrix}.$$

Case (ii). Consider $\delta \in [0.3423, 0.5758)$. In this case, only products B and C carried by retailer 2 have positive demand in equilibrium. The null price for product A to shut down demand is

$$\bar{p}_1(p_2,p_3) = \frac{1}{4}\left[5 + (1-\delta)p_2 + (1-\delta)p_3\right]$$

and the adjusted raw demand system for products B and C is

$$\begin{pmatrix} d_2(p_2,p_3) \\ d_3(p_2,p_3) \end{pmatrix} = \begin{pmatrix} \frac{5(5+\delta)}{4} \\ \frac{5(5+\delta)}{4} \end{pmatrix} - \begin{pmatrix} \frac{15+\delta^2}{4} & -\frac{5-4\delta-\delta^2}{4} \\ -\frac{5+4\delta-\delta^2}{4} & \frac{15+\delta^2}{4} \end{pmatrix} \begin{pmatrix} p_2 \\ p_3 \end{pmatrix}.$$

The component-wise smallest equilibrium prices are

$$\begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} = \begin{pmatrix} \frac{9}{8} - \frac{5\delta/2-25/4}{\delta^2+5} - \frac{\delta}{2} \\ \frac{5\delta/4+25/4}{\delta^2+5} - \frac{\delta}{5} + 1 \\ \frac{(4\delta^2+45)(\delta+5)}{20(\delta^2+5)} \end{pmatrix}.$$

The equilibrium demand volumes are

$$\begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{5(\delta-1)}{8} - \frac{25(\delta-1)}{4(\delta^2+5)} - \frac{7\delta^2}{10} \\ \frac{5(\delta+3)}{8} + \frac{25(\delta-1)}{4(\delta^2+5)} - \frac{7\delta^2}{10} \end{pmatrix}.$$

The equilibrium profits are

$$\begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{112\delta^7+460\delta^6+20\delta^5+3225\delta^4+1800\delta^3-7250\delta^2-2500\delta+3125}{800(\delta^2+5)^2} \\ \frac{-112\delta^7+660\delta^6-2020\delta^5+3225\delta^4-9300\delta^3+12750\delta^2+15000\delta+3125}{800(\delta^2+5)^2} \end{pmatrix}.$$

The equilibrium profits for retailers are

$$\begin{pmatrix} \pi_{R1}^* \\ \pi_{R2}^* \end{pmatrix} = \begin{pmatrix} \pi_1^* \\ \pi_2^* + \pi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{125(\delta+2)}{8(\delta^2+5)} - \frac{5\delta}{2} + \frac{7\delta^2}{5} - \frac{95}{16} \end{pmatrix}.$$

Case (iii). Consider $\delta \in [0.5758, 1)$. In this case, only product C carried by retailer 2 has positive demand in equilibrium. The null prices for products A and B to shut down demand is

$$\begin{pmatrix} \bar{p}_1(p_3) \\ \bar{p}_2(p_3) \end{pmatrix} = \frac{5+(1-\delta)p_3}{15+\delta^2} \begin{pmatrix} 5-\delta \\ 5+\delta \end{pmatrix}$$

and the adjusted raw demand system for product C is

$$d_3(p_3) = \frac{125+50\delta+5\delta^2-(50+14\delta^2)p_3}{15+\delta^2}.$$

The component-wise smallest equilibrium prices are

$$\begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} = \begin{pmatrix} \frac{33}{28} - \frac{25\delta/2-375/14}{7\delta^2+25} - \frac{17\delta/2-5/2}{\delta^2+15} \\ -\frac{33}{28} + \frac{250}{7(7\delta^2+25)} - \frac{3\delta/2-65/2}{\delta^2+15} \\ \frac{225+50\delta+33\delta^2}{100+28\delta^2} \end{pmatrix}.$$

The equilibrium demand volumes are

$$\begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{25\delta+185}{\delta^2+15} - \frac{23}{2} \end{pmatrix}.$$

The equilibrium profits are

$$\begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{(-23\delta^2+50\delta+25)^2}{8(\delta^2+15)(7\delta^2+25)} \end{pmatrix}.$$

The equilibrium profits for retailers are

$$\begin{pmatrix} \pi_{R1}^* \\ \pi_{R2}^* \end{pmatrix} = \begin{pmatrix} \pi_1^* \\ \pi_2^* + \pi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(-23\delta^2+50\delta+25)^2}{8(\delta^2+15)(7\delta^2+25)} \end{pmatrix}.$$

The maximum of retailer 2's profit is achieved at $\delta = 0.822$.

