
Online Appendix

“Pricing and Matching with Forward-looking Buyers and Sellers”

The rest of proofs can be found in a supplemental note that is posted on the authors’ websites.

A. Proof of Lemma 1

The proof of Lemma (B) is facilitated by an auxiliary two-sided dynamic mechanism design problem.

A.1. An Optimal Two-sided Dynamic Mechanism Design Benchmark

In this subsection, we formally define a two-sided dynamic mechanism design problem. We restrict ourselves to direct mechanisms. A mechanism specifies a product allocation and money transfer rule that we encode as follows.

Buyer ϕ is assigned with

$$y_\phi \triangleq (\tau_\phi, a_\phi, s_\phi, m_\phi, p_\phi),$$

where $\tau_\phi \in [t_\phi, T]$ is the time that the intermediary decides on whether to accept buyer ϕ ’s demand request, $a_\phi \in \{0, 1\}$ is an indicator for whether buyer ϕ ’s demand request is accepted, s_ϕ is the time that buyer ϕ is discharged from the system, m_ϕ is an indicator for whether buyer ϕ ’s demand request is honored at time s_ϕ , and p_ϕ is the price that buyer ϕ pays to the intermediary at time s_ϕ .

Similarly, seller ψ is assigned with

$$y_\psi \triangleq (\tau_\psi, a_\psi, s_\psi, m_\psi, p_\psi),$$

where $\tau_\psi \in [t_\psi, T]$ is the time that the intermediary decides on whether to accept seller ψ ’s supply request, $a_\psi \in \{0, 1\}$ is an indicator for whether seller ψ ’s supply request is accepted, s_ψ is the time that seller ψ is discharged from the system, m_ψ is an indicator for whether seller ψ ’s supply request is honored at time s_ψ , and p_ψ is the price that the intermediary pays to seller ψ at time s_ψ .

Denote by $y^t \triangleq \{y_\phi, y_\psi : \tau_\phi \leq t, \tau_\psi \leq t\}$ the set of decisions by the intermediary made up to time t . Finally, denote the intermediary’s information set by \mathcal{H}_t , the filtration generated by the buyers’ and sellers’ arrivals up to time t and assignment decisions prior to time t , i.e., $\mathcal{H}_t = \sigma(H^t, y^{t-})$. A *feasible mechanism* satisfies the following properties:

1. (Causality on the demand side) τ_ϕ and s_ϕ are stopping times with respect to the filtration \mathcal{H}_t . Moreover, a_ϕ is \mathcal{H}_{τ_ϕ} -measurable, and m_ϕ and p_ϕ are \mathcal{H}_{s_ϕ} -measurable.²⁹
2. (Causality on the supply side) τ_ψ and s_ψ are stopping times with respect to the filtration \mathcal{H}_t . Moreover, a_ψ is \mathcal{H}_{τ_ψ} -measurable, and m_ψ and p_ψ are \mathcal{H}_{s_ψ} -measurable.³⁰
3. (Demand-supply balancing condition) At each point of time, the number of products buyers receive equals the number of products sellers deliver, i.e.,

$$\sum_{\phi \in H^t} \mathbf{1}\{s_\phi = t, m_\phi = 1\} = \sum_{\psi \in H^t} \mathbf{1}\{s_\psi = t, m_\psi = 1\}, \quad \forall t \in [0, T]. \quad (1)$$

We denote by \mathcal{Y} , the class of all feasible mechanisms y^T . The intermediary collects profit

$$\Pi(y^T) \triangleq \sum_{\phi \in H^T} p_\phi - \sum_{\psi \in H^T} p_\psi.$$

The utility garnered by buyer ϕ when she reports her true type as $\hat{\phi}$ (either truthfully or manipulatively) is then given by $U^d(\phi, y_{\hat{\phi}})$. When buyer ϕ manipulates her report on her private type, she can only reveal her arrival no earlier than her true arrival (i.e., $t_{\hat{\phi}} \geq t_\phi$).