F.2. Centralized Solution

Case (i): Consider $\delta \in [0, \frac{\sqrt{705}-25}{8} \approx 0.1940)$. The optimal prices are

$$w + [R + T(R)]^{-1}(a - Rw) = \begin{pmatrix} \frac{9}{4} - \frac{2\delta}{5} \\ \frac{9}{4} \\ \frac{9}{4} + \frac{2\delta}{5} \end{pmatrix}.$$

The optimal demand volumes are

$$R^T[R + R^T]^{-1}(a - Rw) = \begin{pmatrix} -\frac{2\delta^2}{5} - \frac{5\delta}{2} + \frac{1}{2} \\ -\frac{4\delta^2}{5} + \frac{1}{2} \\ -\frac{2\delta^2}{5} + \frac{5\delta}{2} + \frac{1}{2} \end{pmatrix}.$$

The optimal profits are

$$\begin{pmatrix} \frac{(8\delta-5)(4\delta^2+25\delta-5)}{200} \\ \frac{1}{8} - \frac{\delta^2}{5} \\ \frac{(8\delta+5)(-4\delta^2+25\delta+5)}{200} \end{pmatrix},$$

and the optimal total profit is $\frac{8\delta^2}{5} + \frac{3}{8}$.

Case (ii): Consider $\delta \in [0.1939, 0.5758)$. In this case, only products B and C have positive demand in the optimal solution. The form of optimal prices is the same as case (ii) of the decentralized case in which retailer 2 carrying products B and C is the only remaining firm in the market. Then the optimal prices are

$$\begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} = \begin{pmatrix} \frac{9}{8} - \frac{5\delta/2 - 25/4}{\delta^2+5} - \frac{\delta}{2} \\ \frac{5\delta/4 + 25/4}{\delta^2+5} - \frac{\delta}{5} + 1 \\ \frac{(4\delta^2+45)(\delta+5)}{20(\delta^2+5)} \end{pmatrix}.$$

The optimal demand volumes are

$$\begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{5(\delta-1)}{8} - \frac{25(\delta-1)}{4(\delta^2+5)} - \frac{7\delta^2}{10} \\ \frac{5(\delta+3)}{8} + \frac{25(\delta-1)}{4(\delta^2+5)} - \frac{7\delta^2}{10} \end{pmatrix}.$$

The optimal profits are

$$\begin{pmatrix} \pi_1^* \\ \pi_2^* \\ \pi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{112\delta^7+460\delta^6+20\delta^5+3225\delta^4+1800\delta^3-7250\delta^2-2500\delta+3125}{800(\delta^2+5)^2} \\ \frac{-112\delta^7+660\delta^6-2020\delta^5+3225\delta^4-9300\delta^3+12750\delta^2+15000\delta+3125}{800(\delta^2+5)^2} \end{pmatrix}.$$

The optimal total profit is

$$\pi_1^* + \pi_2^* + \pi_3^* = \frac{125(\delta+2)}{8(\delta^2+5)} - \frac{5\delta}{2} + \frac{7\delta^2}{5} - \frac{95}{16}.$$

Case (iii). Consider $\delta \in [0.5758, 1)$. In this case, only product C carried by retailer 2 has positive demand in the optimal solution. The form of optimal prices are the same as case (iii) of the decentralized case. The optimal prices prices are

$$\begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} = \begin{pmatrix} \frac{33}{28} - \frac{25\delta/2-375/14}{7\delta^2+25} - \frac{17\delta/2-5/2}{\delta^2+15} \\ -\frac{33}{28} + \frac{250}{7(7\delta^2+25)} - \frac{3\delta/2-65/2}{\delta^2+15} \\ \frac{225+50\delta+33\delta^2}{100+28\delta^2} \end{pmatrix}.$$