Analogously, the utility garnered by seller ψ when she reports her true type as $\hat{\psi}$ (either truthfully or manipulatively) is then given by $U^s(\psi, y_{\hat{\psi}})$. When seller ψ manipulates her report on her private type, she can only reveal her arrival no earlier than her true arrival (i.e., $t_{\hat{\psi}} \geq t_\psi$).

The intermediary now faces the following optimization problem that seeks an optimal two-sided dynamic mechanism:

²⁹ At each time t , when the intermediary determines whether stopping times τ_ϕ and s_ϕ are equal to or greater than t , the information available to her consists of the collection of the types of all buyers and sellers who arrive up to time t and all decisions that the intermediary makes before time t . At time τ_ϕ , when the intermediary determines whether to accept buyer ϕ 's demand request, a_ϕ , the information available to the intermediary consists of the collection of the types of all buyers and sellers who arrive up to time τ_ϕ and all decisions that the intermediary makes before time τ_ϕ . At time s_ϕ , when the intermediary determines whether buyer ϕ 's demand request is honored, m_ϕ , and how much to charge buyer ϕ , p_ϕ , the information available to the intermediary consists of the collection of the types of all buyers and sellers who arrive up to time s_ϕ and all decisions that the intermediary makes before time s_ϕ .

³⁰ The interpretations of filtrations \mathcal{H}_t , \mathcal{H}_{τ_ψ} , \mathcal{H}_{s_ψ} are analogous to those on the demand side.

$$\begin{aligned}
& \max_{y^T \in \mathcal{Y}} \mathbb{E} [\Pi(y^T)] \\
& \text{s.t. } \mathbb{E}_{-\phi} [U^d(\phi, y_\phi)] \geq \mathbb{E}_{-\phi} [U^d(\phi, y_{\hat{\phi}})], \forall \phi, \hat{\phi}, \text{ s.t. } t_{\hat{\phi}} \in [t_\phi, T], \quad (\text{ICd}) \\
& \mathbb{E}_{-\phi} [U^d(\phi, y_\phi)] \geq 0, \quad \forall \phi, \quad (\text{IRd}) \quad (\text{B}') \\
& \mathbb{E}_{-\psi} [U^s(\psi, y_\psi)] \geq \mathbb{E}_{-\psi} [U^s(\psi, y_{\hat{\psi}})], \forall \psi, \hat{\psi}, \text{ s.t. } t_{\hat{\psi}} \in [t_\psi, T], \quad (\text{ICs}) \\
& \mathbb{E}_{-\psi} [U^s(\psi, y_\psi)] \geq 0, \quad \forall \psi. \quad (\text{IRs})
\end{aligned}$$

While expectation $\mathbb{E}_{-\phi}[\cdot]$ is taken with respect to $H^T \setminus \{\phi\}$, expectation $\mathbb{E}_{-\psi}[\cdot]$ is taken with respect to $H^T \setminus \{\psi\}$. Denote by J^* the optimal value obtained in problem (B'). We establish the following result that is essentially due to the revelation principle.

LEMMA 3 (BENCHMARK). *For any pricing policy $\pi \in \Pi$ and matching policy $M \in \mathcal{M}$, we have*

$$J^{\pi, M} \leq J^*.$$

Proof of Lemma 3. Consider the class of pricing and matching mechanisms, $\tilde{\mathcal{Y}}(\subset \mathcal{Y})$, each of which, say y^T , corresponds to a given pricing policy $\pi \in \Pi$ and a given matching policy $M \in \mathcal{M}$ such that $y_\phi = y_\phi^{\pi, M}$ and $y_\psi = y_\psi^{\pi, M}$. Now consider the optimization problem:

$$\begin{aligned}
& \max_{y^T \in \tilde{\mathcal{Y}}} \mathbb{E} [\Pi(y^T)] \\
& \text{s.t. } \mathbb{E}_{-\phi} \left[U^d(\phi, y_\phi) | \pi_{t_\phi}^d, \mathcal{N}_{t_\phi}^s \right] \geq \mathbb{E}_{-\phi} \left[U^d(\phi, y_{\hat{\phi}}) | \pi_{t_\phi}^d, \mathcal{N}_{t_\phi}^s \right], \forall \phi, \hat{\phi}, \text{ s.t. } t_{\hat{\phi}} \in [t_\phi, T], \quad (\text{IC-d}) \\
& \mathbb{E}_{-\phi} \left[U^d(\phi, y_\phi) | \pi_{t_\phi}^d, \mathcal{N}_{t_\phi}^s \right] \geq 0, \quad \forall \phi, \quad (\text{IR-d}) \quad (2) \\
& \mathbb{E}_{-\psi} \left[U^s(\psi, y_\psi) | \pi_{t_\psi}^s \right] \geq \mathbb{E}_{-\psi} \left[U^s(\psi, y_{\hat{\psi}}) | \pi_{t_\psi}^s \right], \forall \psi, \hat{\psi}, \text{ s.t. } t_{\hat{\psi}} \in [t_\psi, T], \quad (\text{IC-s}) \\
& \mathbb{E}_{-\psi} \left[U^s(\psi, y_\psi) | \pi_{t_\psi}^s \right] \geq 0, \quad \forall \psi. \quad (\text{IR-s})
\end{aligned}$$

Denote by J^{π^*, M^*} the optimal value for problem (2). For any pricing policy $\pi \in \Pi$ and matching policy $M \in \mathcal{M}$, we must have $J^{\pi, M} \leq J^{\pi^*, M^*}$. This is because, given a pricing policy $\pi \in \Pi$ and a matching policy $M \in \mathcal{M}$, its corresponding mechanism in $\tilde{\mathcal{Y}}$ is feasible. Consider the mechanism y^T where the intermediary commits to “simulating” each buyer’s stopping and purchasing rules (i.e., $(\tau_\phi, a_\phi) = (\tau_\phi^{\pi, M}, a_\phi^{\pi, M})$ for each buyer ϕ) and each seller’s stopping and selling rules (i.e., $(\tau_\psi, a_\psi) = (\tau_\psi^{\pi, M}, a_\psi^{\pi, M})$ for each seller ψ). Since by definition $(\tau_\phi^{\pi, M}, a_\phi^{\pi, M})$ is buyer ϕ ’s best response to herself and the pricing policy π and the matching policy M , (IR-d) and (IC-d) are satisfied in problem (2). Similarly, since by definition $(\tau_\psi^{\pi, M}, a_\psi^{\pi, M})$ is seller ψ ’s best response to herself and the

pricing policy π and the matching policy M , (IR-s) and (IC-s) are satisfied in problem (2). As a result, we verify that $J^{\pi, M} \leq J^{\pi^*, M^*}$ for any π and M .

Moreover, problem (B') is a relaxation of problem (2) in that a) problem (B') requires the constraints to be satisfied in expectation, while problem (2) requires the constraints to be satisfied along every sample path; b) problem (B') optimizes over a larger set than problem (2), since $\tilde{\mathcal{Y}} \subset \mathcal{Y}$. Hence, $J^{\pi^*, M^*} \leq J^*$. Together with $J^{\pi, M} \leq J^{\pi^*, M^*}$, we have $J^{\pi, M} \leq J^*$ for any π and M . \square