The optimal demand volumes are

$$\begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{25\delta+185}{\delta^2+15} - \frac{23}{2} \end{pmatrix}.$$

The optimal total profit is

$$\pi_1^* + \pi_2^* + \pi_3^* = \frac{(-23\delta^2+50\delta+25)^2}{8(\delta^2+15)(7\delta^2+25)}.$$

G. Robustness Check for Example 2

For all possible combinations of wholesale prices on the grid $w_i^o \in \{0.4, 0.8, 1.2, 1.6, 2\}, i = 1, 2, 3$, we compute the following performance measures while varying $\delta \in [0, 1)$: the profit for retailer 1's product A, the profit for retailer 2's product B, the profit for retailer 2's product C, the aggregate decentralized profit, the aggregate centralized profit and the efficiency ratio. For all 6 performance measures, we display below the histograms for the largest percentage increase and decrease due to asymmetry, across the 125 scenarios.

1. The profit for retailer 1's product A: For all 125 scenarios, this profit measure decreases with δ. See Figure G.1 for the maximum percentage decrease.

2. The profit for retailer 2's product B: For all 125 scenarios, this profit measure decreases with δ. See Figure G.2 for the maximum percentage decrease.

3. The profit for retailer 2's product C: For all 125 scenarios, this profit measure increases with δ. See Figure G.3 for the maximum percentage increase.

4. The aggregate profit of the decentralized system: See Figure G.4.

5. The aggregate profit of the centralized system: See Figure G.5.

6. The efficiency ratio: See Figure G.6.

Figure G.1 Profit for Retailer 1's Product A

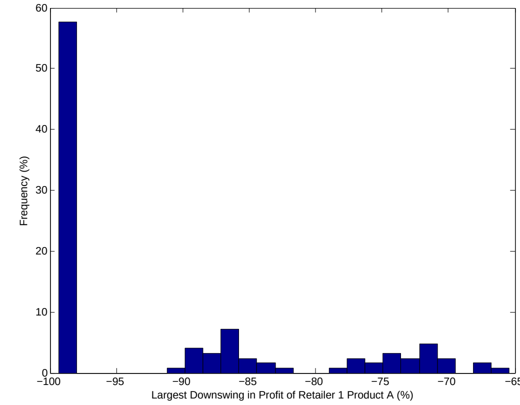


Figure G.2 Profit for Retailer 2's Product B

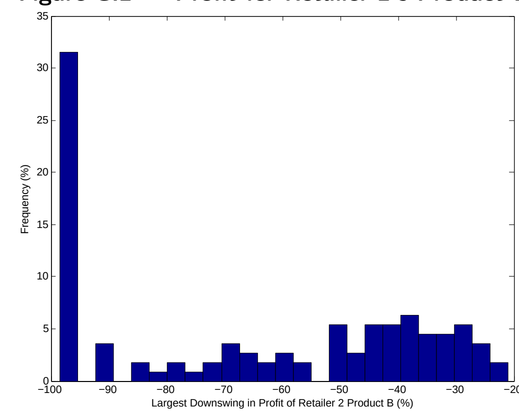
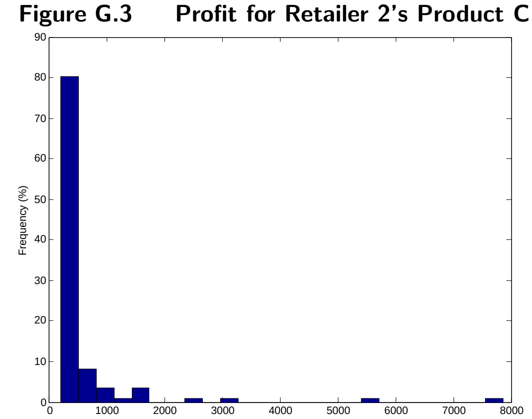