A.2. Upper Bound on the Mechanism Design Benchmark

Recall that every buyer's and seller's type is two dimensional. Therefore, we still face the analytical difficulty of computing the optimal value of the two-sided dynamic mechanism design problem (B'). In order to use this benchmark to analyze the performance of any pricing and matching policy, rather than precisely computing the optimal value of this benchmark, our approach is to further establish a tractable upper bound by solving a *relaxed* two-sided dynamic mechanism design problem wherein every buyer's and seller's arrival time is assumed to be known by the intermediary. Hence, in the relaxed problem, every buyer's and seller's type is reduced to be one dimensional, i.e., the relaxed problem is subject to the following one-dimensional incentive compatibility constraints:

$$\mathbb{E}_{-\phi} [U^d(\phi, y_\phi)] \geq \mathbb{E}_{-\phi} [U^d(\phi, y_{\phi_{v'}})] \text{ with } \phi_{v'} \triangleq (t_\phi, v'), \forall \phi, v', \quad (\text{ICd}')$$

$$\mathbb{E}_{-\psi} [U^s(\psi, y_\psi)] \geq \mathbb{E}_{-\psi} [U^s(\psi, y_{\psi_{c'}})] \text{ with } \psi_{c'} \triangleq (t_\psi, c'), \forall \psi, c'. \quad (\text{ICs}')$$

By applying the Myersonian approach (Myerson 1981) to this relaxed mechanism design problem and making further relaxations, we can complete the proof of Lemma 1.

Proof of Lemma 1. Consider the following optimization problem with the optimal value denoted by \bar{J}^1 :

$$\begin{aligned} \max_{\{y^T \in \mathcal{Y}\}} \quad & \mathbb{E} \left[\sum_{\phi, \psi \in H^T} V^d(v_\phi) - V^s(c_\psi) - b(s_\phi - t_\phi) - h(s_\psi - t_\psi) \right] \\ \text{s.t.} \quad & \sum_{\phi \in H^t} \mathbf{1}\{s_\phi = t, m_\phi = 1\} = \sum_{\psi \in H^t} \mathbf{1}\{s_\psi = t, m_\psi = 1\}, \quad \forall t \in [0, T]. \end{aligned} \quad (\text{B}'')$$

Combining Lemmas S.2 and S.4, we immediately have $J^* \leq \bar{J}^1$.

Now, we consider an optimization problem that has the same definition as (B'') except that we assume that the intermediary is clairvoyant that she knows buyers' and sellers' arrival processes

H^T at time 0, i.e., the intermediary's every feasible mechanism y^T is adapted to $\sigma(H^T)$. We denote by \bar{J}^2 the optimal value of this new optimization problem. Hence, we have $\bar{J}^1 \leq \bar{J}^2$.

Note that if buyer ϕ and seller ψ are matched, then we have $s_\phi, s_\psi \geq \max\{t_\phi, t_\psi\}$. Therefore, we have $\bar{J}^2 \leq \mathbb{E}[\bar{J}(H^T)]$.

Therefore, all analyses above jointly imply

$$J^* \leq \bar{J}^1 \leq \bar{J}^2 \leq \mathbb{E}[\bar{J}(H^T)].$$

Therefore, this result and Lemma 3 jointly complete the proof of this lemma. \square

B. The Effects of Market Conditions on Pricing Heuristic

First, we explore the effects of the variation of the buyer valuation distribution and the seller cost distribution on prices p^* and w^* , as well as the intermediary's matching quantity μ^* and her profit \bar{J}^* .

THEOREM 4 (COMPARATIVE STATICS ON VALUATION/COST DISTRIBUTION). *Consider a market wherein $\lambda^d T = \lambda^s T = 1$, buyers' valuations are uniformly distributed on $[V - \theta^d, V + \theta^d]$ with $\theta^d \geq 0$ and sellers' costs are uniformly distributed on $[C - \theta^s, C + \theta^s]$ with $\theta^s \geq 0$ and $C < V$. Then,*

- (i) p^* is decreasing in $\theta^d \in \left[0, \max\left\{\left(\frac{V-C}{3} - \theta^s\right)^+, \left(\sqrt{(V-C)\theta^s} - \theta^s\right)^+\right\}\right]$ and increasing in $\theta^d \geq \max\left\{\left(\frac{V-C}{3} - \theta^s\right)^+, \left(\sqrt{(V-C)\theta^s} - \theta^s\right)^+\right\}$. In addition, p^* is increasing in θ^s .
- (ii) w^* is increasing in $\theta^s \in \left[0, \max\left\{\left(\frac{V-C}{3} - \theta^d\right)^+, \left(\sqrt{(V-C)\theta^d} - \theta^d\right)^+\right\}\right]$ and decreasing in $\theta^s \geq \max\left\{\left(\frac{V-C}{3} - \theta^d\right)^+, \left(\sqrt{(V-C)\theta^d} - \theta^d\right)^+\right\}$. In addition, w^* is decreasing in θ^d .
- (iii) μ^* is decreasing in $\theta^d + \theta^s$.
- (iv) \bar{J}^* is decreasing in $\theta^d + \theta^s \in [0, V - C]$ and increasing in $\theta^d + \theta^s \geq V - C$.

Parameters θ^d and θ^s measure the degree of variations of buyers' valuations and sellers' costs, respectively. Theorem 4(i) has the following interpretations. First, consider the scenario that buyers' valuations are concentrated (θ^d is small). Suppose buyers' valuations become slightly more dispersed (θ^d increases). Because buyers' valuations are still highly concentrated, if the intermediary charges buyers more, then she fails to capture a large number of buyers and thus she loses substantial revenue from buyers. However, if the intermediary slightly lowers the price for buyers, then she can capture a vast number of buyers without losing too much revenue from each matched buyer. Therefore, the intermediary tends to charge buyers less. Second, consider the scenario that

buyers' valuations are dispersed (θ^d is large). Suppose buyers' valuations become even more dispersed (θ^d increases). If the intermediary charges buyers less, then she both loses revenue from each matched buyer and fails to attract substantially more buyers to purchase because of highly dispersed buyers' valuations. However, if the intermediary slightly raises the price for buyers, then she can collect more revenue from each matched buyer without losing too many buyers who decide not to buy. Therefore, the intermediary tends to charge buyers more. Third, consider the scenario that sellers' costs become more dispersed (θ^s increases). Suppose the intermediary charges buyers less. Then more buyers consider to purchase the product. To achieve the market clearance, the intermediary has to incentivize more sellers to sell the product. Because sellers' costs are more dispersed, the intermediary has to substantially increase the compensation for sellers. As a result, the net profit from matching one buyer and one seller is too low or even negative. Therefore, the intermediary charges buyers more when sellers' costs are more dispersed.

We can make analogous interpretations for Theorem 4(ii) about the effects of buyer valuation variation and seller cost variation on the wage that the intermediary compensates sellers.

In Theorem 4(iii), if the sellers' costs become more dispersed, then following from Theorem 4(i), the intermediary raises the price for buyers. As a result, the number of buyers who are matched reduces. An analogous interpretation can be made if buyers' valuations become more dispersed.

Theorem 4(iv) has the following interpretations. First, consider the scenario that buyers' valuations are well concentrated. Suppose buyers' valuations become slightly more dispersed (θ^d increases). Because buyers' valuations are still well concentrated, slightly reducing the price for buyers can maintain almost the same number of matches. However, the intermediary is less profitable from each match by doing so. Therefore, the intermediary's total profit decreases. Second, consider the scenario that buyers' valuations are highly dispersed. Suppose buyers' valuations become even more dispersed (θ^d increases). Although the total number of matches decreases, it allows the intermediary to set a sufficiently high price to attract only high-value buyers to purchase and set a sufficiently low compensation to attract only low-cost sellers to sell. Therefore, the benefit from a higher net profit from each match outperforms the reduced number of matches. Therefore, the intermediary's total profit increases. The analogous arguments can be made on the effect of the seller cost variation.

Second, we explore the effects of the buyer valuation distribution (characterized by $F^d(\cdot)$) and the seller cost distribution (characterized by $F^s(\cdot)$) on prices p^* and w^* , as well as the intermediary's matching quantity μ^* and her profit \bar{J}^* .