Figure G.3 Profit for Retailer 2's Product C



H. Robustness Check for Example 3

For the 125 wholesale price vectors described, we compute the following performance measures while varying $\delta \in (0, 1]$: the profit for retailer 1, the profit for retailer 2, the profit for retailer 3, the aggregate decentralized profit, the aggregate centralized profit and the efficiency ratio. For all 6 performance measures, we display below the histograms for the largest percentage increase and

Figure G.4 Profit for Decentralized System

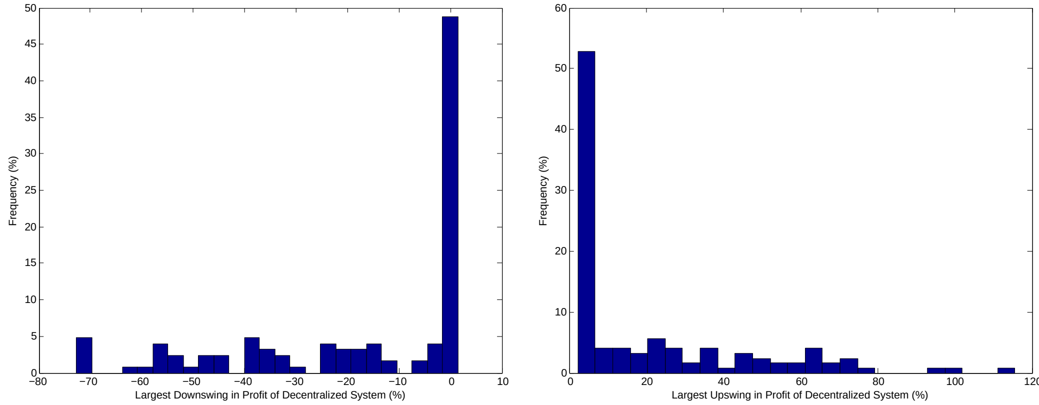


Figure G.5 Profit for Centralized System

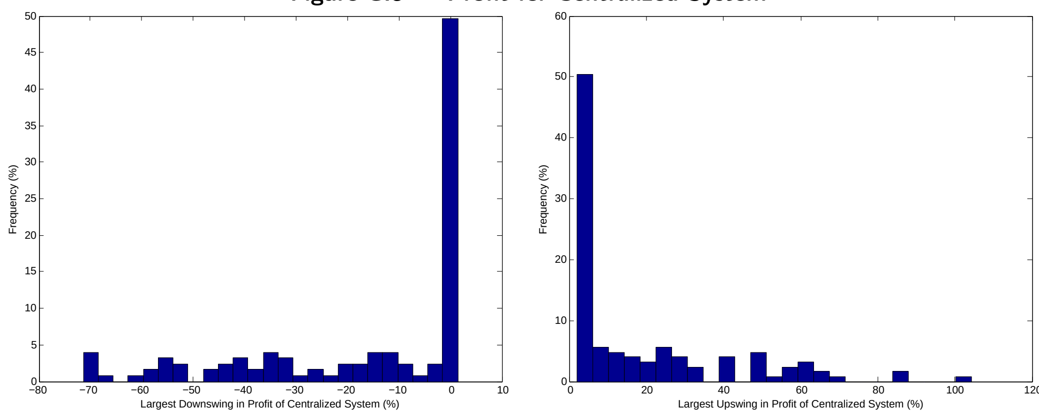
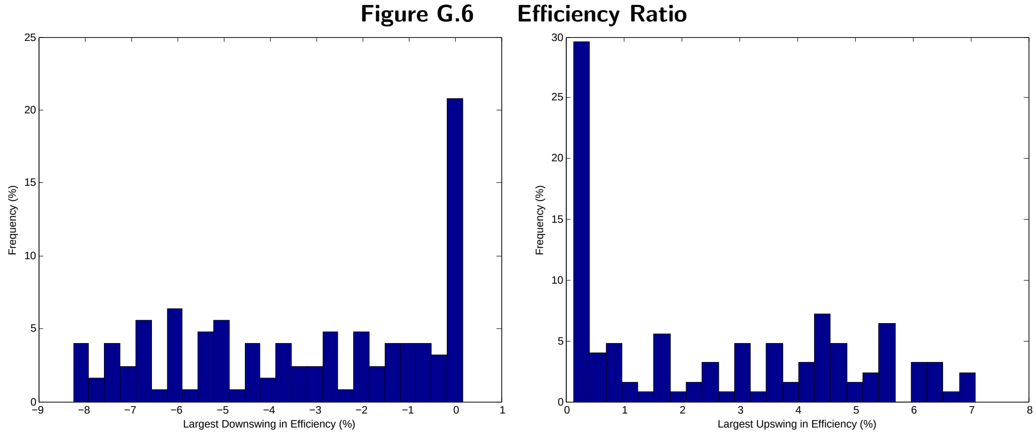


Figure G.6 Efficiency Ratio



decrease due to asymmetry, across the 125 scenarios.

1. The profit for retailer 1: For all 125 scenarios, this profit measure decreases with δ. See Figure H.1 for the maximum percentage decrease.

2. The profit for retailer 2: For all 125 scenarios, this profit measure decreases with δ. See Figure H.2 for the maximum percentage decrease.

Figure H.1 Profit for Retailer 1

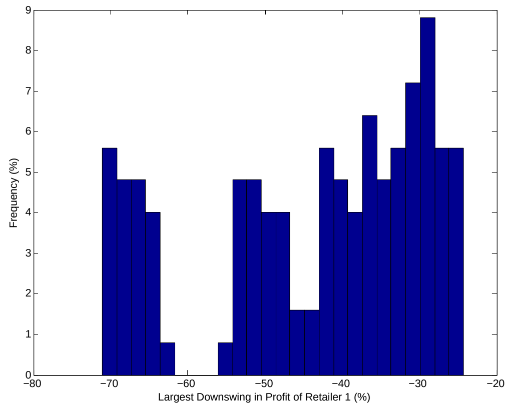
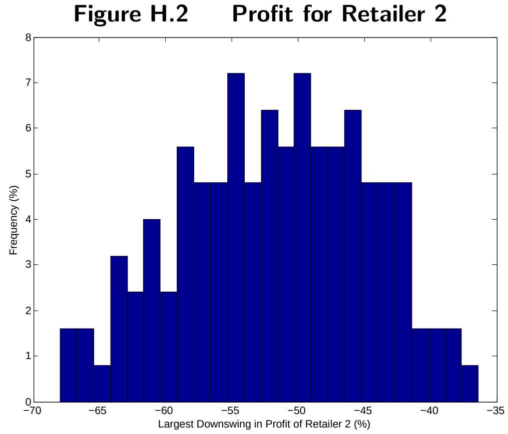
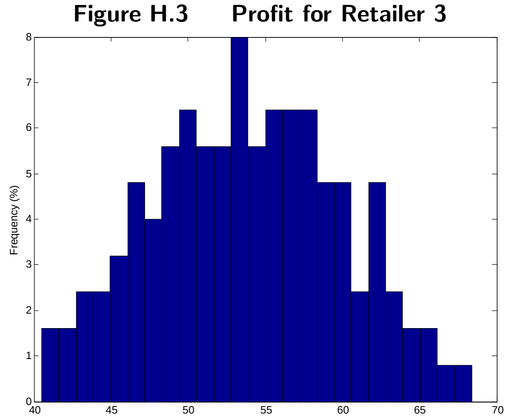


Figure H.2 Profit for Retailer 2



3. The profit for retailer 3: For all 125 scenarios, this profit measure increases with δ. See Figure H.3 for the maximum percentage increase.