THEOREM 5 (COMPARATIVE STATICS ON VALUATION/COST DISTRIBUTION).

(i) Consider a family of buyer valuation distribution functions $F_\theta^d(\cdot)$ parameterized by $\theta > 0$, where $F_{\theta_1}^d(\theta_1 v) = F_{\theta_2}^d(\theta_2 v)$ for all $v \in [0, \bar{v}]$ and $\theta_1, \theta_2 > 0$. Assume the generalized failure rate $\frac{v f_\theta^d(v)}{F_\theta^d(v)}$ is increasing in v . Then, p^* and w^* are increasing in θ . Moreover, μ^* and \bar{J}^* are increasing in θ .

(ii) Consider a family of seller cost distribution functions $F_\theta^s(\cdot)$ parameterized by $\theta > 0$, where $F_{\theta_1}^s(\theta_1 c) = F_{\theta_2}^s(\theta_2 c)$ for all $c \in [0, \bar{c}]$ and $\theta_1, \theta_2 > 0$. Assume $\frac{c f_\theta^s(c)}{F_\theta^s(c)}$ is decreasing in c . Then, p^* and w^* are increasing in θ . Moreover, μ^* and \bar{J}^* are decreasing in θ .

In Theorem 5(i), the condition that $\frac{v f_\theta^d(v)}{F_\theta^d(v)}$ is increasing is the commonly used Increasing Generalized Failure Rate (IGFR) condition. The condition $F_{\theta_1}^d(\theta_1 v) = F_{\theta_2}^d(\theta_2 v)$ for all $v \in [0, \bar{v}]$ is satisfied by many commonly used distributions, e.g., the family of uniform distributions, Uniform $[0, \theta]$, and Weibull and Gamma distributions, Weibull, Gamma (α, θ) . This condition $F_{\theta_1}^d(\theta_1 v) = F_{\theta_2}^d(\theta_2 v)$ implies that as θ increases, a buyer is more likely to have a higher product valuation. Therefore, when θ increases, on the demand side, the intermediary can raise up the price for buyers, p^* , to a certain level, such that she can both collect a higher revenue from each buyer and attract more buyers to request the product (μ^* increases). Meanwhile, in order to match the increasing number of demand requests, on the supply side, the intermediary increases payments to sellers (w^* increases) to encourage more sellers to deliver the product. As θ increases, since buyer product valuations are more likely to be high and the intermediary matches more pairs of buyers and sellers (μ^* increases), the intermediary makes a higher profit (\bar{J}^* increases).

In Theorem 5(ii), the condition that $\frac{c f_\theta^s(c)}{F_\theta^s(c)}$ is non-increasing in c is an analogous condition on the supply side to the IGFR condition assumed for the demand side. The condition $F_{\theta_1}^s(\theta_1 c) = F_{\theta_2}^s(\theta_2 c)$ implies that as θ decreases, a seller is more likely to incur a lower product production and delivery cost. Therefore, when θ decreases, on the supply side, the intermediary cuts down the price for sellers, w^* , to a certain level, such that she can both pay less to each seller and attract more sellers to deliver the product (μ^* increases). Meanwhile, in order to match the increasing number of supply requests, on the demand side, the intermediary reduces the price for buyers (p^* decreases) to encourage more buyers to request the product. As θ decreases, since seller production and delivery costs are more likely to be low and the intermediary matches more pairs of buyers and sellers (μ^* increases), the intermediary makes a higher profit (\bar{J}^* increases).

In summary, in a thick market with large volumes of demand and supply (for which the fluid approximation tends to apply), while both a stochastically larger willingness-to-pay distribution and a smaller willingness-to-sell distribution tend to result in a higher total matching quantity and a higher total profit level for the intermediary, their implications on prices can go either way depending on whether the change is more favorable to the buy or the sell side.