Figure H.3 Profit for Retailer 3



4. The aggregate profit of the decentralized system: See Figure H.4.

5. The aggregate profit of the centralized system: See Figure H.5.

Figure H.4 Profit for Decentralized System

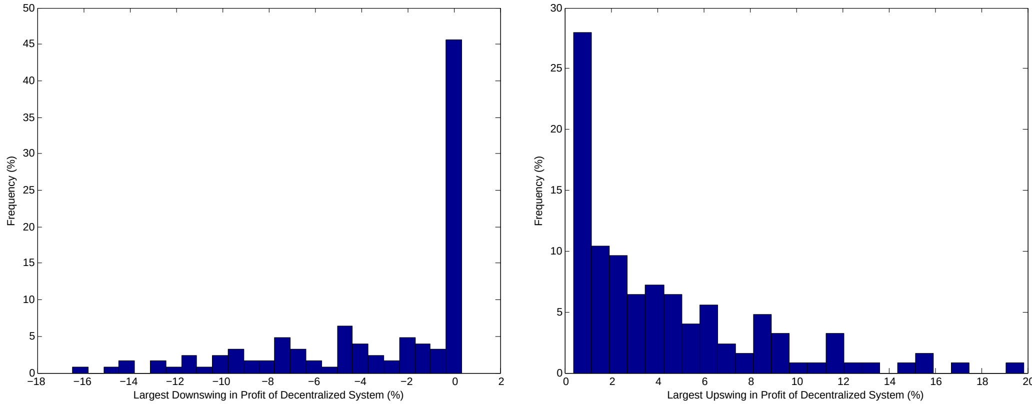
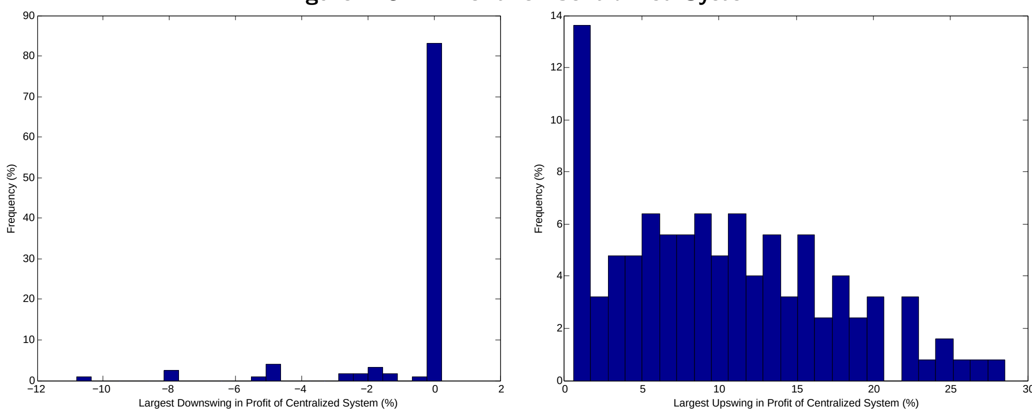
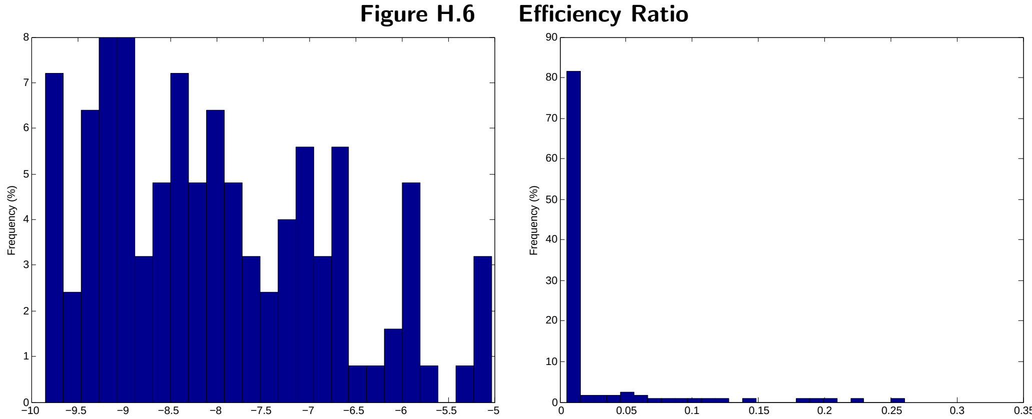


Figure H.5 Profit for Centralized System



6. The efficiency ratio: See Figure H.6.

Figure H.6 Efficiency Ratio



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